

# L'Hôpital's Rule and Indeterminate Forms

CREATED BY TYNAN LAZARUS

October 29, 2018

Now that we have the power of the derivative, we can use it as a way to compute limits that we didn't have the ability to understand before. Early on, we could compute limits of rational functions quite easily. However, we couldn't deal with mixing a ratio of different kinds of functions, like a polynomial and an exponential. L'Hôpital's Rule allows us to evaluate these kinds of limits without much effort. It also allows us to deal with different indeterminate forms. We will see through some examples just how weird  $\infty$  can act and why these indeterminate forms bring about contradictions in our intuition.

## 1.1 The Definition

Theorem (L'Hôpital's Rule): Let  $f(x)$  and  $g(x)$  be differentiable on an interval  $I$  containing  $a$ , and that  $g'(a) \neq 0$  on  $I$  for  $x \neq a$ . Suppose that

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0} \quad \text{or} \quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}.$$

Then as long as the limits exist, we have that

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

There is an analogous version for when  $a$  is  $\infty$  or  $-\infty$ . What this theorem essentially says is that if you tried to compute the limit of a ratio of functions, but you get the indeterminate forms  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ , then you can compute the limit of the ratio of the derivatives of those functions instead. However, take caution that it is not necessarily a short cut. When encountering limits that we have seen before, it may be faster to use a different technique than L'Hôpital's Rule. Also note that we are **not** taking a quotient rule. We just take the derivatives of the top and the bottom of the fraction and leave them there.

## 1.2 How it Works

Before we could compute the derivative of  $\sin(x)$  or  $\cos(x)$ , we had to figure out two trig limits. We found that  $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$  by using some geometry tricks with sectors and whatnot. Now that we know how to compute derivatives, we can use L'Hôpital's Rule to check that this is correct.

In order to use L'Hôpital's Rule, we need to check that it is in the right form and that we get one of the indeterminate forms required. As usual with limits, we attempt to just plug in the value and see if we get a number. If we did get a real number, then we are done. Here we can see that if we try to plug in  $x = 0$  in the limit, we get that  $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = \frac{0}{0}$ , which is an indeterminate form.

Therefore, we can apply L'Hôpital's Rule. Whenever we do so, we will use a " $\stackrel{\text{L'H}}{=}$ " to denote that we have used the rule and " $=$ " to denote our usual simplification. So, applying L'Hôpital's Rule, we get  $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{\cos(x)}{1}$ . However, this second expression is a limit of a continuous function,

so we can just plug  $x = 0$  and get that  $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{\cos(x)}{1} = \cos(0) = 1$ , verifying what we already know to be true.

We can do the same with our other trig limit,  $\lim_{x \rightarrow 0} \frac{\cos(x) - 1}{x} = 0$ . First, we have to check that L'Hôpital's Rule even applies. If we tried to plug in  $x = 0$  we would get  $\frac{\cos(0) - 1}{0} = \frac{0}{0}$ , which is one of our indeterminate forms. L'Hôpital's Rule does apply to this form, so we get that  $\lim_{x \rightarrow 0} \frac{\cos(x) - 1}{x} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{-\sin(x)}{1} = -\sin(0) = 0$ .

## 1.3 Examples with Indeterminate Forms

### 1.3.1 $\frac{0}{0}$ Form

**Question:** Why is  $\frac{0}{0}$  indeterminate? In general,  $\frac{0}{\text{stuff}} = 0$ , and  $\frac{\text{stuff}}{0}$  acts like  $\infty$ . So the top pulls the limit down towards zero, and the bottom pulls it up to infinity. So who wins?

- Let's say we want to compute  $\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 4}$ . We can see that if we try to plug in  $x = 2$ , we get  $\frac{0}{0}$ . Therefore we can apply L'Hôpital's Rule to get

$$\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 4} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 2} \frac{1}{2x} = \frac{1}{2(2)} = \frac{1}{4}.$$

But this is a limit that we could've computed in the first week of the course; we don't even need the relative canon that is L'Hôpital to swat this little limit. Earlier we would've just factored the bottom and gotten

$$\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{x - 2}{(x - 2)(x + 2)} = \lim_{x \rightarrow 2} \frac{1}{x + 2} = \frac{1}{4}.$$

Either way is just as quick because this is a simple limit.

- For a more interesting example, let's try to compute  $\lim_{x \rightarrow 0} \frac{\ln(\sec(x))}{3x^2}$ . We see a limit, so our first instinct is to plug in the limiting value  $x = 0$ . When we do this, we get  $\frac{\ln(\sec(0))}{3(0)^2} = \frac{\ln(1)}{0} = \frac{0}{0}$ . This is one of the indeterminate forms that L'Hôpital's Rule can help us with. So, we use it to get

$$\lim_{x \rightarrow 0} \frac{\ln(\sec(x))}{3x^2} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{\frac{1}{\sec(x)} \cdot \sec(x) \tan(x)}{6x} = \lim_{x \rightarrow 0} \frac{\tan(x)}{6x}.$$

Trying to take this limit also results in  $\frac{0}{0}$ . So did L'Hôpital fail us? Not quite. All L'Hôpital tells us is that the limit of the original ratio is that same as the limit of the ratio of the derivatives. We got  $\frac{0}{0}$ , which is what L'Hôpital's Rule is designed for, so let's use it again! Thus, we get

$$\lim_{x \rightarrow 0} \frac{\ln(\sec(x))}{3x^2} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{\tan(x)}{6x} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{\sec^2(x)}{6} = \frac{1}{6}.$$

Therefore, our original limit has a value of  $1/6$ . This problem shows us that you may need to use L'Hôpital's Rule multiple times before we get an answer. However, we do need to check that we are in the correct indeterminate form each time before we can apply it.

### 1.3.2 $\frac{\infty}{\infty}$ Form

**Question:** Why is  $\frac{\infty}{\infty}$  indeterminate? Usually  $\frac{\infty}{\text{stuff}}$  acts like  $\infty$  and  $\frac{\text{stuff}}{\infty}$  goes to 0. So the top pulls the limit up to infinity and the bottom tries to pull it down to 0. So who wins?

- Consider the following limit,  $\lim_{x \rightarrow \infty} \frac{2x^2}{e^{3x}}$ . Since this ratio is of a polynomial and an exponential function, we can't solve it with any of the usual techniques from earlier in the course. We can see that if we could plug in larger and larger values that  $2x^2$  diverges up to infinity, and so does  $e^{3x}$ . Thus, this limit looks like  $\frac{\infty}{\infty}$ , which L'Hôpital's Rule can handle. We get that

$$\lim_{x \rightarrow \infty} \frac{2x^2}{e^{3x}} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{4x}{3e^{3x}}.$$

But we can see that this second limit is also  $\frac{\infty}{\infty}$ , so we can apply L'Hôpital's Rule again to get

$$\lim_{x \rightarrow \infty} \frac{2x^2}{e^{3x}} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{4x}{3e^{3x}} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{4}{9e^{3x}} = 0.$$

Remember that we can apply L'Hôpital's Rule as many times as is needed. However, this can backfire.

- Consider  $\lim_{x \rightarrow -\infty} \frac{x^{67} - 3x^{40} + x + 1}{x^{12} + 2x^{64} - x^2 - 5}$ . If we try the limit, we see that we get  $\frac{-\infty}{\infty}$ . We could blindly try to use L'Hôpital's Rule, but when we do that we only reduce the degrees of the numerator and denominator by one. We would still be left with large powers on the top and the bottom, and it would still be some sort of  $\frac{\pm\infty}{\pm\infty}$ . In fact, we would have to do L'Hôpital's Rule 64 times before we get an answer that is not in an indeterminate form!

This problem can be much more easily done with our old technique. We can see that we get

$$\lim_{x \rightarrow -\infty} \frac{x^{67} - 3x^{40} + x + 1}{x^{12} + 2x^{64} - x^2 - 5} \cdot \frac{1/x^{64}}{1/x^{64}} = \lim_{x \rightarrow -\infty} \frac{x^3 - \frac{3}{x^{24}} + \frac{1}{x^{63}} + \frac{1}{x^{64}}}{\frac{1}{x^{52}} + 2 - \frac{1}{x^{62}} - \frac{5}{x^{64}}} = -\infty.$$

- Now let's compute  $\lim_{x \rightarrow 0^+} \frac{\ln(e^x - 1)}{\ln(x)}$ . When we try to plug in  $x = 0$  (or rather, smaller and smaller positive numbers), we know that  $\ln(x)$  approaches  $-\infty$ . Thus, this limit looks like  $\frac{-\infty}{-\infty}$ . Thus, by L'Hôpital's Rule we get

$$\lim_{x \rightarrow 0^+} \frac{\ln(e^x - 1)}{\ln(x)} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{e^x - 1} \cdot e^x}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{xe^x}{e^x - 1}.$$

Now when we try to plug in  $x = 0$ , we get the indeterminate form  $\frac{0}{0}$ . So we can use L'Hôpital's Rule again. Now we get

$$\lim_{x \rightarrow 0^+} \frac{\ln(e^x - 1)}{\ln(x)} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} \frac{xe^x}{e^x - 1} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} \frac{xe^x + e^x}{e^x} = \lim_{x \rightarrow 0^+} x + 1 = 1.$$

From this example we can see that sometimes the indeterminate forms can change as we use L'Hôpital's Rule and simplify. As long as we are careful and check at each step whether we can use L'Hôpital's Rule or not, we can still get to the answer.

### 1.3.3 $0 \cdot \infty$ Form

**Question:** Why is  $0 \cdot \infty$  indeterminate? Usually  $0 \cdot (\text{stuff}) = 0$  and  $(\text{stuff}) \cdot \infty = \infty$ . So one piece tries to pull the limit down to zero, and the other tries to pull it up to  $\infty$ . Does one side win? Or do they sort of balance each other out and we get an answer of something like 7?

- As a first example, let's compute  $\lim_{x \rightarrow \infty} x \sin\left(\frac{1}{x}\right)$ . We can see that as  $x$  goes off to infinity,  $1/x$  goes to zero and  $\sin(0) = 0$ . So we have the form  $\infty \cdot 0$ . However, this isn't a form that L'Hôpital's Rule can be used on. In order to determine what value the limit approaches we have to first put it in the correct form. The trick that we will use is a way to rewrite  $x$ . Recall that  $x = \frac{1}{\frac{1}{x}}$ . Using this, we can rewrite the given limit as follows,

$$\lim_{x \rightarrow \infty} x \sin\left(\frac{1}{x}\right) = \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x}} \sin\left(\frac{1}{x}\right) = \lim_{x \rightarrow \infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}.$$

Now if we look at this limit and consider  $x$  tending to  $\infty$ , we see the top approaches 0 and the bottom also approaches 0. Thus, we are now in the correct form for L'Hôpital's Rule. Thus, we get

$$\lim_{x \rightarrow \infty} x \sin\left(\frac{1}{x}\right) = \lim_{x \rightarrow \infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{\cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right)}{\frac{-1}{x^2}} = \lim_{x \rightarrow \infty} \cos\left(\frac{1}{x}\right) = \cos(0) = 1.$$

We can see that sometimes we'll need to do some manipulation of the terms in the limit before we can use L'Hôpital's Rule.

- Now let's compute  $\lim_{x \rightarrow 0^+} x^3 \ln(x)$ . As  $x$  approaches zero, we can see that we get the form  $0 \cdot -\infty$ . Following the same trick as last time, we can compute that the value of this limit is

$$\lim_{x \rightarrow 0^+} x^3 \ln(x) = \lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^{-3}} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-3x^{-4}} = \lim_{x \rightarrow 0^+} \frac{x^3}{-3} = 0.$$

- For a slightly trickier example, consider  $\lim_{t \rightarrow \frac{\pi}{2}^-} \tan(t) \sin\left(t - \frac{\pi}{2}\right)$ . We can see that this is of the  $\infty \cdot 0$  type, but we can't use the same trick as last time. When in doubt with trigonometric functions, turn everything back into sines and cosines. Here, we get

$$\lim_{t \rightarrow \frac{\pi}{2}^-} \tan(t) \sin\left(t - \frac{\pi}{2}\right) = \lim_{t \rightarrow \frac{\pi}{2}^-} \frac{\sin(t)}{\cos(t)} \sin\left(t - \frac{\pi}{2}\right),$$

which is in the  $\frac{0}{0}$  form now. Thus, we can now use L'Hôpital's Rule. Therefore, we calculate

$$\begin{aligned} \lim_{t \rightarrow \frac{\pi}{2}^-} \tan(t) \sin\left(t - \frac{\pi}{2}\right) &= \lim_{t \rightarrow \frac{\pi}{2}^-} \frac{\sin(t) \sin\left(t - \frac{\pi}{2}\right)}{\cos(t)} \\ &\stackrel{\text{L'H}}{=} \lim_{t \rightarrow \frac{\pi}{2}^-} \frac{\cos(t) \sin\left(t - \frac{\pi}{2}\right) + \sin(t) \cos\left(t - \frac{\pi}{2}\right)}{-\sin(t)} \\ &= \frac{\cos\left(\frac{\pi}{2}\right) \sin(0) + \sin\left(\frac{\pi}{2}\right) \cos(0)}{-\sin\left(\frac{\pi}{2}\right)} = -1. \end{aligned}$$

### 1.3.4 $\infty - \infty$ Form

**Question:** Why is  $\infty - \infty$  indeterminate? In general  $\infty - (\text{stuff}) = \infty$ , but  $(\text{stuff}) - \infty = -\infty$ . So who wins? Or do they balance out and we get something like  $-\pi$ ?

- To start with, let's look at  $\lim_{x \rightarrow \infty} \ln(4x^2 - 6) - \ln(-x + 3x^2 + 5)$ . We know that the end behavior of  $\ln(x)$  approaches infinity as  $x$  gets larger and larger. Since the insides of both logs approaches infinity, the limit looks like  $\infty - \infty$ . To move forward with computing the limit, we can use our logarithm rules to simplify and get

$$\lim_{x \rightarrow \infty} \ln(4x^2 - 6) - \ln(-x + 3x^2 + 5) = \lim_{x \rightarrow \infty} \ln \left( \frac{4x^2 - 6}{-x + 3x^2 + 5} \right).$$

Now recall that  $\ln(x)$  is a continuous function on its domain. That means we can pass the limit to the inside of the function, and only have to worry about what happens with the rational function on the inside. In other words,

$$\lim_{x \rightarrow \infty} \ln \left( \frac{4x^2 - 6}{-x + 3x^2 + 5} \right) = \ln \left( \lim_{x \rightarrow \infty} \frac{4x^2 - 6}{-x + 3x^2 + 5} \right).$$

Notice that the inside limit is something we spent a lot of time understanding how to compute at the beginning of the class! So we can use our usual limit techniques to compute this. In fact, this is also in the  $\frac{\infty}{\infty}$  form, so we could even use L'Hôpital's Rule from the earlier section!

Thus, we finally conclude that the answer we are looking for is

$$\lim_{x \rightarrow \infty} \ln \left( \frac{4x^2 - 6}{-x + 3x^2 + 5} \right) = \ln \left( \lim_{x \rightarrow \infty} \frac{4x^2 - 6}{-x + 3x^2 + 5} \right) = \ln \left( \lim_{x \rightarrow \infty} \frac{4 - \frac{6}{x^2}}{\frac{-1}{x} + 3 + \frac{5}{x^2}} \right) = \ln \left( \frac{4}{3} \right).$$

- For a second example, we'll compute  $\lim_{x \rightarrow 1^+} \frac{1}{x-1} - \frac{1}{\ln x}$ . To see that this really is in  $\infty - \infty$  form, notice that as  $x$  approaches 1 from the right,  $\ln(x)$  will approach zero from the right. Thus, the denominator of both pieces of the limit approaches zero from the right, and we know from our parent functions that  $\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$ . Therefore we really are in the  $\infty - \infty$  form. This time only one side has a logarithm, so we can't use our log rules right off the bat. However, this really just looks like a fraction subtracted from another fraction, and we know how to simplify that with a common denominator. So we get

$$\lim_{x \rightarrow 1^+} \frac{1}{x-1} - \frac{1}{\ln x} = \lim_{x \rightarrow 1^+} \frac{\ln(x) - (x-1)}{(x-1)\ln(x)}.$$

When we look at this limit, we see that we are now in  $\frac{0}{0}$  form, which is perfect for L'Hôpital's! Notice that on the bottom there are two functions of  $x$  that are being multiplied, so when we do the derivative we will need the product rule. Thus,

$$\begin{aligned} \lim_{x \rightarrow 1^+} \frac{1}{x-1} - \frac{1}{\ln x} &= \lim_{x \rightarrow 1^+} \frac{\ln(x) - (x-1)}{(x-1)\ln(x)} \\ &\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 1^+} \frac{\frac{1}{x} - 1}{\ln(x) + (x-1)\frac{1}{x}} \\ &\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 1^+} \frac{\frac{-1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{-1}{1+1} = \frac{-1}{2}. \end{aligned}$$

### 1.3.5 $\infty^0$ Form

**Question:** Why is  $\infty^0$  indeterminate? In general  $\infty$  raised to any positive power should be equal to  $\infty$ ,  $\infty$  raised to a negative power is 0, and anything raised to the zero should be equal to 1. So who wins?

- An example of this form is the limit  $\lim_{x \rightarrow \infty} (\ln(x))^{1/x}$ . Notice that as  $x$  gets large,  $\ln(x)$  also gets large, and that  $\frac{1}{x}$  gets small. So the power converges to 0 but the function on the inside diverges to  $\infty$ . The issue with this form is that we can't do much with manipulating the exponent. So we would really like to get the  $\frac{1}{x}$  out of the exponent so that we can deal with it more effectively. We will exploit our log rules here to bring that power down. However, we have to be careful! We can't just use the log rule  $\ln(a^b) = b \ln(a)$  right now because it's not in that form! First, we need to transform it into that form. As with all our other techniques, we need to change the equation without changing the outcome. Before we have been multiplying by weird forms of 1, or we have added weird forms of zero. This time, we will use the fact that  $e^x$  and  $\ln(x)$  are inverses of each other. Recall that  $e^{\ln(x)} = x$ . That is, if we apply both  $\ln(x)$  and  $e^x$  to a function, we end up with exactly the same thing. For us, that means we will do the following

$$\lim_{x \rightarrow \infty} (\ln(x))^{1/x} = \lim_{x \rightarrow \infty} e^{\ln((\ln(x))^{1/x})}.$$

Now we can use our log rule to bring the power down, but notice that everything will be happening in the exponent of  $e$ . Since  $e^x$  is a continuous function, we can also push the limit up into the exponent. Thus, we have

$$\lim_{x \rightarrow \infty} e^{\ln((\ln(x))^{1/x})} = e^{\lim_{x \rightarrow \infty} \frac{1}{x} \cdot \ln(\ln(x))} = e^{\lim_{x \rightarrow \infty} \frac{\ln(\ln(x))}{x}}.$$

Now we only need to worry about the limit that is in the exponent. In this case, as  $x$  goes to  $\infty$ , both the top and bottom go to  $\infty$ , so we are in the proper form for L'Hôpital's Rule. Thus, we have

$$\begin{aligned} \lim_{x \rightarrow \infty} e^{\ln((\ln(x))^{1/x})} &= e^{\lim_{x \rightarrow \infty} \frac{\ln(\ln(x))}{x}} \\ &\stackrel{\text{L'H}}{=} e^{\lim_{x \rightarrow \infty} \frac{1}{\ln(x)} \cdot \frac{1}{x}} \\ &= e^0 = 1. \end{aligned}$$

- Now let's compute  $\lim_{x \rightarrow \infty} x^{1/\ln(x)}$ . Recall that as  $x$  tends to infinity, so does  $\ln(x)$ . So  $\frac{1}{\ln(x)}$  approaches zero as  $x$  goes to infinity. Thus, we are in the form  $\infty^0$ . Again we need to be able to get into that exponent, so we use the same trick with  $e^x$  and  $\ln(x)$ . Then we get

$$\begin{aligned} \lim_{x \rightarrow \infty} x^{1/\ln(x)} &= \lim_{x \rightarrow \infty} e^{\ln(x^{1/\ln(x)})} \\ &= e^{\lim_{x \rightarrow \infty} \ln(x^{1/\ln(x)})} \\ &= e^{\lim_{x \rightarrow \infty} \frac{1}{\ln(x)} \ln(x)} \\ &= e^{\lim_{x \rightarrow \infty} 1} = e. \end{aligned}$$

It turns out we don't even technically need L'Hôpital's Rule here because the logarithms cancel before we even need to take derivatives. However, we can use it twice to get the same value as the cancellation.

### 1.3.6 $1^\infty$ Form

**Question:** Why is  $1^\infty$  indeterminate? Usually 1 raised to any power is just equal to 1. But fractions raised to the  $\infty$  goes to zero, and numbers larger than 1 raised to the  $\infty$  should go off to  $\infty$ . So where does  $1^\infty$  go?

- Consider the limit  $\lim_{x \rightarrow 0^+} (e^x + x)^{1/x}$ . Since  $e^0 = 1$ , we can see that this limit is of the form we want,  $1^\infty$ . We can see here that we are in the same situation as last time: we have an exponent that has an  $x$  in it and we need to move it around to be able to deal with it. We can try the same technique as last time and exploit the inverse property of  $e^x$  and  $\ln(x)$ , namely that  $e^{\ln(x)} = x$ . Thus, we get

$$\lim_{x \rightarrow 0^+} (e^x + x)^{1/x} = \lim_{x \rightarrow 0^+} e^{\ln((e^x + x)^{1/x})} = \lim_{x \rightarrow 0^+} e^{\frac{\ln(e^x + x)}{x}}.$$

Here notice that as  $x$  approaches zero, the top of the fraction in the exponent approaches  $\ln(1) = 0$ , and the denominator approaches zero as well. Thus, we are in the earlier case of  $\frac{0}{0}$ . Therefore, we can move forward and use L'Hôpital's Rule to get

$$\begin{aligned} \lim_{x \rightarrow 0^+} (e^x + x)^{1/x} &= \lim_{x \rightarrow 0^+} e^{\frac{\ln(e^x + x)}{x}} \\ &\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} e^{\frac{1}{e^x + x} \cdot (e^x + 1)} \\ &= e^{\lim_{x \rightarrow 0^+} \frac{e^x + 1}{e^x + x}} = e^2. \end{aligned}$$

- For a second example, let's compute  $\lim_{x \rightarrow \infty} \left(\frac{x+2}{x-1}\right)^x$ . Here we can easily see that the exponent goes to infinity as  $x$  grows large, and notice that  $\frac{x+2}{x-1}$  converges to 1 by L'Hôpital's Rule. Thus we are in the form  $1^\infty$ . As before, we use our logarithm trick to get

$$\lim_{x \rightarrow \infty} \left(\frac{x+2}{x-1}\right)^x = \lim_{x \rightarrow \infty} e^{x \ln\left(\frac{x+2}{x-1}\right)}.$$

Notice that we now have the form  $\infty \cdot 0$  in the exponent, so we can use our earlier tricks to solve this exponent. Thus,

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(\frac{x+2}{x-1}\right)^x &= \lim_{x \rightarrow \infty} e^{x \ln\left(\frac{x+2}{x-1}\right)} \\ &= \lim_{x \rightarrow \infty} e^{\frac{\ln\left(\frac{x+2}{x-1}\right)}{1/x}} \\ &\stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} e^{\frac{\frac{x-1}{x+2} \cdot \frac{(x-1) - (x+2)}{(x-1)^2}}{-1/x^2}} \\ &= \lim_{x \rightarrow \infty} e^{\frac{3x^2}{(x+2)(x-1)}} = e^3. \end{aligned}$$

### 1.3.7 $0^0$ Form

**Question:** Why is  $0^0$  indeterminate? In general zero raised to any positive power is just zero, but but anything raised to the zero should be equal to 1. So which is it?

- Consider  $\lim_{x \rightarrow 0^+} x^{\frac{-1}{\ln(x)}}$ . To check that this is in the right form, we need to look at the exponent. as  $x$  approaches 0,  $\ln(x)$  approaches negative infinity. Then 1 divided by something that approaches infinity goes to zero. So we are in  $0^0$  form. We again have functions in the exponent, so we'll use the logarithm trick. Thus,

$$\begin{aligned} \lim_{x \rightarrow 0^+} x^{\frac{-1}{\ln(x)}} &= \lim_{x \rightarrow 0^+} e^{\ln(x^{-1/\ln(x)})} \\ &= \lim_{x \rightarrow 0^+} e^{\frac{\ln(x)}{-\ln(x)}} \\ &\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} e^{\frac{1/x}{-1/x}} \\ &\stackrel{\text{L'H}}{=} e^{1/-1} = e^{-1}. \end{aligned}$$

We could have just simplified  $\frac{\ln(x)}{-\ln(x)}$  to be  $-1$  right off the bat, but we used L'Hôpital's Rule to show that it still works.

- For another example, we'll compute  $\lim_{x \rightarrow 0^+} x^{x^{10}}$ . Since we don't have a ratio, we use our usual transformation to get one to show up. To start,

$$\lim_{x \rightarrow 0^+} x^{x^{10}} = \lim_{x \rightarrow 0^+} e^{\ln(x^{x^{10}})} = \lim_{x \rightarrow 0^+} e^{x^{10} \ln(x)}.$$

We now have a  $0 \cdot \infty$  form, which we dealt with above. To continue, we get

$$\begin{aligned} \lim_{x \rightarrow 0^+} x^{x^{10}} &= \lim_{x \rightarrow 0^+} e^{x^{10} \ln(x)} \\ &= \lim_{x \rightarrow 0^+} e^{\frac{\ln(x)}{x^{-10}}} \\ &\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} e^{\frac{1/x}{-10x^{-11}}} \\ &= \lim_{x \rightarrow 0^+} e^{\frac{-x^{10}}{10}} = 1. \end{aligned}$$

- As our last example, consider  $\lim_{x \rightarrow 0^+} x^{\sin(x)}$ . Since  $\sin(0) = 0$ , we are in the form  $0^0$ . As before, we get

$$\lim_{x \rightarrow 0^+} x^{\sin(x)} = \lim_{x \rightarrow 0^+} e^{\sin(x) \ln(x)} = \lim_{x \rightarrow 0^+} e^{\frac{\ln(x)}{\csc(x)}} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} e^{\frac{1/x}{-\csc(x) \cot(x)}}.$$

This is in the form  $\frac{\infty}{\infty}$ , but before we blindly keep using L'Hôpital's rule, we should simplify the expression. Thus, we get

$$\lim_{x \rightarrow 0^+} e^{\frac{1/x}{-\csc(x) \cot(x)}} = \lim_{x \rightarrow 0^+} e^{\frac{-\sin(x) \sin(x)}{x \cos(x)}},$$

which is in  $\frac{0}{0}$  form. However, we should recognize the limit  $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$ . When we incorporate this into the calculation, we don't even need L'Hôpital's Rule any more (but we could if desired). Thus, we get our final answer of

$$\lim_{x \rightarrow 0^+} e^{\frac{-\sin(x) \sin(x)}{x \cos(x)}} = e^{-1 \cdot \frac{0}{1}} = 1.$$