### THEOREM 6—L'Hôpital's Rule

Suppose that $f(a) = g(a) = 0$, that $f$ and $g$ are differentiable on an open interval $I$ containing $a$, and that $g'(x) \neq 0$ on $I$ if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

**EXAMPLE 1** The following limits involve $0/0$ indeterminate forms, so we apply l'Hôpital's Rule. In some cases, it must be applied repeatedly.

**(a)** $$\lim_{x \to 0} \frac{3x - \sin x}{x} = \lim_{x \to 0} \frac{3 - \cos x}{1} = \frac{3 - \cos x}{1}\bigg|_{x=0} = 2$$

**(b)** $$\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x} = \lim_{x \to 0} \frac{\dfrac{1}{2\sqrt{1 + x}}}{1} = \frac{1}{2}$$

**(c)** $$\lim_{x\to 0}\frac{\sqrt{1+x}-1-x/2}{x^2}$$ $\frac{0}{0}$; apply l'Hôpital's Rule.

$$= \lim_{x\to 0}\frac{(1/2)(1+x)^{-1/2}-1/2}{2x}$$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$$= \lim_{x\to 0}\frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8}$$ Not $\frac{0}{0}$; limit is found.

**(d)** $\lim_{x \to 0} \dfrac{x - \sin x}{x^3}$ $\quad \dfrac{0}{0}$; apply l'Hôpital's Rule.

$= \lim_{x \to 0} \dfrac{1 - \cos x}{3x^2}$ $\quad$ Still $\dfrac{0}{0}$; apply l'Hôpital's Rule again.

$= \lim_{x \to 0} \dfrac{\sin x}{6x}$ $\quad$ Still $\dfrac{0}{0}$; apply l'Hôpital's Rule again.

$= \lim_{x \to 0} \dfrac{\cos x}{6} = \dfrac{1}{6}$ $\quad$ Not $\dfrac{0}{0}$; limit is found.

Here is a summary of the procedure we followed in Example 1.

### Using L'Hôpital's Rule

To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, we continue to differentiate $f$ and $g$, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

**EXAMPLE 2** Be careful to apply l'Hôpital's Rule correctly:

$$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\sin x}{1 + 2x} \qquad \text{Not } \frac{0}{0}$$

It is tempting to try to apply l'Hôpital's Rule again, which would result in

$$\lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2},$$

but this is not the correct limit. l'Hôpital's Rule can be applied only to limits that give indeterminate forms, and $\lim_{x \to 0} (\sin x)/(1 + 2x)$ does not give an indeterminate form. Instead, this limit is $0/1 = 0$, and the correct answer for the original limit is 0. ■

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**EXAMPLE 3** In this example the one-sided limits are different.

(a) $\displaystyle \lim_{x \to 0^+} \frac{\sin x}{x^2}$ $\qquad \frac{0}{0}$

$$= \lim_{x \to 0^+} \frac{\cos x}{2x} = \infty \qquad \text{Positive for } x > 0$$

(b) $\displaystyle \lim_{x \to 0^-} \frac{\sin x}{x^2}$ $\qquad \frac{0}{0}$

$$= \lim_{x \to 0^-} \frac{\cos x}{2x} = -\infty \qquad \text{Negative for } x < 0$$

## Indeterminate Forms $\infty/\infty$, $\infty \cdot 0$, $\infty - \infty$

Sometimes when we try to evaluate a limit as $x \to a$ by substituting $x = a$ we get an indeterminant form like $\infty/\infty$, $\infty \cdot 0$, or $\infty - \infty$, instead of $0/0$. We first consider the form $\infty/\infty$.

More advanced treatments of calculus prove that l'Hôpital's Rule applies to the indeterminate form $\infty/\infty$, as well as to $0/0$. If $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists. In the notation $x \to a$, $a$ may be either finite or infinite. Moreover, $x \to a$ may be replaced by the one-sided limits $x \to a^+$ or $x \to a^-$.

**EXAMPLE 4** Find the limits of these $\infty/\infty$ forms:

**(a)** $\lim_{x \to \pi/2} \frac{\sec x}{1 + \tan x}$ **(b)** $\lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}}$ **(c)** $\lim_{x \to \infty} \frac{e^x}{x^2}$.

**Solution**

**(a)** The numerator and denominator are discontinuous at $x = \pi/2$, so we investigate the one-sided limits there. To apply l'Hôpital's Rule, we can choose $I$ to be any open interval with $x = \pi/2$ as an endpoint.

$$\lim_{x \to (\pi/2)^-} \frac{\sec x}{1 + \tan x}$$ $\frac{\infty}{\infty}$ from left, apply l'Hôpital's Rule

$$= \lim_{x \to (\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \to (\pi/2)^-} \sin x = 1$$

The right-hand limit is 1 also, with $(-\infty)/(-\infty)$ as the indeterminate form. Therefore, the two-sided limit is equal to 1.

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**(b)** $$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0 \qquad \frac{1/x}{1/\sqrt{x}} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$$

**(c)** $$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$ ■

Next we turn our attention to the indeterminate forms $\infty \cdot 0$ and $\infty - \infty$. Sometimes these forms can be handled by using algebra to convert them to a $0/0$ or $\infty/\infty$ form. Here again we do not mean to suggest that $\infty \cdot 0$ or $\infty - \infty$ is a number. They are only notations for functional behaviors when considering limits. Here are examples of how we might work with these indeterminate forms.

**EXAMPLE 5** Find the limits of these $\infty \cdot 0$ forms:

**(a)** $\lim_{x\to\infty}\left(x \sin \frac{1}{x}\right)$ **(b)** $\lim_{x\to 0^+} \sqrt{x} \ln x$

**Solution**

**(a)** $$\lim_{x\to\infty}\left(x \sin \frac{1}{x}\right) = \lim_{h\to 0^+}\left(\frac{1}{h} \sin h\right) = \lim_{h\to 0^+} \frac{\sin h}{h} = 1$$ $\infty \cdot 0$; let $h = 1/x$.

**(b)** $$\lim_{x\to 0^+} \sqrt{x} \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/\sqrt{x}}$$ $\infty \cdot 0$ converted to $\infty/\infty$

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**(a)** $$\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right) = \lim_{h\to 0^+}\left(\frac{1}{h}\sin h\right) = \lim_{h\to 0^+}\frac{\sin h}{h} = 1$$ $\infty \cdot 0$; let $h = 1/x$.

**(b)** $$\lim_{x\to 0^+}\sqrt{x}\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/\sqrt{x}}$$ $\infty \cdot 0$ converted to $\infty/\infty$

$$= \lim_{x\to 0^+}\frac{1/x}{-1/2x^{3/2}}$$ l'Hôpital's Rule applied

$$= \lim_{x\to 0^+}(-2\sqrt{x}) = 0$$ ■

**EXAMPLE 6** Find the limit of this $\infty - \infty$ form:

$$\lim_{x \to 0} \left( \frac{1}{\sin x} - \frac{1}{x} \right).$$

**Solution** If $x \to 0^+$, then $\sin x \to 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to \infty - \infty.$$

Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty - (-\infty) = -\infty + \infty.$$

Neither form reveals what happens in the limit. To find out, we first combine the fractions:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}. \qquad \text{Common denominator is } x \sin x.$$

Then we apply l'Hôpital's Rule to the result:

$$\lim_{x \to 0} \left( \frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} \qquad \text{Still } \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0.$$

## Indeterminate Powers

Limits that lead to the indeterminate forms $1^\infty$, $0^0$, and $\infty^0$ can sometimes be handled by first taking the logarithm of the function. We use l'Hôpital's Rule to find the limit of the logarithm expression and then exponentiate the result to find the original function limit. This procedure is justified by the continuity of the exponential function and Theorem 10 in Section 2.5, and it is formulated as follows. (The formula is also valid for one-sided limits.)

> If $\lim_{x\to a} \ln f(x) = L$, then
>
> $$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L.$$
>
> Here $a$ may be either finite or infinite.

**EXAMPLE 7** Apply l'Hôpital's Rule to show that $\lim_{x \to 0^+} (1 + x)^{1/x} = e$.

**Solution** The limit leads to the indeterminate form $1^\infty$. We let $f(x) = (1 + x)^{1/x}$ and find $\lim_{x \to 0^+} \ln f(x)$. Since

$$\ln f(x) = \ln(1 + x)^{1/x} = \frac{1}{x}\ln(1 + x),$$

l'Hôpital's Rule now applies to give

$$\lim_{x \to 0^+} \ln f(x) = \lim_{x \to 0^+} \frac{\ln(1 + x)}{x} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{1 + x}}{1} \qquad \text{l'Hôpital's Rule applied}$$

$$= \frac{1}{1} = 1.$$

Therefore, $\lim_{x \to 0^+} (1 + x)^{1/x} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{\ln f(x)} = e^1 = e.$ ■

**EXAMPLE 8** Find $\lim_{x\to\infty} x^{1/x}$.

**Solution** The limit leads to the indeterminate form $\infty^0$. We let $f(x) = x^{1/x}$ and find $\lim_{x\to\infty} \ln f(x)$. Since

$$\ln f(x) = \ln x^{1/x} = \frac{\ln x}{x},$$

l'Hôpital's Rule gives

$$\begin{aligned}\lim_{x\to\infty} \ln f(x) &= \lim_{x\to\infty} \frac{\ln x}{x} && \frac{\infty}{\infty}\\ &= \lim_{x\to\infty} \frac{1/x}{1} && \text{l'Hôpital's Rule applied}\\ &= \frac{0}{1} = 0.\end{aligned}$$

Therefore $\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1.$ ■