31. L'Hopital's Rule

31.1. Limit of indeterminate type

Some limits for which the substitution rule does not apply can be found by using inspection. For instance,

$$\lim_{x\to 0}\frac{\cos x}{x^2}\qquad\left(\frac{\text{about }1}{\text{small pos.}}\right)$$
$$=\infty$$

On the other hand, we have seen (8) that inspection cannot be used to find the limit of a fraction when both numerator and denominator go to 0. The examples given were

$$\lim_{x\to 0^+}\frac{x^2}{x},\qquad \lim_{x\to 0^+}\frac{x}{x^2},\qquad \lim_{x\to 0^+}\frac{x}{x}.$$

In each case, both numerator and denominator go to 0. If we *had* a way to use inspection to decide the limit in this case, then it would have to give the same answer in all three cases. Yet, the first limit is 0, the second is ∞ and the third is 1 (as can be seen by canceling x's).

We say that each of the above limits is **indeterminate of type** $\frac{0}{0}$. A useful way to remember that one cannot use inspection in this case is to imagine that the numerator going to 0 is trying to make the fraction small, while the denominator going to 0 is trying to make the fraction large. There is a struggle going on. In the first case above, the numerator wins (limit is 0); in the second case, the denominator wins (limit is ∞); in the third case, there is a compromise (limit is 1).

Changing the limits above so that x goes to infinity instead gives a different indeterminate

type. In each of the limits

$$\lim_{x \to \infty} \frac{x^2}{x}, \qquad \lim_{x \to \infty} \frac{x}{x^2}, \qquad \lim_{x \to \infty} \frac{x}{x}.$$

both numerator and denominator go to infinity. The numerator going to infinity is trying to make the fraction large, while the denominator going to infinity is trying to make the fraction small. Again, there is a struggle. Once again, we can cancel x's to see that the first limit is ∞ (numerator wins), the second limit is 0 (denominator wins), and the third limit is 1 (compromise). The different answers show that one cannot use inspection in this case. Each of these limits is indeterminate of type $\frac{\infty}{\infty}$.

Sometimes limits of indeterminate types $\frac{0}{0}$ or $\frac{\infty}{\infty}$ can be determined by using some algebraic technique, like canceling between numerator and denominator as we did above (see also 12). Usually, though, no such algebraic technique suggests itself, as is the case for the limit

$$\lim_{x \to 0} \frac{x^2}{\sin x},$$

which is indeterminate of type $\frac{0}{0}$. Fortunately, there is a general rule that can be applied, namely, l'Hôpital's rule.

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31.2. L'Hôpital's rule

L'HÔPITAL'S RULE. If the limit

$$\lim \frac{f(x)}{g(x)}$$

is of indeterminate type $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)},$$

provided this last limit exists. Here, lim stands for $\lim_{x \to a}$, $\lim_{x \to a^{\pm}}$, or $\lim_{x \to \pm\infty}$.

The pronunciation is lō-pē-täl. Evidently, this result is actually due to the mathematician Bernoulli rather than to l'Hôpital. The verification of l'Hôpital's rule (omitted) depends on the mean value theorem.

31.2.1 Example Find $\lim_{x \to 0} \dfrac{x^2}{\sin x}$.

Solution As observed above, this limit is of indeterminate type $\frac{0}{0}$, so l'Hôpital's rule applies. We have

$$\lim_{x \to 0} \frac{x^2}{\sin x} \left(\frac{0}{0} \right) \overset{\text{l'H}}{=} \lim_{x \to 0} \frac{2x}{\cos x} = \frac{0}{1} = 0,$$

where we have first used l'Hôpital's rule and then the substitution rule. □

The solution of the previous example shows the notation we use to indicate the type of an indeterminate limit and the subsequent use of l'Hôpital's rule.

31.2.2 Example   Find $\lim\limits_{x \to -\infty} \dfrac{3x - 2}{e^{x^2}}$.

Solution   We have

$$\lim_{x \to -\infty} \frac{3x - 2}{e^{x^2}} \quad \left(\frac{-\infty}{\infty}\right) \quad \overset{\text{l'H}}{=} \quad \lim_{x \to -\infty} \frac{3}{e^{x^2}(2x)} \qquad \left(\frac{3}{\text{large neg.}}\right)$$
$$= 0.$$

□

31.3. Common mistakes

Here are two pitfalls to avoid:

- L'Hôpital's rule should not be used if the limit is not indeterminate (of the appropriate type). For instance, the following limit is *not* indeterminate; in fact, the substitution rule applies to give the limit:

$$\lim_{x \to 0} \frac{\sin x}{x + 1} = \frac{0}{1} = 0.$$

An application of l'Hôpital's rule gives the wrong answer:

$$\lim_{x \to 0} \frac{\sin x}{x + 1} \overset{\text{l'H}}{=} \lim_{x \to 0} \frac{\cos x}{1} = \frac{1}{1} = 1 \qquad \text{(wrong)}.$$

- Although l'Hôpital's rule involves a quotient $f(x)/g(x)$ as well as derivatives, the quotient rule of differentiation is not involved. The expression in l'Hôpital's rule is

$$\frac{f'(x)}{g'(x)} \qquad \text{and not} \qquad \left(\frac{f(x)}{g(x)}\right)'.$$

31.4. Examples

31.4.1 Example   Find $\lim\limits_{\theta \to 0} \dfrac{\sin \theta}{\theta}$.

Solution   We have

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \;\left(\frac{0}{0}\right) \;\overset{\text{l'H}}{=}\; \lim_{\theta \to 0} \frac{\cos \theta}{1} = \frac{1}{1} = 1.$$

(In 19, we had to work pretty hard to determine this important limit. It is tempting to go back and replace that argument with this much easier one, but unfortunately we used this limit to derive the formula for the derivative of $\sin \theta$, which is used here in the application of l'Hôpital's rule, so that would make for a circular argument.)   □

Sometimes repeated use of l'Hôpital's rule is called for:

31.4.2 Example   Find $\lim\limits_{x \to \infty} \dfrac{3x^2 + x + 4}{5x^2 + 8x}$.

Solution   We have

$$\lim_{x \to \infty} \frac{3x^2 + x + 4}{5x^2 + 8x} \;\left(\frac{\infty}{\infty}\right) \;\overset{\text{l'H}}{=}\; \lim_{x \to \infty} \frac{6x + 1}{10x + 8} \;\left(\frac{\infty}{\infty}\right) \;\overset{\text{l'H}}{=}\; \lim_{x \to \infty} \frac{6}{10} = \frac{6}{10} = \frac{3}{5}.$$

□

For the limit at infinity of a rational function (i.e., polynomial over polynomial) as in the preceding example, we also have the method of dividing numerator and denominator by the highest power of the variable in the denominator (see). That method is probably preferable to using l'Hôpital's rule repeatedly, especially if the degrees of the polynomials are large. Sometimes though, we have no alternate approach:

31.4.3 Example Find $\lim\limits_{x \to 0} \dfrac{e^x - 1 - x - x^2/2}{x^3}$.

Solution We have

$$
\lim_{x \to 0} \frac{e^x - 1 - x - x^2/2}{x^3} \quad \left(\frac{0}{0}\right) \overset{\text{l'H}}{=} \lim_{x \to 0} \frac{e^x - 1 - x}{3x^2} \quad \left(\frac{0}{0}\right)
$$
$$
\overset{\text{l'H}}{=} \lim_{x \to 0} \frac{e^x - 1}{6x} \quad \left(\frac{0}{0}\right)
$$
$$
\overset{\text{l'H}}{=} \lim_{x \to 0} \frac{e^x}{6} = \frac{e^0}{6} = \frac{1}{6}.
$$

□

There are other indeterminate types, to which we now turn. The strategy for each is to transform the limit into either type $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ and then use l'Hôpital's rule.

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31.5. Indeterminate product

Type $\infty \cdot 0$. The limit

$$\lim_{x \to \infty} x e^{-x} \quad (\infty \cdot 0)$$

cannot be determined by using inspection. The first factor going to ∞ is trying to make the expression large, while the second factor going to 0 is trying to make the expression small. There is a struggle going on. We say that this limit is indeterminate of type $\infty \cdot 0$.

The strategy for handling this type is to rewrite the product as a quotient and then use l'Hôpital's rule.

31.5.1 Example Find $\lim\limits_{x \to \infty} x e^{-x}$.

Solution As noted above, this limit is indeterminate of type $\infty \cdot 0$. We rewrite the expression as a fraction and then use l'Hôpital's rule:

$$\lim_{x \to \infty} x e^{-x} = \lim_{x \to \infty} \frac{x}{e^x} \quad \left(\frac{\infty}{\infty}\right)$$

$$\overset{\text{l'H}}{=} \lim_{x \to \infty} \frac{1}{e^x} \quad \left(\frac{1}{\text{large pos.}}\right)$$

$$= 0.$$

$\square$

31.5.2 Example Find $\lim\limits_{x \to 0^+} (\cot 2x)(\sin 6x)$.

Solution   First

$$\lim_{x \to 0^+} \cot 2x = \lim_{x \to 0^+} \frac{\cos 2x}{\sin 2x} \quad \left(\frac{\text{about } 1}{\text{small pos.}} \right)$$
$$= \infty.$$

Therefore, the given limit is indeterminate of type $\infty \cdot 0$. We rewrite as a fraction and then use l'Hôpital's rule:

$$\lim_{x \to 0^+} (\cot 2x)(\sin 6x) = \lim_{x \to 0^+} \frac{\sin 6x}{\tan 2x} \quad \left(\frac{0}{0} \right)$$
$$\overset{\text{l'H}}{=} \lim_{x \to 0^+} \frac{6 \cos 6x}{2 \sec^2 2x}$$
$$= \frac{6}{2} = 3.$$

$\square$

31.6. Indeterminate difference

Type $\infty - \infty$.   The substitution rule cannot be used on the limit

$$\lim_{x \to \frac{\pi}{2}^-} (\tan x - \sec x) \quad (\infty - \infty)$$

since $\tan \pi/2$ is undefined. Nor can one determine this limit by using inspection. The first term going to infinity is trying to make the expression large and positive, while the second term going to negative infinity is trying to make the expression large and negative. There is a struggle going on. We say that this limit is indeterminate of type $\infty - \infty$.

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The strategy for handling this type is to combine the terms into a single fraction and then use l'Hôpital's rule.

31.6.1 Example Find $\lim\limits_{x \to \frac{\pi}{2}^-} (\tan x - \sec x)$.

Solution As observed above, this limit is indeterminate of type $\infty - \infty$. We combine the terms and then use l'Hôpital's rule:

$$
\begin{aligned}
\lim_{x \to \frac{\pi}{2}^-} (\tan x - \sec x) &= \lim_{x \to \frac{\pi}{2}^-} \left(\frac{\sin x}{\cos x} - \frac{1}{\cos x} \right) \\
&= \lim_{x \to \frac{\pi}{2}^-} \frac{\sin x - 1}{\cos x} \quad \left(\frac{0}{0} \right) \\
&\overset{\text{l'H}}{=} \lim_{x \to \frac{\pi}{2}^-} \frac{\cos x}{-\sin x} \\
&= \frac{0}{-1} = 0.
\end{aligned}
$$

$\square$

31.6.2 Example Find $\lim\limits_{x \to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$.

Solution The limit is indeterminate of type $\infty - \infty$. We combine the terms and then use

l'Hôpital's rule:

$$\lim_{x\to 1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to 1^+}\frac{x\ln x-x+1}{(x-1)\ln x}\quad\left(\frac{0}{0}\right)$$

$$\overset{\text{l'H}}{=}\lim_{x\to 1^+}\frac{\ln x+x(1/x)-1}{\ln x+(x-1)(1/x)}$$

$$=\lim_{x\to 1^+}\frac{\ln x}{\ln x+1-1/x}\quad\left(\frac{0}{0}\right)$$

$$\overset{\text{l'H}}{=}\lim_{x\to 1^+}\frac{1/x}{1/x+1/x^2}$$

$$=\frac{1}{2}.$$

$\square$

31.7. Indeterminate powers

Type ∞^0. The limit

$$\lim_{x\to\infty}x^{1/x}\quad(\infty^0)$$

cannot be determined by using inspection. The base going to infinity is trying to make the expression large, while the exponent going to 0 is trying to make the expression equal to 1. There is a struggle going on. We say that this limit is indeterminate of type ∞^0.

The strategy for handling this type (as well as the types 1^∞ and 0^0 yet to be introduced) is to first find the limit of the natural logarithm of the expression (ultimately using l'Hôpital's rule) and then use an inverse property of logarithms to get the original limit.

31.7.1 Example Find $\lim\limits_{x\to\infty} x^{1/x}$.

Solution As was noted above, this limit is of indeterminate type ∞^0. We first find the limit of the natural logarithm of the given expression:

$$\lim_{x\to\infty} \ln x^{1/x} = \lim_{x\to\infty} (1/x)\ln x \quad (0\cdot\infty)$$

$$= \lim_{x\to\infty} \frac{\ln x}{x} \quad \left(\frac{\infty}{\infty}\right)$$

$$\overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{1/x}{1}$$

$$= 0.$$

Therefore,

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln x^{1/x}} = e^0 = 1,$$

where we have used the inverse property of logarithms $y = e^{\ln y}$ and then the previous computation. (The next to the last equality also uses continuity of the exponential function.) □

Type 1^∞. The limit

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x \quad (1^\infty)$$

cannot be determined by using inspection. The base going to 1 is trying to make the expression equal to 1, while the exponent going to infinity is trying to make the expression go to ∞ (raising a number greater than 1 to ever higher powers produces ever larger results). We say that this limit is indeterminate of type 1^∞.

31.7.2   Example   Find $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x}$.

Solution   As was noted above, this limit is indeterminate of type 1^{∞}. We first find the limit of the natural logarithm of the given expression:

$$
\begin{aligned}
\lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^{x} &= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \\
&= \lim_{x\to\infty}\frac{\ln\left(1+1/x\right)}{x^{-1}} \qquad \left(\frac{0}{0}\right) \\
&\overset{\text{l'H}}{=} \lim_{x\to\infty}\frac{\dfrac{1}{1+1/x}\left(-x^{-2}\right)}{-x^{-2}} \\
&= \lim_{x\to\infty}\frac{1}{1+1/x} = 1.
\end{aligned}
$$

Therefore,

$$
\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = \lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^{x}} = e^{1} = e.
$$

$\square$

Type 0^{0}.   The limit

$$
\lim_{x\to 0}(\tan x)^{x} \qquad \left(0^{0}\right)
$$

cannot be determined by using inspection. The base going to 0 is trying to make the expression small, while the exponent going to 0 is trying to make the expression equal to 1. We say that this limit is indeterminate of type 0^{0}.

31.7.3 **Example** Find $\lim\limits_{x \to 0^+} (\tan x)^x$.

Solution As was noted above, this limit is indeterminate of type 0^0. We first find the limit of the natural logarithm of the given expression:

$$
\lim_{x \to 0^+} \ln(\tan x)^x = \lim_{x \to 0^+} x \ln \tan x
$$
$$
= \lim_{x \to 0^+} \frac{\ln \tan x}{x^{-1}} \qquad \left(\frac{-\infty}{\infty} \right)
$$
$$
\overset{\text{l'H}}{=} \lim_{x \to 0^+} \frac{\cot x \ \sec^2 x}{-x^{-2}}
$$
$$
= \lim_{x \to 0^+} \frac{-x^2}{\sin x \cos x} \qquad \left(\frac{0}{0} \right)
$$
$$
\overset{\text{l'H}}{=} \lim_{x \to 0^+} \frac{-2x}{\cos^2 x - \sin^2 x}
$$
$$
= \frac{0}{1} = 0.
$$

Therefore,
$$
\lim_{x \to 0^+} (\tan x)^x = \lim_{x \to 0^+} e^{\ln(\tan x)^x} = e^0 = 1.
$$

$\square$

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Between the two applications of l'Hôpital's rule in the last example, we used algebra to simplify the expression. Indiscriminate repeated use of l'Hôpital's rule with no attempt at simplification can give rise to unwieldy expressions, so it is best to make the expression as simple as possible before each application of the rule.

31.8. Summary

We summarize the indeterminate types:

> INDETERMINATE TYPES.
>
> $$\frac{0}{0}, \quad \frac{\pm\infty}{\pm\infty}, \quad \pm\infty \cdot 0, \quad \infty - \infty, \quad \infty^0, \quad 1^{\pm\infty}, \quad 0^0.$$

Limits with the following forms are *not* indeterminate since the parts work together toward a common goal (no struggle):

$$\frac{0}{\infty}, \quad \frac{\infty}{0}, \quad \infty \cdot \infty, \quad \infty + \infty, \quad \infty^\infty, \quad 0^\infty.$$

The last of these, 0^∞, might require some explanation. The base is approaching 0, so it can be made close enough to 0 that it is at least less than 1 in absolute value. But raising a number less than 1 in absolute value to ever higher powers produces numbers ever closer to 0 (for instance, $\frac{1}{2}$ raised to the powers $1, 2, 3, \ldots$ produces $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$). Therefore, both base and exponent work together to produce a limit of 0.

Due to the great variety of forms limits can take on, it is recommended that the reader employ the sort of imagery we have used (of parts either struggling against each other or working together toward a common goal) as an aid to keeping straight which forms are indeterminate and which are not.

31.8.1 Example   Find $\lim\limits_{x \to 0^+} (1 - e^x)^{1/x}$.

Solution   This limit has the form 0^{∞}, which is not indeterminate (both base and exponent are working together to make the expression small). The limit is 0.   □

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$31-$ Exercises

31 – 1   Find $\lim\limits_{x\to 0} \dfrac{e^{3x}-1}{x}$.

31 – 2   Find $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^2}$.

31 – 3   Find $\lim\limits_{x\to 0^+} x\ln x$.

31 – 4   Find $\lim\limits_{x\to 0} \dfrac{e^x - x - 1}{\cos x - 1}$.

31 – 5   Find $\lim\limits_{x\to 0^+} \left(\dfrac{1}{\sin x} - \dfrac{1}{x} \right)$.

31 – 6   Find $\lim\limits_{x\to\pi/2} \dfrac{1 - \sin x}{\csc x}$.

$31-7$ Find $\lim\limits_{x \to 0^+} (\sin x)^x$.

$31-8$ Find $\lim\limits_{x \to \infty} (e^x + x)^{1/x}$.

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