4

APPLICATIONS OF DIFFERENTIATION

APPLICATIONS OF DIFFERENTIATION

The methods we have learned in this chapter for finding extreme values have practical applications in many areas of life.

- A businessperson wants to minimize costs and maximize profits.
- A traveler wants to minimize transportation time.
- Fermat’s Principle in optics states that light follows the path that takes the least time.

4.7

Optimization Problems

In this section, we will learn:

How to solve problems involving maximization and minimization of factors.

OPTIMIZATION PROBLEMS

In this section (and the next), we solve such problems as:

- Maximizing areas, volumes, and profits
- Minimizing distances, times, and costs

OPTIMIZATION PROBLEMS

In solving such practical problems, the greatest challenge is often to convert the word problem into a mathematical optimization problem—by setting up the function that is to be maximized or minimized.

Let’s recall the problem-solving principles discussed in Chapter 1 and adapt them to this situation.

STEPS IN SOLVING OPTIMIZATION PROBLEMS

Thus, there are six steps involved in solving optimization problems.

These are as follows.

1. UNDERSTAND THE PROBLEM

Read the problem carefully until it is clearly understood.

Ask yourself:

- What is the unknown?
- What are the given quantities?
- What are the given conditions?

2. DRAW A DIAGRAM

In most problems, it is useful to draw a diagram and identify the given and required quantities on the diagram.

3. INTRODUCE NOTATION

Assign a symbol to the quantity that is to be maximized or minimized.

- Let's call it *Q* for now.

3. INTRODUCE NOTATION

Also, select symbols (*a*, *b*, *c*, . . . , *x*, *y*) for other unknown quantities and label the diagram with these symbols.

- It may help to use initials as suggestive symbols.
- Some examples are: *A* for area, *h* for height, and *t* for time.

4. EXPRESS *Q* IN TERMS OF THE VARIABLES

Express *Q* in terms of some of the other symbols from Step 3.

5. EXPRESS *Q* IN TERMS OF ONE VARIABLE

If *Q* has been expressed as a function of more than one variable in Step 4, use the given information to find relationships—in the form of equations—among these variables.

Then, use the equations to eliminate all but one variable in the expression for *Q*.

5. EXPRESS *Q* IN TERMS OF ONE VARIABLE

Thus, Q will be expressed as

a function of one variable x, say,

$Q = f(x)$.

- Write the domain of this function.

6. FIND THE ABSOLUTE MAX./MIN. VALUE OF *f*

Use the methods of Sections 4.1 and 4.3 to find the absolute maximum or minimum value of *f*.

- In particular, if the domain of *f* is a closed interval, then the Closed Interval Method in Section 4.1 can be used.

Example 1

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river.

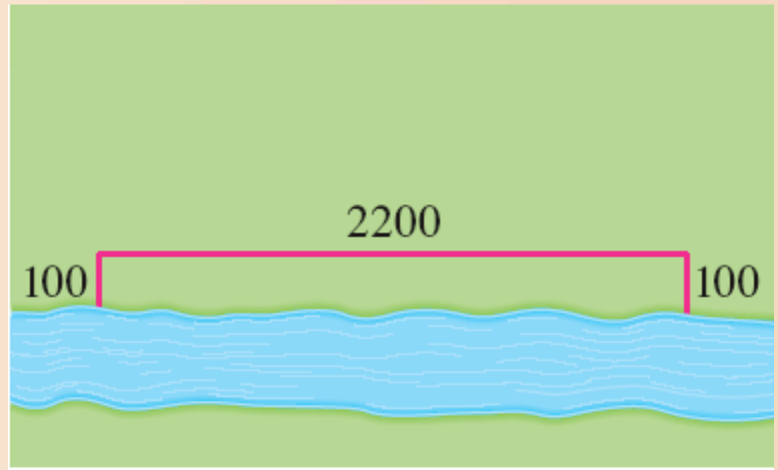
- What are the dimensions of the field that has the largest area?

In order to get a feeling for what is happening in the problem, let's experiment with some special cases.

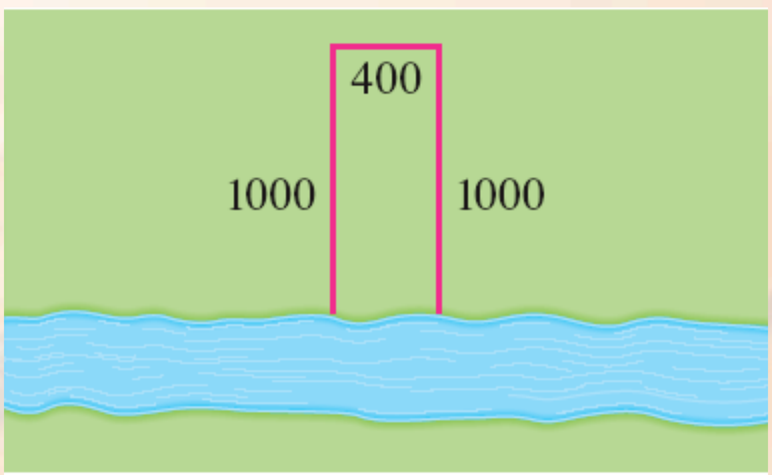
OPTIMIZATION PROBLEMS

Example 1

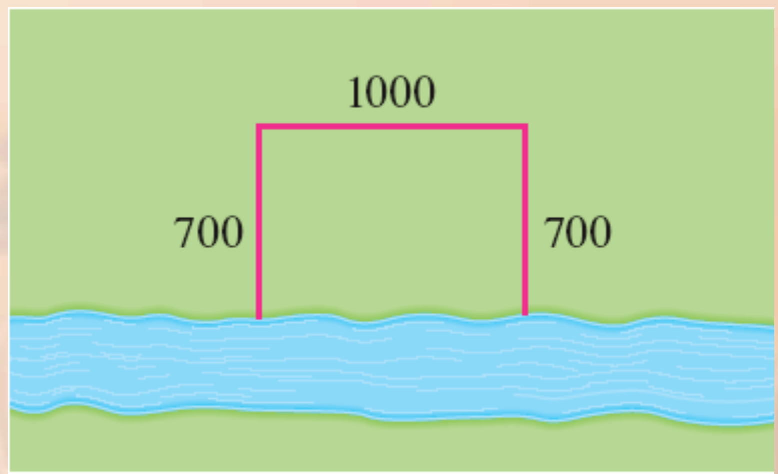
Here are three possible ways of laying out the 2400 ft of fencing.



Area = $100 \cdot 2200 = 220{,}000 \text{ ft}^2$



Area = $1000 \cdot 400 = 400{,}000 \text{ ft}^2$



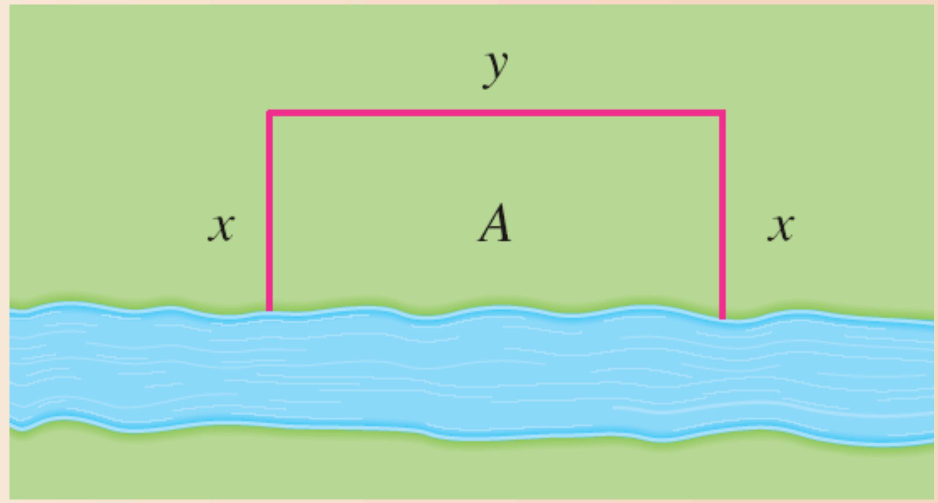
Area = $700 \cdot 1000 = 700{,}000 \text{ ft}^2$

We see that when we try shallow, wide fields or deep, narrow fields, we get relatively small areas.

- It seems plausible that there is some intermediate configuration that produces the largest area.

Example 1

This figure illustrates the general case.



We wish to maximize the area *A* of the rectangle.

- Let *x* and *y* be the depth and width of the rectangle (in feet).
- Then, we express *A* in terms of *x* and *y*: $A = xy$

We want to express A as a function of just one variable.

- So, we eliminate y by expressing it in terms of x.
- To do this, we use the given information that the total length of the fencing is 2400 ft.
- Thus, $2x + y = 2400$

From that equation, we have:

$$y = 2400 - 2x$$

This gives:

$$A = x(2400 - 2x) = 2400x - 2x^2$$

- Note that $x \geq 0$ and $x \leq 1200$ (otherwise $A < 0$).

So, the function that we wish to maximize is: $A(x) = 2400x - 2x^2 \qquad 0 \leq x \leq 1200$

- The derivative is: $A'(x) = 2400 - 4x$
- So, to find the critical numbers, we solve: $2400 - 4x = 0$
- This gives: $x = 600$

The maximum value of A must occur either at that critical number or at an endpoint of the interval.

- $A(0) = 0$; $A(600) = 720{,}000$; and $A(1200) = 0$
- So, the Closed Interval Method gives the maximum value as:

$$A(600) = 720{,}000$$

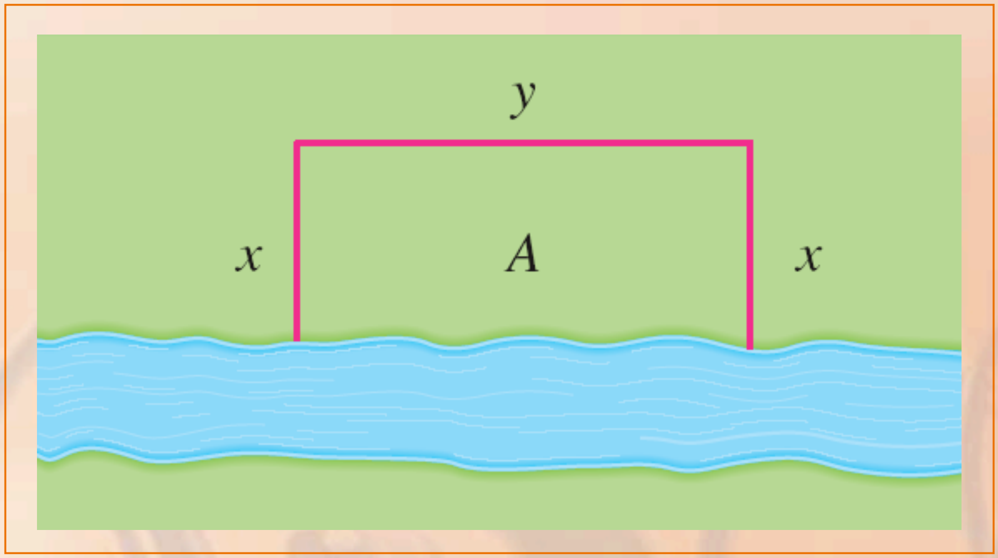
Alternatively, we could have observed that

$$A''(x) = -4 < 0 \text{ for all } x$$

So, A is always concave downward and the local maximum at $x = 600$ must be an absolute maximum.

Thus, the rectangular field should be:

- 600 ft deep
- 1200 ft wide



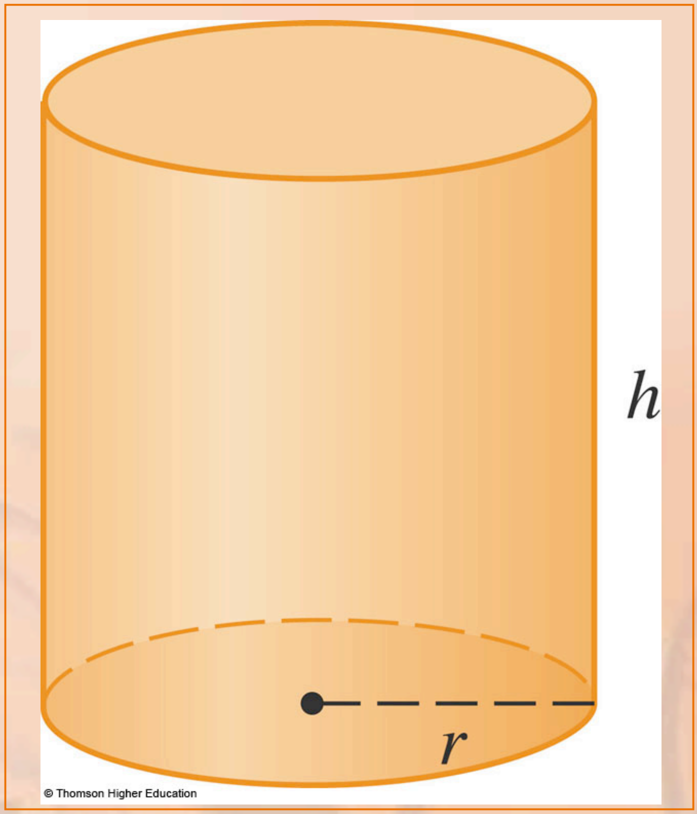
A cylindrical can is to be made to hold 1 L of oil.

- Find the dimensions that will minimize the cost of the metal to manufacture the can.

OPTIMIZATION PROBLEMS

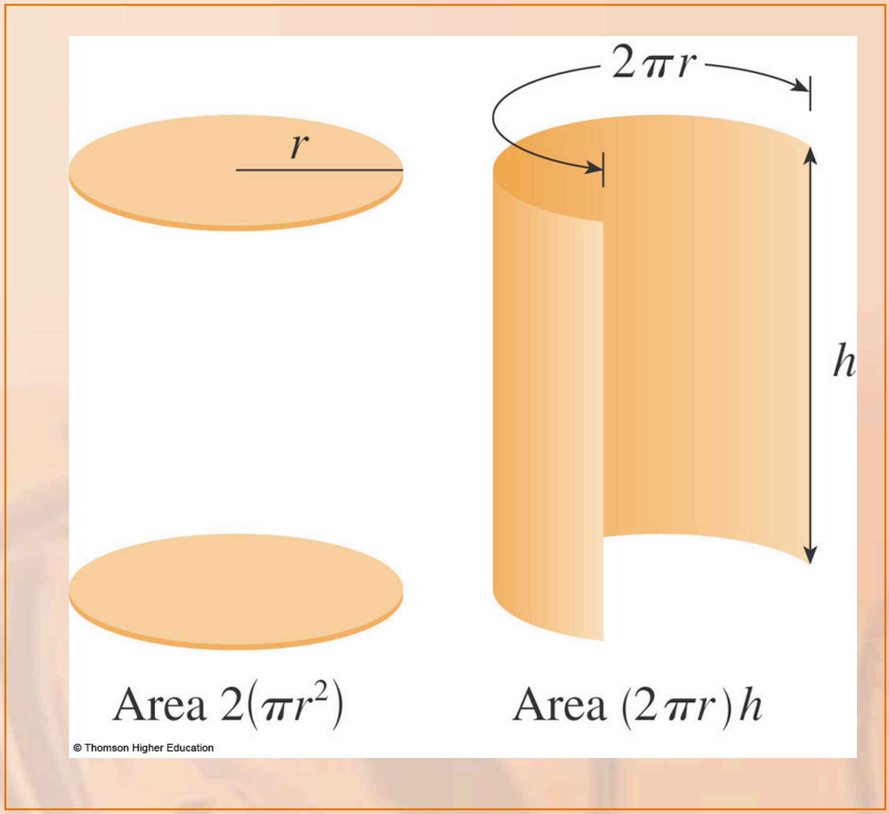
Example 2

Draw the diagram as in this figure, where *r* is the radius and *h* the height (both in centimeters).

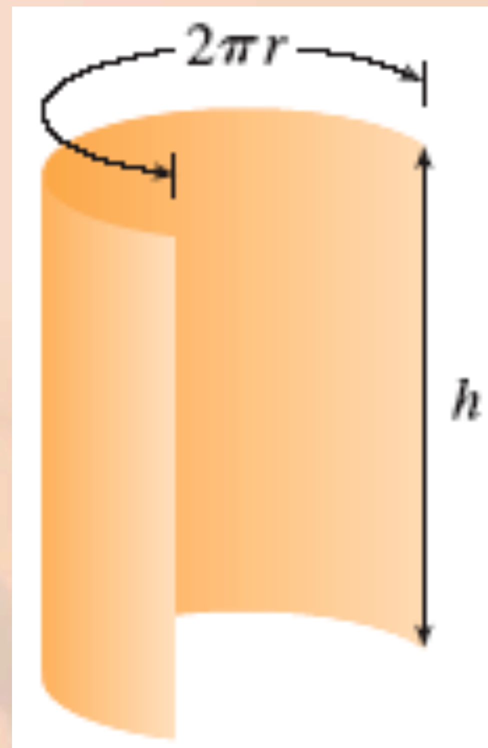


Example 2

To minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides.)



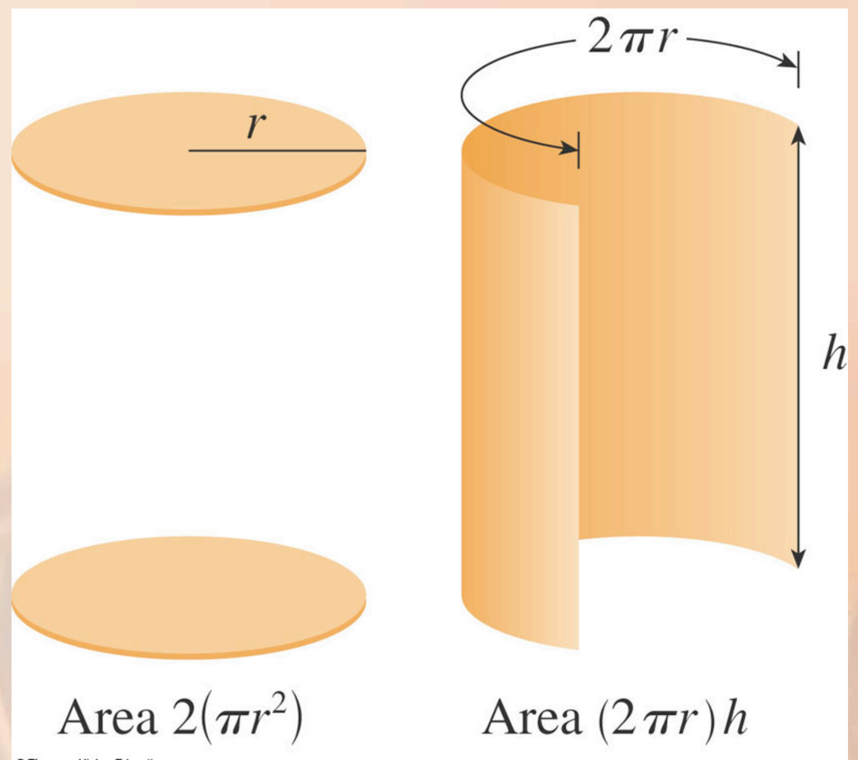
We see that the sides are made from a rectangular sheet with dimensions $2\pi r$ and h.



Example 2

So, the surface area is:

$$A = 2\pi r^2 + 2\pi rh$$



To eliminate h, we use the fact that the volume is given as 1 L, which we take to be 1000 cm^3.

- Thus, $\pi r^2 h = 1000$
- This gives $h = 1000/(\pi r^2)$

Substituting this in the expression for *A* gives:

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

So, the function that we want to minimize is:

$$A(r) = 2\pi r^2 + \frac{2000}{r} \qquad r > 0$$

To find the critical numbers, we differentiate:

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

- Then, $A'(r) = 0$ when $\pi r^3 = 500$
- So, the only critical number is: $r = \sqrt[3]{500/\pi}$

As the domain of *A* is $(0, \infty)$, we can't use the argument of Example 1 concerning endpoints.

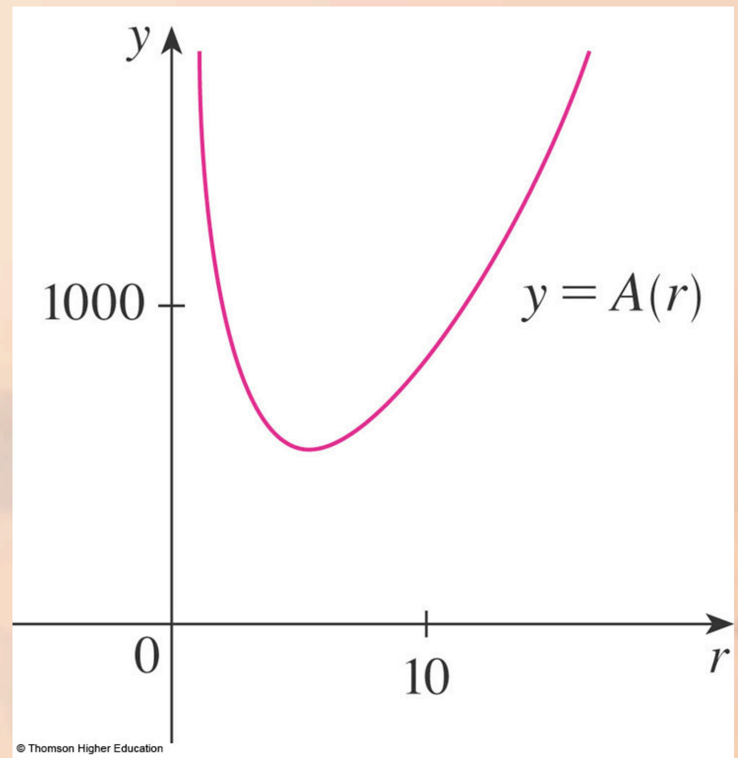
- However, we can observe that $A'(r) < 0$ for $r < \sqrt[3]{500/\pi}$ and $A'(r) > 0$ for $r > \sqrt[3]{500/\pi}$
- So, *A* is decreasing for all *r* to the left of the critical number and increasing for all *r* to the right.
- Thus, $r = \sqrt[3]{500/\pi}$ must give rise to an absolute minimum.

OPTIMIZATION PROBLEMS

Example 2

Alternatively, we could argue that $A(r) \to \infty$ as $r \to 0^+$ and $A(r) \to \infty$ as $r \to \infty$.

- So, there must be a minimum value of $A(r)$, which must occur at the critical number.



The value of *h* corresponding to $r = \sqrt[3]{500/\pi}$ is:

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi (500/\pi)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$

Thus, to minimize the cost of the can,

- The radius should be $r = \sqrt[3]{500/\pi}$ cm
- The height should be equal to twice the radius—namely, the diameter

Note 1

The argument used in the example to justify the absolute minimum is a variant of the First Derivative Test—which applies only to local maximum or minimum values.

- It is stated next for future reference.

FIRST DERIV. TEST FOR ABSOLUTE EXTREME VALUES

Suppose that c is a critical number of a continuous function f defined on an interval.

- If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

- If $f'(x) < 0$ for all $x < c$ and if $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

Note 2

An alternative method for solving optimization problems is to use implicit differentiation.

- Let's look at the example again to illustrate the method.

We work with the same equations

$$A = 2\pi r^2 + 2\pi rh \qquad \pi r^2 h = 100$$

- However, instead of eliminating h, we differentiate both equations implicitly with respect to r :

$$A' = 4\pi r + 2\pi h + 2\pi rh' \qquad 2\pi rh + \pi r^2 h' = 0$$

The minimum occurs at a critical number.

- So, we set $A' = 0$, simplify, and arrive at the equations

$$2r + h + rh' = 0 \qquad 2h + rh' = 0$$

- Subtraction gives: $2r - h = 0$ or $h = 2r$

Example 3

Find the point on the parabola

$$y^2 = 2x$$

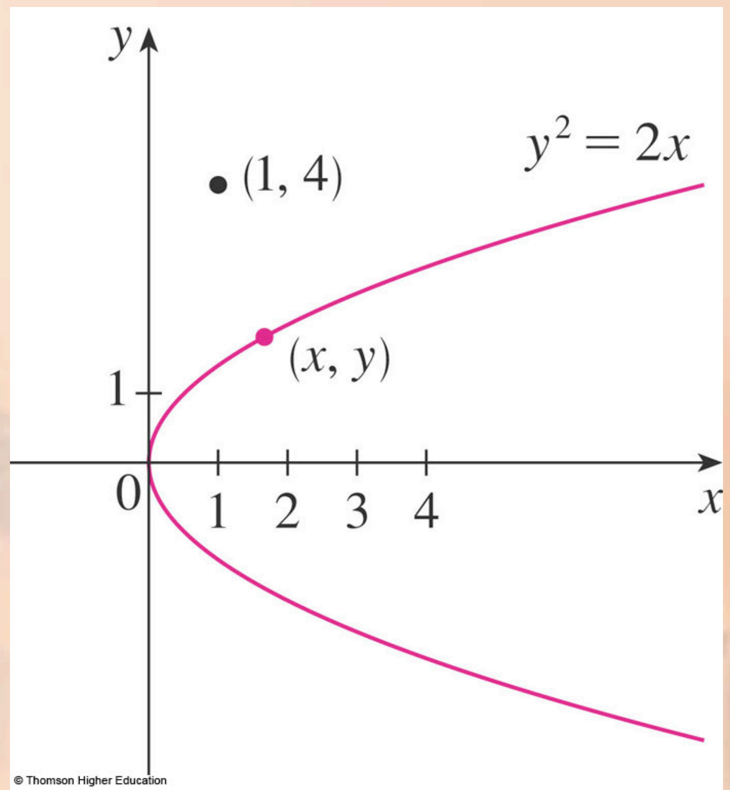
that is closest to the point (1, 4).

OPTIMIZATION PROBLEMS

Example 3

The distance between the point (1, 4) and the point (*x*, *y*) is:

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$



- However, if (x, y) lies on the parabola, then $x = \frac{1}{2} y^2$.
- So, the expression for d becomes:

$$d = \sqrt{(\tfrac{1}{2}y^2 - 1)^2 + (y-4)^2}$$

Alternatively, we could have substituted $y=\sqrt{2x}$ to get *d* in terms of *x* alone.

Instead of minimizing *d*, we minimize its square:

$$d^2 = f(y) = \left(\tfrac{1}{2}y^2 - 1\right)^2 + (y-4)^2$$

- You should convince yourself that the minimum of *d* occurs at the same point as the minimum of d^2.
- However, d^2 is easier to work with.

Differentiating, we obtain:

$$f'(y) = 2\left(\tfrac{1}{2}y^2 - 1\right)y + 2(y-4) = y^3 - 8$$

So, $f'(y) = 0$ when $y = 2$.

Observe that $f'(y) < 0$ when $y < 2$ and $f'(y) > 0$ when $y > 2$.

So, by the First Derivative Test for Absolute Extreme Values, the absolute minimum occurs when $y = 2$.

- Alternatively, we could simply say that, due to the geometric nature of the problem, it's obvious that there is a closest point but not a farthest point.

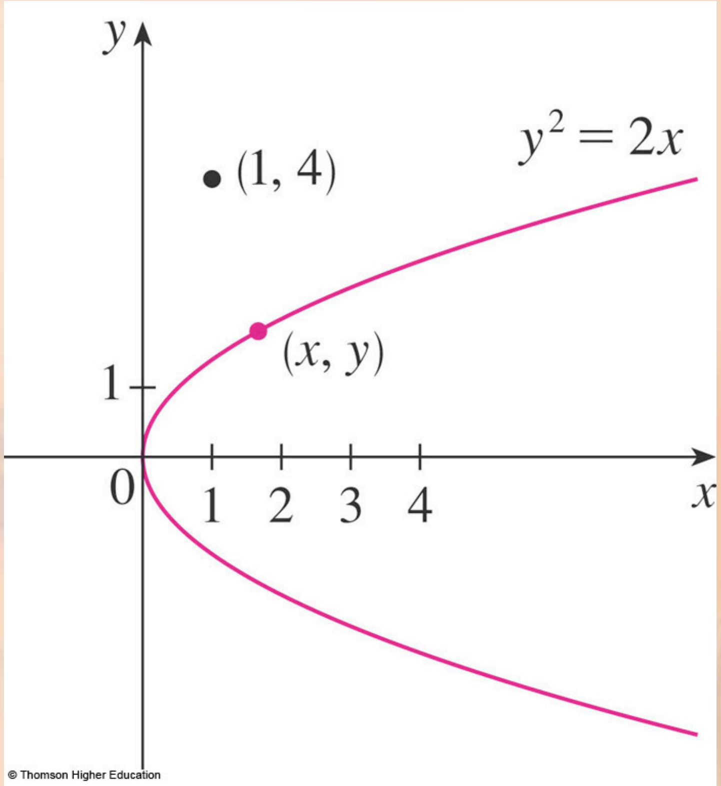
OPTIMIZATION PROBLEMS

Example 3

The corresponding value of x is:

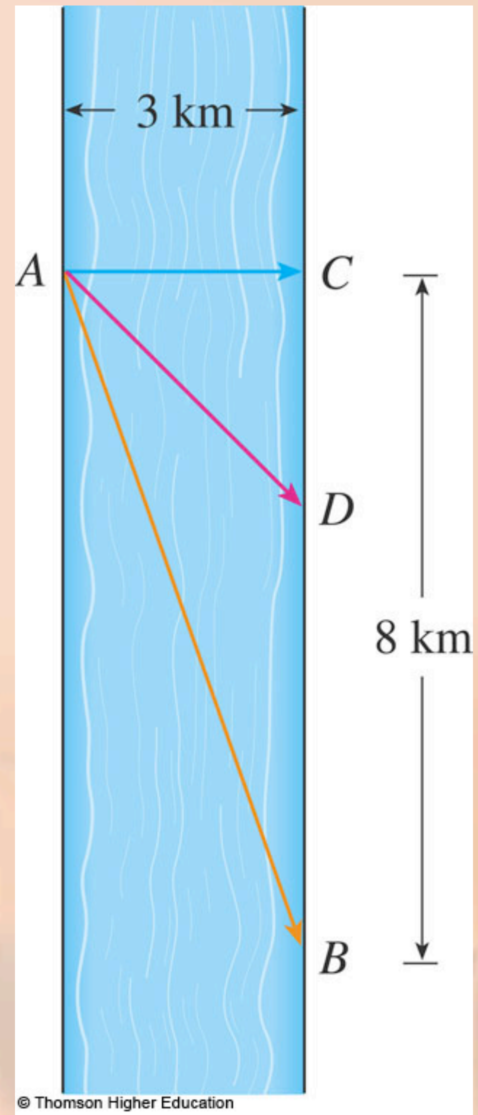
$$x = \tfrac{1}{2} y^2 = 2$$

Thus, the point on $y^2 = 2x$ closest to (1, 4) is (2, 2).



Example 4

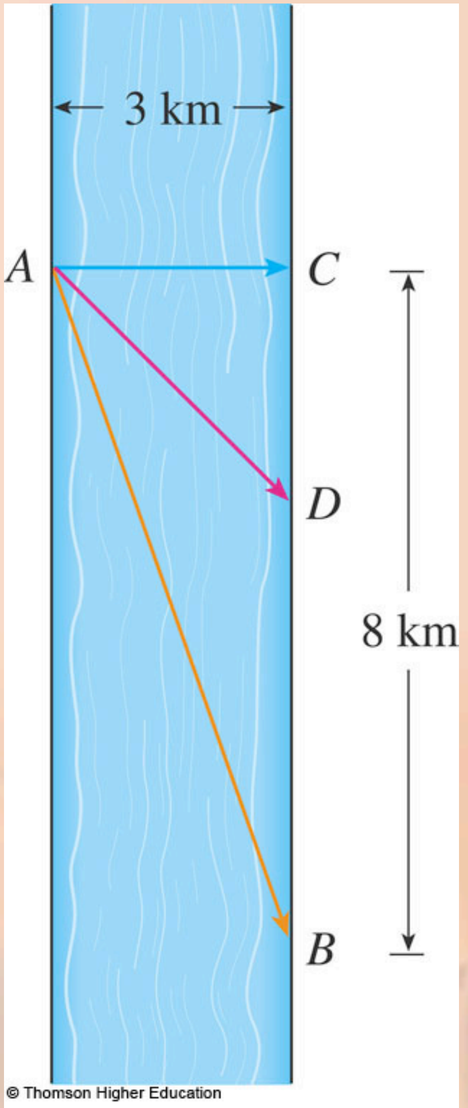
A man launches his boat from point *A* on a bank of a straight river, 3 km wide,
and wants to reach point *B*
(8 km downstream on the opposite bank) as quickly as possible.



Example 4

He could proceed in any
of three ways:

- Row his boat directly across the river to point *C* and then run to *B*
- Row directly to *B*
- Row to some point *D* between *C* and *B* and then run to *B*

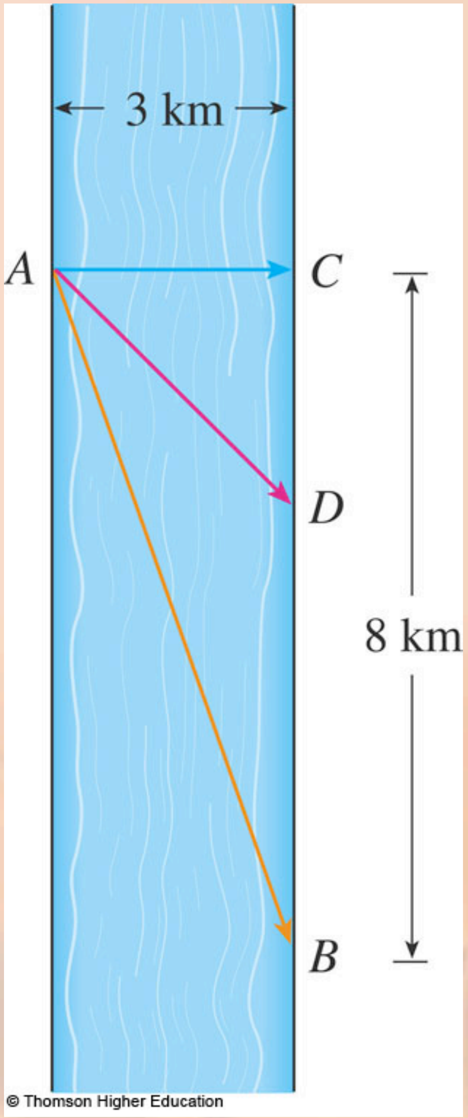


OPTIMIZATION PROBLEMS

Example 4

If he can row 6 km/h and run 8 km/h, where should he land to reach *B* as soon as possible?

- We assume that the speed of
  the water is negligible compared with the speed at which he rows.



If we let x be the distance from C to D, then:

- The running distance is: $|DB| = 8 - x$
- The Pythagorean Theorem gives the rowing distance as: $|AD| = \sqrt{x^2 + 9}$

We use the equation $\text{time} = \dfrac{\text{distance}}{\text{rate}}$

- Then, the rowing time is: $\sqrt{x^2+9}/6$
- The running time is: (8 – *x*)/8
- So, the total time *T* as a function of *x* is:

$$T(x) = \frac{\sqrt{x^2+9}}{6} + \frac{8-x}{8}$$

The domain of this function T is [0, 8].

- Notice that if $x = 0$, he rows to C, and if $x = 8$, he rows directly to B.

- The derivative of T is: $T'(x) = \dfrac{x}{6\sqrt{x^2+9}} - \dfrac{1}{8}$

Thus, using the fact that $x \geq 0$,

we have: $$T'(x) = 0 \Leftrightarrow \frac{x}{6\sqrt{x^2+9}} = \frac{1}{8}$$

$$\Leftrightarrow 4x = 3\sqrt{x^2+9}$$

$$\Leftrightarrow 16x^2 = 9(x^2+9)$$

$$\Leftrightarrow 7x^2 = 81 \Leftrightarrow x = \frac{9}{\sqrt{7}}$$

- The only critical number is: $9/\sqrt{7}$

To see whether the minimum occurs at this critical number or at an endpoint of the domain [0, 8], we evaluate *T* at all three points:

$$T(0) = 1.5$$

$$T\left(\frac{9}{\sqrt{7}}\right) = 1 + \frac{\sqrt{7}}{8} \approx 1.33$$

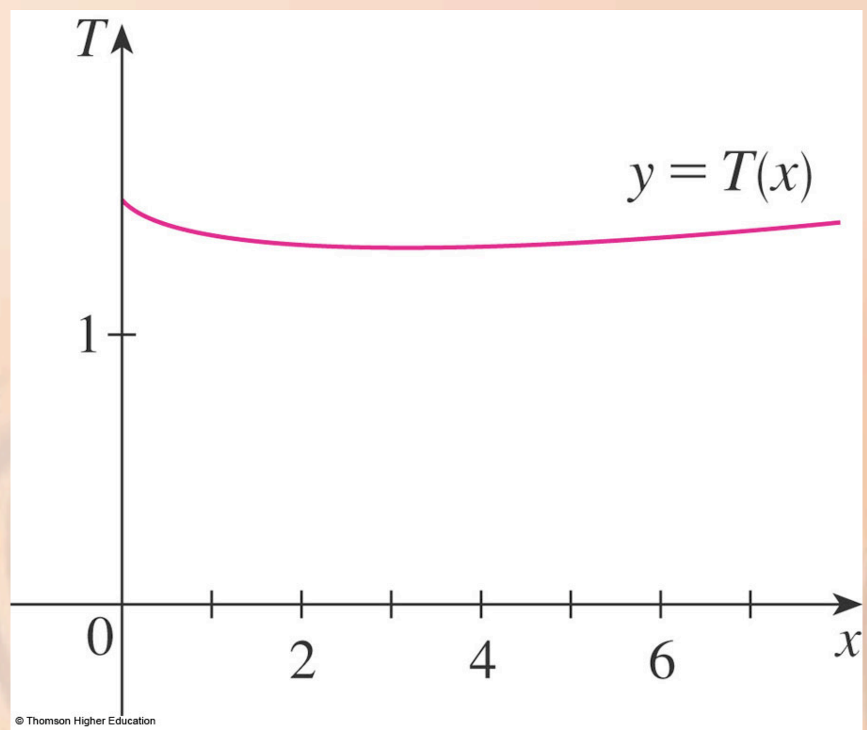
$$T(8) = \frac{\sqrt{73}}{6} \approx 1.42$$

OPTIMIZATION PROBLEMS

Example 4

Since the smallest of these values of *T* occurs when $x = 9/\sqrt{7}$, the absolute minimum value of *T* must occur there.

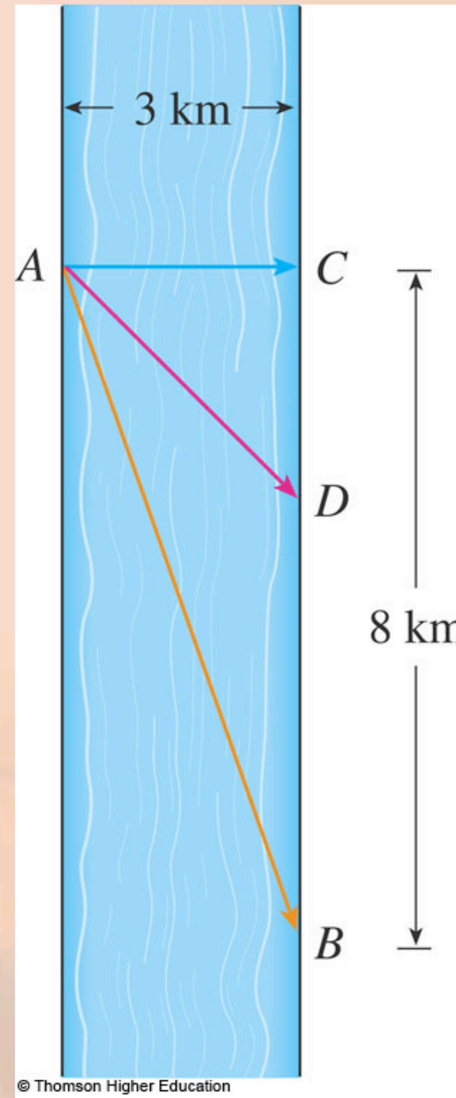
- The figure illustrates this calculation by showing the graph of *T*.



OPTIMIZATION PROBLEMS

Example 4

Thus, the man should land the boat at a point $9/\sqrt{7}$ (≈ 3.4 km) downstream from his starting point.



Example 5

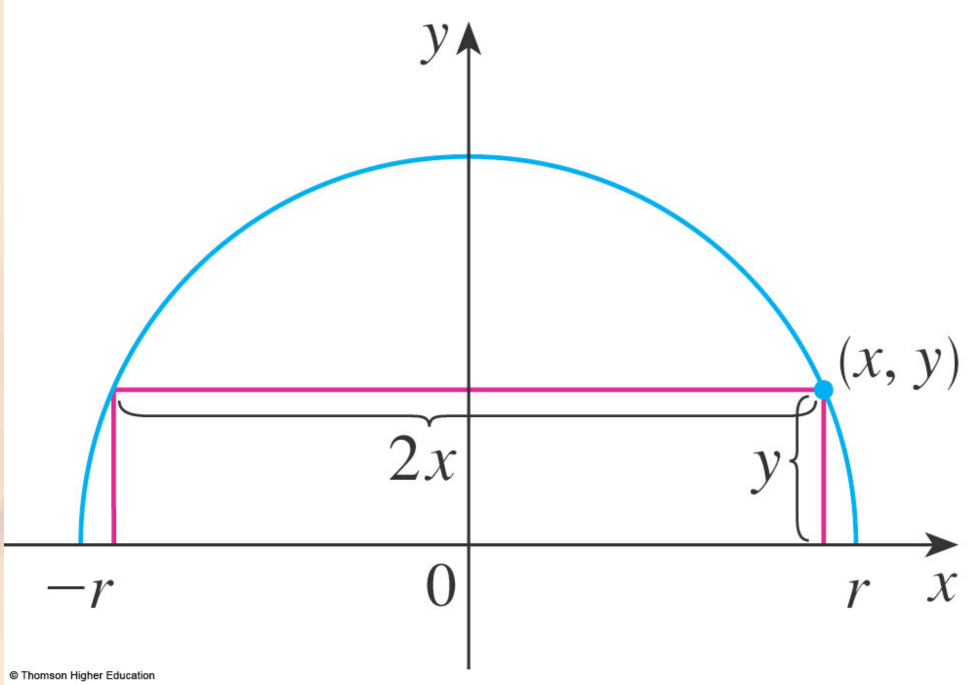
Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

OPTIMIZATION PROBLEMS

E. g. 5—Solution 1

Let's take the semicircle to be the upper half of the circle $x^2 + y^2 = r^2$ with center the origin.

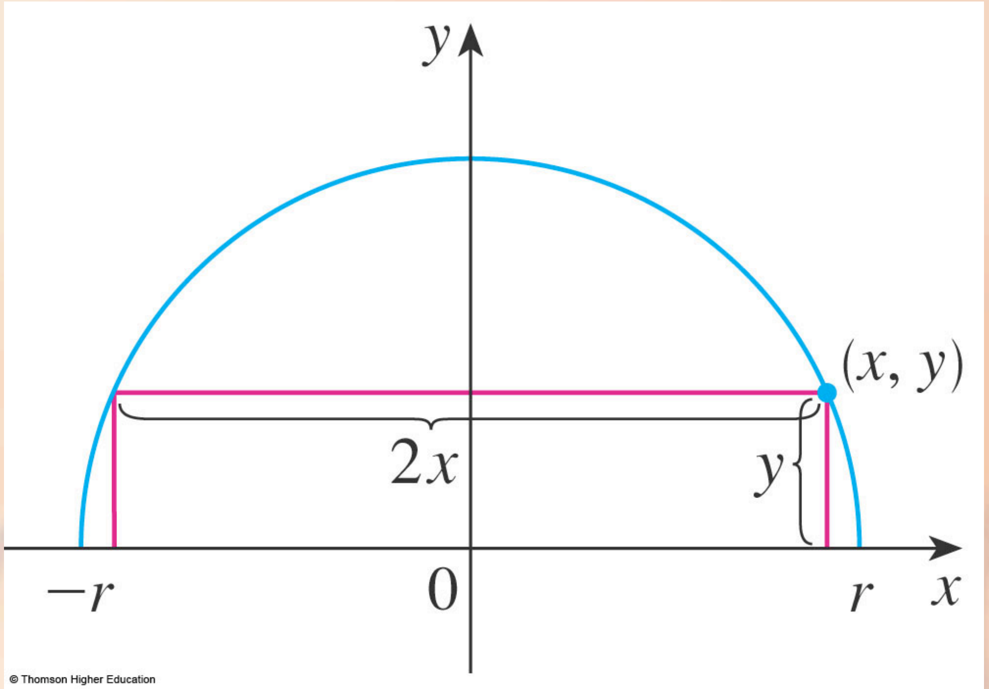
- Then, the word inscribed means that the rectangle has two vertices on the semicircle and two vertices on the x-axis.



E. g. 5—Solution 1

Let (x, y) be the vertex that lies in the first quadrant.

- Then, the rectangle has sides of lengths $2x$ and y.
- So, its area is:
  $A = 2xy$



To eliminate *y*, we use the fact that (x, y) lies on the circle $x^2 + y^2 = r^2$.

- So, $y = \sqrt{r^2 - x^2}$
- Thus, $A = 2x\sqrt{r^2 - x^2}$

The domain of this function is $0 \le x \le r$.

Its derivative is:

$$A' = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}$$

- This is 0 when $2x^2 = r^2$, that is $x = r/\sqrt{2}$, (since $x \ge 0$).

This value of *x* gives a maximum value of *A*, since *A*(0) = 0 and *A*(*r*) = 0 .

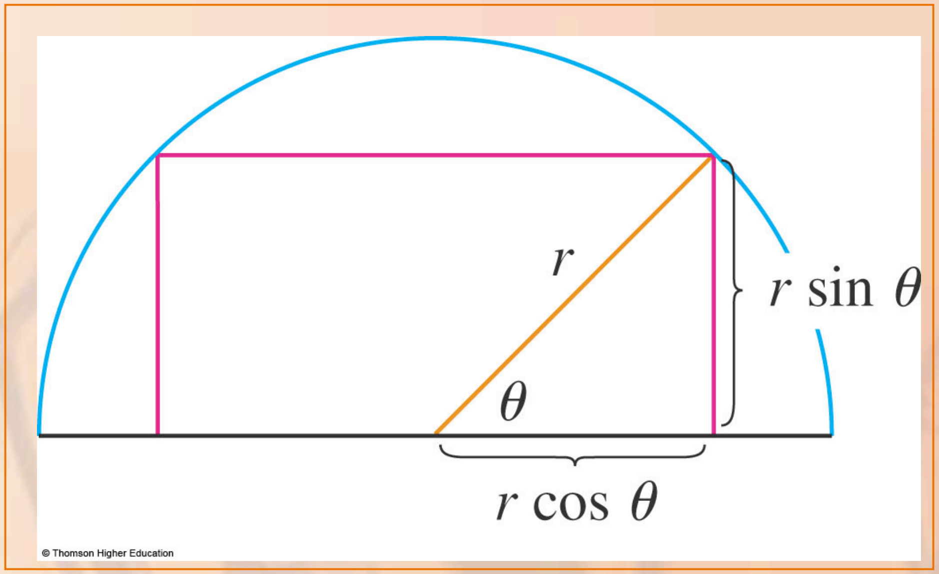
Thus, the area of the largest inscribed rectangle is:

$$A\left(\frac{r}{\sqrt{2}}\right) = 2\frac{r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}} = r^2$$

A simpler solution is possible if we think of using an angle as a variable.

Let θ be the angle shown here.

- Then, the area of the rectangle is:
  $$A(\theta) = (2r\cos\theta)(r\sin\theta)$$
  $$= r^2(2\sin\theta\cos\theta)$$
  $$= r^2\sin 2\theta$$



We know that $\sin 2\theta$ has a maximum value of 1 and it occurs when $2\theta = \pi/2$.

- So, $A(\theta)$ has a maximum value of r^2 and it occurs when $\theta = \pi/4$.

Notice that this trigonometric solution doesn't involve differentiation.

- In fact, we didn't need to use calculus at all.

Let us now look at optimization problems in business and economics.

MARGINAL COST FUNCTION

In Section 3.7, we introduced the idea of marginal cost.

- Recall that if $C(x)$, the cost function, is the cost of producing x units of a certain product, then the marginal cost is the rate of change of C with respect to x.

- In other words, the marginal cost function is the derivative, $C'(x)$, of the cost function.

Now, let's consider marketing.

- Let $p(x)$ be the price per unit that the company can charge if it sells x units.
- Then, p is called the demand function (or price function), and we would expect it to be a decreasing function of x.

REVENUE FUNCTION

If x units are sold and the price per unit is $p(x)$, then the total revenue is:

$$R(x) = xp(x)$$

- This is called the revenue function.

MARGINAL REVENUE FUNCTION

The derivative *R'* of the revenue function is called the marginal revenue function.

- It is the rate of change of revenue with respect to the number of units sold.

MARGINAL PROFIT FUNCTION

If x units are sold, then the total profit is

$$P(x) = R(x) - C(x)$$

and is called the profit function.

The marginal profit function is P', the derivative of the profit function.

MINIMIZING COSTS AND MAXIMIZING REVENUES

In Exercises 53–58, you are asked to use the marginal cost, revenue, and profit functions to minimize costs and maximize revenues and profits.

MAXIMIZING REVENUE

Example 6

A store has been selling 200 DVD burners a week at $350 each. A market survey indicates that, for each $10 rebate offered to buyers, the number of units sold will increase by 20 a week.

- Find the demand function and the revenue function.
- How large a rebate should the store offer to maximize its revenue?

DEMAND & REVENUE FUNCTIONS **Example 6**

If x is the number of DVD burners sold per week, then the weekly increase in sales is $x - 200$.

- For each increase of 20 units sold, the price is decreased by $10.

So, for each additional unit sold, the decrease in price will be 1/20 x 10 and the demand function is:

$$p(x) = 350 - (10/20)(x - 200)$$

$$= 450 - \tfrac{1}{2}x$$

REVENUE FUNCTION **Example 6**

The revenue function is:

$$R(x) = xp(x)$$
$$= 450x - \tfrac{1}{2}x^2$$

MAXIMIZING REVENUE

Example 6

Since $R'(x) = 450 - x$, we see that $R'(x) = 0$ when $x = 450$.

- This value of x gives an absolute maximum by the First Derivative Test (or simply by observing that the graph of R is a parabola that opens downward).

MAXIMIZING REVENUE

Example 6

The corresponding price is:

$$p(450) = 450 - \tfrac{1}{2}(450) = 225$$

The rebate is: $350 - 225 = 125$

- Therefore, to maximize revenue, the store should offer a rebate of $125.