$$= \lim_{x\to 0} \frac{4\sin 2x}{-4x^2\sin 2x + 4x\cos 2x + 8x\cos 2x + 4\sin 2x + 4\sin x\cos x}$$

$$= \lim_{x\to 0} \frac{4\sin 2x}{(6-4x^2)\sin 2x + 12x\cos 2x}$$

$$= \lim_{x\to 0} \frac{8\cos 2x}{(12-8x^2)\cos 2x - 8x\sin 2x + 12\cos 2x - 24x\sin 2x} = \frac{1}{3}.$$

Numerically we find:

x	1	0.1	0.01
$\frac{1}{\sin^2 x} - \frac{1}{x^2}$	0.412283	0.334001	0.333340

80. In the following cases, check that $x = c$ is a critical point and use Exercise 75 to determine whether $f(c)$ is a local minimum or a local maximum.

(a) $f(x) = x^5 - 6x^4 + 14x^3 - 16x^2 + 9x + 12 \quad (c = 1)$

(b) $f(x) = x^6 - x^3 \quad (c = 0)$

SOLUTION

(a) Let $f(x) = x^5 - 6x^4 + 14x^3 - 16x^2 + 9x + 12$. Then $f'(x) = 5x^4 - 24x^3 + 42x^2 - 32x + 9$, so $f'(1) = 5 - 24 + 42 - 32 + 9 = 0$ and $c = 1$ is a critical point. Now,

$$f''(x) = 20x^3 - 72x^2 + 84x - 32 \text{ so } f''(1) = 0;$$

$$f'''(x) = 60x^2 - 144x + 84 \text{ so } f'''(1) = 0;$$

$$f^{(4)}(x) = 120x - 144 \text{ so } f^{(4)}(1) = -24 \neq 0.$$

Thus, $n = 4$ is even and $f^{(4)} < 0$, so $f(1)$ is a local maximum.

(b) Let $f(x) = x^6 - x^3$. Then, $f'(x) = 6x^5 - 3x^2$, so $f'(0) = 0$ and $c = 0$ is a critical point. Now,

$$f''(x) = 30x^4 - 6x \text{ so } f''(0) = 0;$$

$$f'''(x) = 120x - 6 \text{ so } f'''(0) = -6 \neq 0.$$

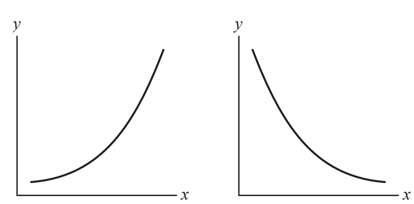
Thus, $n = 3$ is odd, so $f(0)$ is neither a local minimum nor a local maximum.

4.6 Graph Sketching and Asymptotes

Preliminary Questions

1. Sketch an arc where f' and f'' have the sign combination $++$. Do the same for $-+$.

SOLUTION An arc with the sign combination $++$ (increasing, concave up) is shown below at the left. An arc with the sign combination $-+$ (decreasing, concave up) is shown below at the right.



2. If the sign combination of f' and f'' changes from $++$ to $+-$ at $x = c$, then (choose the correct answer):

(a) $f(c)$ is a local min

(b) $f(c)$ is a local max

(c) c is a point of inflection

SOLUTION Because the sign of the second derivative changes at $x = c$, the correct response is **(c)**: c is a point of inflection.

3. The second derivative of the function $f(x) = (x-4)^{-1}$ is $f''(x) = 2(x-4)^{-3}$. Although $f''(x)$ changes sign at $x = 4$, $f(x)$ does not have a point of inflection at $x = 4$. Why not?

SOLUTION The function *f* does not have a point of inflection at $x = 4$ because $x = 4$ is not in the domain of *f*.

Exercises

1. Determine the sign combinations of f' and f'' for each interval A–G in Figure 1.

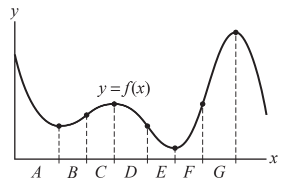


FIGURE 1

SOLUTION

- In A, f is decreasing and concave up, so $f' < 0$ and $f'' > 0$.
- In B, f is increasing and concave up, so $f' > 0$ and $f'' > 0$.
- In C, f is increasing and concave down, so $f' > 0$ and $f'' < 0$.
- In D, f is decreasing and concave down, so $f' < 0$ and $f'' < 0$.
- In E, f is decreasing and concave up, so $f' < 0$ and $f'' > 0$.
- In F, f is increasing and concave up, so $f' > 0$ and $f'' > 0$.
- In G, f is increasing and concave down, so $f' > 0$ and $f'' < 0$.

2. State the sign change at each transition point A–G in Figure 2. Example: $f'(x)$ goes from $+$ to $-$ at A.

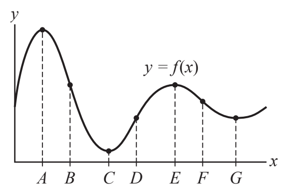


FIGURE 2

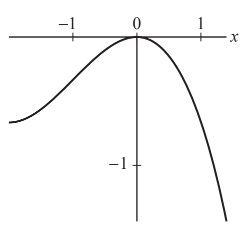
SOLUTION

- At A, the graph changes from increasing to decreasing, so f' goes from $+$ to $-$.
- At B, the graph changes from concave down to concave up, so f'' goes from $-$ to $+$.
- At C, the graph changes from decreasing to increasing, so f' goes from $-$ to $+$.
- At D, the graph changes from concave up to concave down, so f'' goes from $+$ to $-$.
- At E, the graph changes from increasing to decreasing, so f' goes from $+$ to $-$.
- At F, the graph changes from concave down to concave up, so f'' goes from $-$ to $+$.
- At G, the graph changes from decreasing to increasing, so f' goes from $-$ to $+$.

In Exercises 3–6, draw the graph of a function for which f' and f'' take on the given sign combinations.

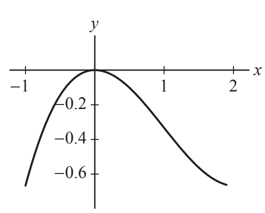
3. $++$, $+-$, $--$

SOLUTION This function changes from concave up to concave down at $x = -1$ and from increasing to decreasing at $x = 0$.



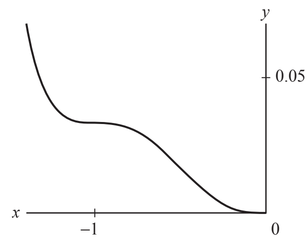
4. $+-$, $--$, $-+$

SOLUTION This function changes from increasing to decreasing at $x = 0$ and from concave down to concave up at $x = 1$.



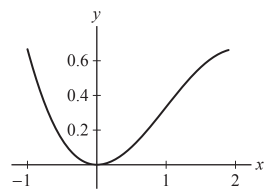
5. $-+,\quad --,\quad -+$

SOLUTION The function is decreasing everywhere and changes from concave up to concave down at $x = -1$ and from concave down to concave up at $x = -\frac{1}{2}$.



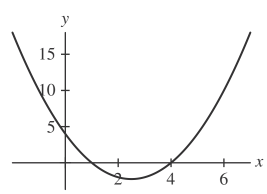
6. $-+,\quad ++,\quad +-$

SOLUTION This function changes from decreasing to increasing at $x = 0$ and from concave up to concave down at $x = 1$.



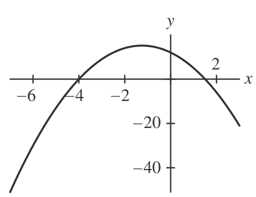
7. Sketch the graph of $y = x^2 - 5x + 4$.

SOLUTION Let $f(x) = x^2 - 5x + 4$. Then $f'(x) = 2x - 5$ and $f''(x) = 2$. Hence f is decreasing for $x < 5/2$, is increasing for $x > 5/2$, has a local minimum at $x = 5/2$ and is concave up everywhere.



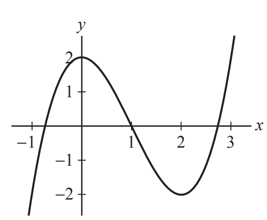
8. Sketch the graph of $y = 12 - 5x - 2x^2$.

SOLUTION Let $f(x) = 12 - 5x - 2x^2$. Then $f'(x) = -5 - 4x$ and $f''(x) = -4$. Hence f is increasing for $x < -5/4$, is decreasing for $x > -5/4$, has a local maximum at $x = -5/4$ and is concave down everywhere.



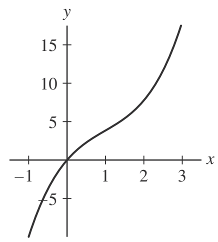
9. Sketch the graph of $f(x) = x^3 - 3x^2 + 2$. Include the zeros of $f(x)$, which are $x = 1$ and $1 \pm \sqrt{3}$ (approximately $-0.73, 2.73$).

SOLUTION Let $f(x) = x^3 - 3x^2 + 2$. Then $f'(x) = 3x^2 - 6x = 3x(x - 2) = 0$ yields $x = 0, 2$ and $f''(x) = 6x - 6$. Thus f is concave down for $x < 1$, is concave up for $x > 1$, has an inflection point at $x = 1$, is increasing for $x < 0$ and for $x > 2$, is decreasing for $0 < x < 2$, has a local maximum at $x = 0$, and has a local minimum at $x = 2$.



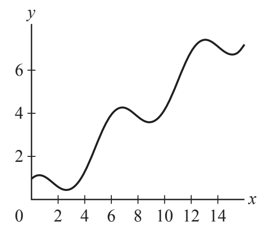
10. Show that $f(x) = x^3 - 3x^2 + 6x$ has a point of inflection but no local extreme values. Sketch the graph.

SOLUTION Let $f(x) = x^3 - 3x^2 + 6x$. Then $f'(x) = 3x^2 - 6x + 6 = 3\left((x - 1)^2 + 1\right) > 0$ for all values of x and $f''(x) = 6x - 6$. Hence f is everywhere increasing and has an inflection point at $x = 1$. It is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.



11. Extend the sketch of the graph of $f(x) = \cos x + \frac{1}{2}x$ in Example 4 to the interval $[0, 5\pi]$.

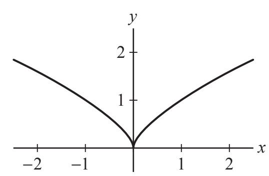
SOLUTION Let $f(x) = \cos x + \frac{1}{2}x$. Then $f'(x) = -\sin x + \frac{1}{2} = 0$ yields critical points at $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}$, and $\frac{29\pi}{6}$. Moreover, $f''(x) = -\cos x$ so there are points of inflection at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$, and $\frac{9\pi}{2}$.



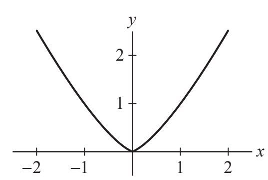
12. Sketch the graphs of $y = x^{2/3}$ and $y = x^{4/3}$.

SOLUTION

- Let $f(x) = x^{2/3}$. Then $f'(x) = \frac{2}{3}x^{-1/3}$ and $f''(x) = -\frac{2}{9}x^{-4/3}$, neither of which exist at $x = 0$. Thus f is decreasing and concave down for $x < 0$ and increasing and concave down for $x > 0$.



- Let $f(x) = x^{4/3}$. Then $f'(x) = \frac{4}{3}x^{1/3}$ and $f''(x) = \frac{4}{9}x^{-2/3}$. Thus f is decreasing and concave up for $x < 0$ and increasing and concave up for $x > 0$.



In Exercises 13–34, find the transition points, intervals of increase/decrease, concavity, and asymptotic behavior. Then sketch the graph, with this information indicated.

13. $y = x^3 + 24x^2$

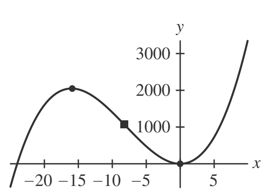
SOLUTION Let $f(x) = x^3 + 24x^2$. Then $f'(x) = 3x^2 + 48x = 3x(x + 16)$ and $f''(x) = 6x + 48$. This shows that f has critical points at $x = 0$ and $x = -16$ and a candidate for an inflection point at $x = -8$.

Interval	$(-\infty, -16)$	$(-16, -8)$	$(-8, 0)$	$(0, \infty)$
Signs of f' and f''	$+-$	$--$	$-+$	$++$

Thus, there is a local maximum at $x = -16$, a local minimum at $x = 0$, and an inflection point at $x = -8$. Moreover,

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \lim_{x\to\infty} f(x) = \infty.$$

Here is a graph of f with these transition points highlighted as in the graphs in the textbook.



14. $y = x^3 - 3x + 5$

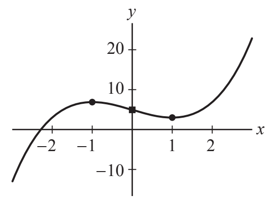
SOLUTION Let $f(x) = x^3 - 3x + 5$. Then $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$. Critical points are at $x = \pm 1$ and the sole candidate point of inflection is at $x = 0$.

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Signs of f' and f''	$+-$	$--$	$-+$	$++$

Thus, $f(-1)$ is a local maximum, $f(1)$ is a local minimum, and there is a point of inflection at $x = 0$. Moreover,

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \lim_{x\to\infty} f(x) = \infty.$$

Here is the graph of f with the transition points highlighted as in the textbook.



15. $y = x^2 - 4x^3$

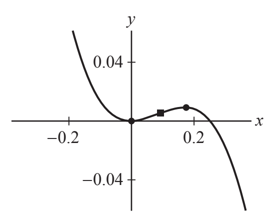
SOLUTION Let $f(x) = x^2 - 4x^3$. Then $f'(x) = 2x - 12x^2 = 2x(1 - 6x)$ and $f''(x) = 2 - 24x$. Critical points are at $x = 0$ and $x = \frac{1}{6}$, and the sole candidate point of inflection is at $x = \frac{1}{12}$.

Interval	$(-\infty, 0)$	$(0, \frac{1}{12})$	$(\frac{1}{12}, \frac{1}{6})$	$(\frac{1}{6}, \infty)$
Signs of f' and f''	$-+$	$++$	$+-$	$--$

Thus, $f(0)$ is a local minimum, $f(\frac{1}{6})$ is a local maximum, and there is a point of inflection at $x = \frac{1}{12}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is the graph of f with transition points highlighted as in the textbook:

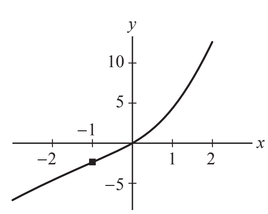


16. $y = \frac{1}{3}x^3 + x^2 + 3x$

SOLUTION Let $f(x) = \frac{1}{3}x^3 + x^2 + 3x$. Then $f'(x) = x^2 + 2x + 3$, and $f''(x) = 2x + 2 = 0$ if $x = -1$. Sign analysis shows that $f'(x) = (x + 1)^2 + 2 > 0$ for all x (so that $f(x)$ has no critical points and is always increasing), and that $f''(x)$ changes from negative to positive at $x = -1$, implying that the graph of $f(x)$ has an inflection point at $(-1, f(-1))$. Moreover,

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \lim_{x\to\infty} f(x) = \infty.$$

A graph with the inflection point indicated appears below:



17. $y = 4 - 2x^2 + \frac{1}{6}x^4$

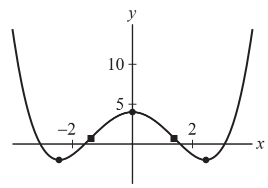
SOLUTION Let $f(x) = \frac{1}{6}x^4 - 2x^2 + 4$. Then $f'(x) = \frac{2}{3}x^3 - 4x = \frac{2}{3}x\left(x^2 - 6\right)$ and $f''(x) = 2x^2 - 4$. This shows that f has critical points at $x = 0$ and $x = \pm\sqrt{6}$ and has candidates for points of inflection at $x = \pm\sqrt{2}$.

Interval	$(-\infty,-\sqrt{6})$	$(-\sqrt{6},-\sqrt{2})$	$(-\sqrt{2},0)$	$(0,\sqrt{2})$	$(\sqrt{2},\sqrt{6})$	$(\sqrt{6},\infty)$
Signs of f' and f''	$-+$	$++$	$+-$	$--$	$-+$	$++$

Thus, f has local minima at $x=\pm\sqrt{6}$, a local maximum at $x=0$, and inflection points at $x=\pm\sqrt{2}$. Moreover,

$$\lim_{x\to\pm\infty} f(x)=\infty.$$

Here is a graph of f with transition points highlighted.



18. $y=7x^4-6x^2+1$

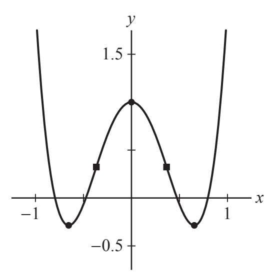
SOLUTION Let $f(x)=7x^4-6x^2+1$. Then $f'(x)=28x^3-12x=4x\left(7x^2-3\right)$ and $f''(x)=84x^2-12$. This shows that f has critical points at $x=0$ and $x=\pm\frac{\sqrt{21}}{7}$ and candidates for points of inflection at $x=\pm\frac{\sqrt{7}}{7}$.

Interval	$(-\infty,-\frac{\sqrt{21}}{7})$	$(-\frac{\sqrt{21}}{7},-\frac{\sqrt{7}}{7})$	$(-\frac{\sqrt{7}}{7},0)$	$(0,\frac{\sqrt{7}}{7})$	$(\frac{\sqrt{7}}{7},\frac{\sqrt{21}}{7})$	$(\frac{\sqrt{21}}{7},\infty)$
Signs of f' and f''	$-+$	$++$	$+-$	$--$	$-+$	$++$

Thus, f has local minima at $x=\pm\frac{\sqrt{21}}{7}$, a local maximum at $x=0$, and inflection points at $x=\pm\frac{\sqrt{7}}{7}$. Moreover,

$$\lim_{x\to\pm\infty} f(x)=\infty.$$

Here is a graph of f with transition points highlighted.

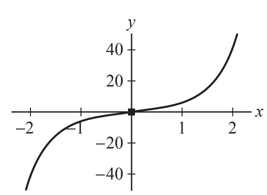


19. $y=x^5+5x$

SOLUTION Let $f(x)=x^5+5x$. Then $f'(x)=5x^4+5=5(x^4+1)$ and $f''(x)=20x^3$. $f'(x)>0$ for all x, so the graph has no critical points and is always increasing. $f''(x)=0$ at $x=0$. Sign analyses reveal that $f''(x)$ changes from negative to positive at $x=0$, so that the graph of $f(x)$ has an inflection point at $(0,0)$. Moreover,

$$\lim_{x\to-\infty} f(x)=-\infty \quad\text{and}\quad \lim_{x\to\infty} f(x)=\infty.$$

Here is a graph of f with transition points highlighted.

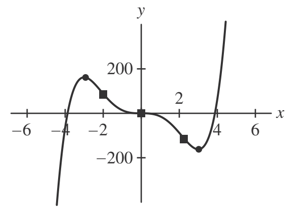


20. $y=x^5-15x^3$

SOLUTION Let $f(x)=x^5-15x^3$. Then $f'(x)=5x^4-45x^2=5x^2(x^2-9)$ and $f''(x)=20x^3-90x=10x(2x^2-9)$. This shows that f has critical points at $x=0$ and $x=\pm3$ and candidate inflection points at $x=0$ and $x=\pm3\sqrt{2}/2$. Sign analyses reveal that $f'(x)$ changes from positive to negative at $x=-3$, is negative on either side of $x=0$ and changes from negative to positive at $x=3$. The graph therefore has a local maximum at $x=-3$ and a local minimum at $x=3$. Further sign analyses show that $f''(x)$ transitions from positive to negative at $x = 0$ and from negative to positive at $x = \pm 3\sqrt{2}/2$. The graph therefore has points of inflection at $x = 0$ and $x = \pm 3\sqrt{2}/2$. Moreover,

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = \infty.$$

Here is a graph of f with transition points highlighted.

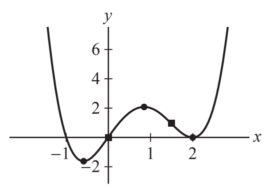


21. $y = x^4 - 3x^3 + 4x$

SOLUTION Let $f(x) = x^4 - 3x^3 + 4x$. Then $f'(x) = 4x^3 - 9x^2 + 4 = (4x^2 - x - 2)(x - 2)$ and $f''(x) = 12x^2 - 18x = 6x(2x - 3)$. This shows that f has critical points at $x = 2$ and $x = \dfrac{1 \pm \sqrt{33}}{8}$ and candidate points of inflection at $x = 0$ and $x = \frac{3}{2}$. Sign analyses reveal that $f'(x)$ changes from negative to positive at $x = \frac{1-\sqrt{33}}{8}$, from positive to negative at $x = \frac{1+\sqrt{33}}{8}$, and again from negative to positive at $x = 2$. Therefore, $f(\frac{1-\sqrt{33}}{8})$ and $f(2)$ are local minima of $f(x)$, and $f(\frac{1+\sqrt{33}}{8})$ is a local maximum. Further sign analyses reveal that $f''(x)$ changes from positive to negative at $x = 0$ and from negative to positive at $x = \frac{3}{2}$, so that there are points of inflection both at $x = 0$ and $x = \frac{3}{2}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is a graph of $f(x)$ with transition points highlighted.



22. $y = x^2(x - 4)^2$

SOLUTION Let $f(x) = x^2(x - 4)^2$. Then

$$f'(x) = 2x(x-4)^2 + 2x^2(x-4) = 2x(x-4)(x-4+x) = 4x(x-4)(x-2)$$

and

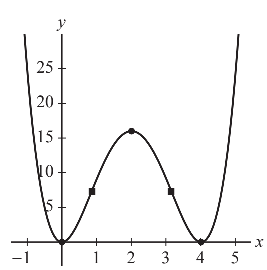
$$f''(x) = 12x^2 - 48x + 32 = 4(3x^2 - 12x + 8).$$

Critical points are therefore at $x = 0$, $x = 4$, and $x = 2$. Candidate inflection points are at solutions of $4(3x^2 - 12x + 8) = 0$, which, from the quadratic formula, are at $2 \pm \frac{\sqrt{48}}{6} = 2 \pm \frac{2\sqrt{3}}{3}$.

Sign analyses reveal that $f'(x)$ changes from negative to positive at $x = 0$ and $x = 4$, and from positive to negative at $x = 2$. Therefore, $f(0)$ and $f(4)$ are local minima, and $f(2)$ a local maximum, of $f(x)$. Also, $f''(x)$ changes from positive to negative at $2 - \frac{2\sqrt{3}}{3}$ and from negative to positive at $2 + \frac{2\sqrt{3}}{3}$. Therefore there are points of inflection at both $x = 2 \pm \frac{2\sqrt{3}}{3}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is a graph of $f(x)$ with transition points highlighted.

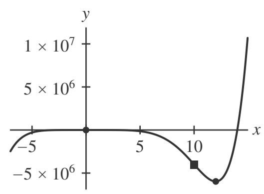


23. $y = x^7 - 14x^6$

SOLUTION Let $f(x) = x^7 - 14x^6$. Then $f'(x) = 7x^6 - 84x^5 = 7x^5(x - 12)$ and $f''(x) = 42x^5 - 420x^4 = 42x^4(x - 10)$. Critical points are at $x = 0$ and $x = 12$, and candidate inflection points are at $x = 0$ and $x = 10$. Sign analyses reveal that $f'(x)$ changes from positive to negative at $x = 0$ and from negative to positive at $x = 12$. Therefore $f(0)$ is a local maximum and $f(12)$ is a local minimum. Also, $f''(x)$ changes from negative to positive at $x = 10$. Therefore, there is a point of inflection at $x = 10$. Moreover,

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = \infty.$$

Here is a graph of f with transition points highlighted.

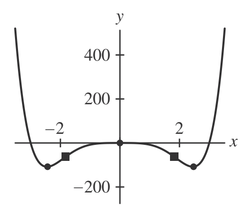


24. $y = x^6 - 9x^4$

SOLUTION Let $f(x) = x^6 - 9x^4$. Then $f'(x) = 6x^5 - 36x^3 = 6x^3(x^2 - 6)$ and $f''(x) = 30x^4 - 108x^2 = 6x^2(5x^2 - 18)$. This shows that f has critical points at $x = 0$ and $x = \pm\sqrt{6}$ and candidate inflection points at $x = 0$ and $x = \pm 3\sqrt{10}/5$. Sign analyses reveal that $f'(x)$ changes from negative to positive at $x = -\sqrt{6}$, from positive to negative at $x = 0$ and from negative to positive at $x = \sqrt{6}$. The graph therefore has a local maximum at $x = 0$ and local minima at $x = \pm\sqrt{6}$. Further sign analyses show that $f''(x)$ transitions from positive to negative at $x = -3\sqrt{10}/5$ and from negative to positive at $x = 3\sqrt{10}/5$. The graph therefore has points of inflection at $x = \pm 3\sqrt{10}/5$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is a graph of f with transition points highlighted.

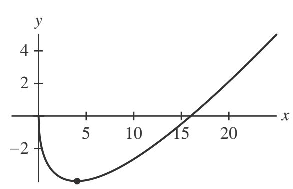


25. $y = x - 4\sqrt{x}$

SOLUTION Let $f(x) = x - 4\sqrt{x} = x - 4x^{1/2}$. Then $f'(x) = 1 - 2x^{-1/2}$. This shows that f has critical points at $x = 0$ (where the derivative does not exist) and at $x = 4$ (where the derivative is zero). Because $f'(x) < 0$ for $0 < x < 4$ and $f'(x) > 0$ for $x > 4$, $f(4)$ is a local minimum. Now $f''(x) = x^{-3/2} > 0$ for all $x > 0$, so the graph is always concave up. Moreover,

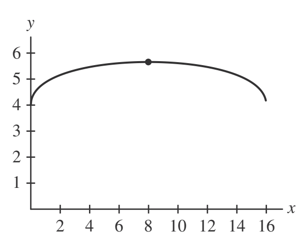
$$\lim_{x\to\infty} f(x) = \infty.$$

Here is a graph of f with transition points highlighted.



26. $y = \sqrt{x} + \sqrt{16 - x}$

SOLUTION Let $f(x) = \sqrt{x} + \sqrt{16 - x} = x^{1/2} + (16 - x)^{1/2}$. Note that the domain of f is $[0, 16]$. Now, $f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}(16 - x)^{-1/2}$ and $f''(x) = -\frac{1}{4}x^{-3/2} - \frac{1}{4}(16 - x)^{-3/2}$. Thus, the critical points are $x = 0$, $x = 8$ and $x = 16$. Sign analysis reveals that $f'(x) > 0$ for $0 < x < 8$ and $f'(x) < 0$ for $8 < x < 16$, so f has a local maximum at $x = 9$. Further, $f''(x) < 0$ on $(0, 16)$, so the graph is always concave down. Here is a graph of f with the transition point highlighted.



27. $y = x(8-x)^{1/3}$

SOLUTION Let $f(x) = x\,(8-x)^{1/3}$. Then

$$f'(x) = x \cdot \tfrac{1}{3}\,(8-x)^{-2/3}\,(-1) + (8-x)^{1/3} \cdot 1 = \frac{24-4x}{3\,(8-x)^{2/3}}$$

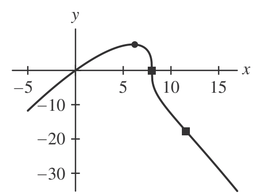
and similarly

$$f''(x) = \frac{4x-48}{9\,(8-x)^{5/3}}.$$

Critical points are at $x = 8$ and $x = 6$, and candidate inflection points are $x = 8$ and $x = 12$. Sign analyses reveal that $f'(x)$ changes from positive to negative at $x = 6$ and $f'(x)$ remains negative on either side of $x = 8$. Moreover, $f''(x)$ changes from negative to positive at $x = 8$ and from positive to negative at $x = 12$. Therefore, f has a local maximum at $x = 6$ and inflection points at $x = 8$ and $x = 12$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = -\infty.$$

Here is a graph of f with the transition points highlighted.



28. $y = (x^2-4x)^{1/3}$

SOLUTION Let $f(x) = (x^2-4x)^{1/3}$. Then

$$f'(x) = \frac{2}{3}(x-2)(x^2-4x)^{-2/3}$$

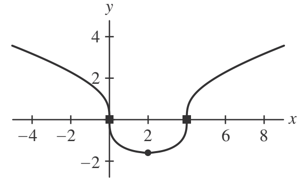
and

$$f''(x) = \frac{2}{3}\left((x^2-4x)^{-2/3} - \frac{4}{3}(x-2)^2(x^2-4x)^{-5/3}\right)$$
$$= \frac{2}{9}(x^2-4x)^{-5/3}\left(3(x^2-4x) - 4(x-2)^2\right) = -\frac{2}{9}(x^2-4x)^{-5/3}(x^2-4x+16).$$

Critical points of $f(x)$ are $x = 2$ (where the derivative is zero) an $x = 0$ and $x = 4$ (where the derivative does not exist); candidate points of inflection are $x = 0$ and $x = 4$. Sign analyses reveal that $f''(x) < 0$ for $x < 0$ and for $x > 4$, while $f''(x) > 0$ for $0 < x < 4$. Therefore, the graph of $f(x)$ has points of inflection at $x = 0$ and $x = 4$. Since $(x^2-x)^{-2/3}$ is positive wherever it is defined, the sign of $f'(x)$ depends solely on the sign of $x-2$. Hence, $f'(x)$ does not change sign at $x = 0$ or $x = 4$, and goes from negative to positive at $x = 2$. $f(2)$ is, in that case, a local minimum. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is a graph of $f(x)$ with the transition points indicated.



29. $y = xe^{-x^2}$

SOLUTION Let $f(x) = xe^{-x^2}$. Then

$$f'(x) = -2x^2e^{-x^2} + e^{-x^2} = (1-2x^2)e^{-x^2},$$

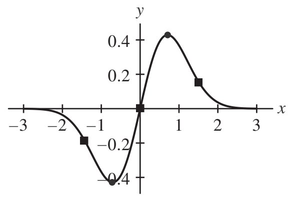
and

$$f''(x) = (4x^3-2x)e^{-x^2} - 4xe^{-x^2} = 2x(2x^2-3)e^{-x^2}.$$

There are critical points at $x = \pm\frac{\sqrt{2}}{2}$, and $x = 0$ and $x = \pm\frac{\sqrt{3}}{2}$ are candidates for inflection points. Sign analysis shows that $f'(x)$ changes from negative to positive at $x = -\frac{\sqrt{2}}{2}$ and from positive to negative at $x = \frac{\sqrt{2}}{2}$. Moreover, $f''(x)$ changes from negative to positive at both $x = \pm\frac{\sqrt{3}}{2}$ and from positive to negative at $x = 0$. Therefore, f has a local minimum at $x = -\frac{\sqrt{2}}{2}$, a local maximum at $x = \frac{\sqrt{2}}{2}$ and inflection points at $x = 0$ and at $x = \pm\frac{\sqrt{3}}{2}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = 0,$$

so the graph has a horizontal asymptote at $y = 0$. Here is a graph of f with the transition points highlighted.



30. $y = (2x^2 - 1)e^{-x^2}$

SOLUTION Let $f(x) = (2x^2 - 1)e^{-x^2}$. Then

$$f'(x) = (2x - 4x^3)e^{-x^2} + 4xe^{-x^2} = 2x(3 - 2x^2)e^{-x^2},$$

and

$$f''(x) = (8x^4 - 12x^2)e^{-x^2} + (6 - 12x^2)e^{-x^2} = 2(4x^4 - 12x^2 + 3)e^{-x^2}.$$

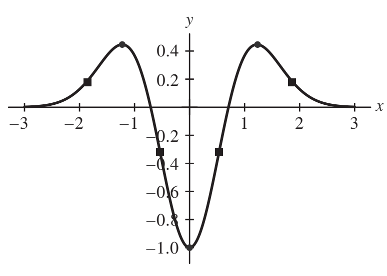
There are critical points at $x = 0$ and $x = \pm\frac{\sqrt{3}}{2}$, and

$$x = -\sqrt{\frac{3+\sqrt{6}}{2}}, \quad x = -\sqrt{\frac{3-\sqrt{6}}{2}}, \quad x = \sqrt{\frac{3-\sqrt{6}}{2}}, \quad x = \sqrt{\frac{3+\sqrt{6}}{2}}$$

are candidates for inflection points. Sign analysis shows that $f'(x)$ changes from positive to negative at $x = \pm\frac{\sqrt{3}}{2}$ and from negative to positive at $x = 0$. Moreover, $f''(x)$ changes from positive to negative at $x = -\sqrt{\frac{3+\sqrt{6}}{2}}$ and at $x = \sqrt{\frac{3-\sqrt{6}}{2}}$ and from negative to positive at $x = -\sqrt{\frac{3-\sqrt{6}}{2}}$ and at $x = \sqrt{\frac{3+\sqrt{6}}{2}}$. Therefore, f has local maxima at $x = \pm\frac{\sqrt{3}}{2}$, a local minimum at $x = 0$ and points of inflection at $x = \pm\sqrt{\frac{3\pm\sqrt{6}}{2}}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = 0,$$

so the graph has a horizontal asymptote at $y = 0$. Here is a graph of f with the transition points highlighted.



31. $y = x - 2\ln x$

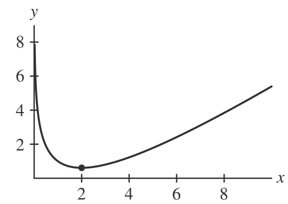
SOLUTION Let $f(x) = x - 2\ln x$. Note that the domain of f is $x > 0$. Now,

$$f'(x) = 1 - \frac{2}{x} \quad \text{and} \quad f''(x) = \frac{2}{x^2}.$$

The only critical point is $x = 2$. Sign analysis shows that $f'(x)$ changes from negative to positive at $x = 2$, so $f(2)$ is a local minimum. Further, $f''(x) > 0$ for $x > 0$, so the graph is always concave up. Moreover,

$$\lim_{x\to\infty} f(x) = \infty.$$

Here is a graph of f with the transition points highlighted.



32. $y = x(4 - x) - 3\ln x$

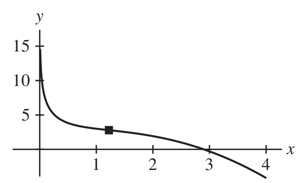
SOLUTION Let $f(x) = x(4 - x) - 3\ln x$. Note that the domain of f is $x > 0$. Now,

$$f'(x) = 4 - 2x - \frac{3}{x} \quad \text{and} \quad f''(x) = -2 + \frac{3}{x^2}.$$

Because $f'(x) < 0$ for all $x > 0$, the graph is always decreasing. On the other hand, $f''(x)$ changes from positive to negative at $x = \sqrt{\frac{3}{2}}$, so there is a point of inflection at $x = \sqrt{\frac{3}{2}}$. Moreover,

$$\lim_{x\to 0+} f(x) = \infty \quad \text{and} \quad \lim_{x\to infty} f(x) = -\infty,$$

so f has a vertical asymptote at $x = 0$. Here is a graph of f with the transition points highlighted.

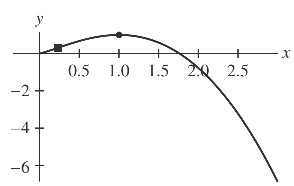


33. $y = x - x^2 \ln x$

SOLUTION Let $f(x) = x - x^2 \ln x$. Then $f'(x) = 1 - x - 2x\ln x$ and $f''(x) = -3 - 2\ln x$. There is a critical point at $x = 1$, and $x = e^{-3/2} \approx 0.223$ is a candidate inflection point. Sign analysis shows that $f'(x)$ changes from positive to negative at $x = 1$ and that $f''(x)$ changes from positive to negative at $x = e^{-3/2}$. Therefore, f has a local maximum at $x = 1$ and a point of inflection at $x = e^{-3/2}$. Moreover,

$$\lim_{x\to\infty} f(x) = -\infty.$$

Here is a graph of f with the transition points highlighted.



34. $y = x - 2\ln(x^2 + 1)$

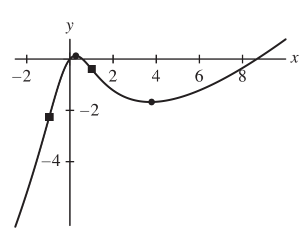
SOLUTION Let $f(x) = x - 2\ln(x^2 + 1)$. Then $f'(x) = 1 - \dfrac{4x}{x^2 + 1}$, and

$$f''(x) = -\frac{(x^2 + 1)(4) - (4x)(2x)}{(x^2 + 1)^2} = \frac{4(x^2 - 1)}{(x^2 + 1)^2}.$$

There are critical points at $x = 2 \pm \sqrt{3}$, and $x = \pm 1$ are candidates for inflection points. Sign analysis shows that $f'(x)$ changes from positive to negative at $x = 2 - \sqrt{3}$ and from negative to positive at $x = 2 + \sqrt{3}$. Moreover, $f''(x)$ changes from positive to negative at $x = -1$ and from negative to positive at $x = 1$. Therefore, f has a local maximum at $x = 2 - \sqrt{3}$, a local minimum at $x = 2 + \sqrt{3}$ and points of inflection at $x = \pm 1$. Finally,

$$\lim_{x\to -\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = \infty.$$

Here is a graph of f with the transition points highlighted.



35. Sketch the graph of $f(x) = 18(x-3)(x-1)^{2/3}$ using the formulas

$$f'(x) = \frac{30\left(x - \frac{9}{5}\right)}{(x-1)^{1/3}}, \qquad f''(x) = \frac{20\left(x - \frac{3}{5}\right)}{(x-1)^{4/3}}$$

SOLUTION

$$f'(x) = \frac{30(x - \frac{9}{5})}{(x-1)^{1/3}}$$

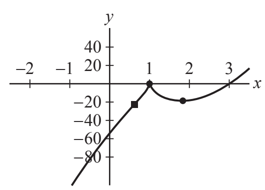
yields critical points at $x = \frac{9}{5}$, $x = 1$.

$$f''(x) = \frac{20(x - \frac{3}{5})}{(x-1)^{4/3}}$$

yields potential inflection points at $x = \frac{3}{5}$, $x = 1$.

Interval	signs of f' and f''
$(-\infty, \frac{3}{5})$	$+-$
$(\frac{3}{5}, 1)$	$++$
$(1, \frac{9}{5})$	$-+$
$(\frac{9}{5}, \infty)$	$++$

The graph has an inflection point at $x = \frac{3}{5}$, a local maximum at $x = 1$ (at which the graph has a cusp), and a local minimum at $x = \frac{9}{5}$. The sketch looks something like this.



36. Sketch the graph of $f(x) = \dfrac{x}{x^2+1}$ using the formulas

$$f'(x) = \frac{1-x^2}{(1+x^2)^2}, \qquad f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}$$

SOLUTION Let $f(x) = \dfrac{x}{x^2+1}$.

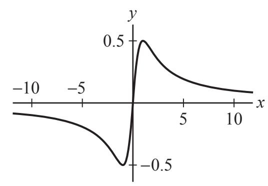
- Because $\lim_{x\to\pm\infty} f(x) = \frac{1}{1} \cdot \lim_{x\to\pm\infty} x^{-1} = 0$, $y = 0$ is a horizontal asymptote for f.
- Now $f'(x) = \dfrac{1-x^2}{\left(x^2+1\right)^2}$ is negative for $x < -1$ and $x > 1$, positive for $-1 < x < 1$, and 0 at $x = \pm 1$. Accordingly, f is decreasing for $x < -1$ and $x > 1$, is increasing for $-1 < x < 1$, has a local minimum value at $x = -1$ and a local maximum value at $x = 1$.
- Moreover,

$$f''(x) = \frac{2x\left(x^2-3\right)}{\left(x^2+1\right)^3}.$$

Here is a sign chart for the second derivative, similar to those constructed in various exercises in Section 4.4. (The legend is on page 408.)

x	$\left(-\infty, -\sqrt{3}\right)$	$-\sqrt{3}$	$\left(-\sqrt{3}, 0\right)$	0	$\left(0, \sqrt{3}\right)$	$\sqrt{3}$	$\left(\sqrt{3}, \infty\right)$
f''	$-$	0	$+$	0	$-$	0	$+$
f	$\frown$	I	$\smile$	I	$\frown$	I	$\smile$

- Here is a graph of $f(x) = \dfrac{x}{x^2+1}$.



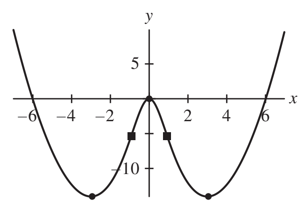
CAS *In Exercises 37–40, sketch the graph of the function, indicating all transition points. If necessary, use a graphing utility or computer algebra system to locate the transition points numerically.*

37. $y = x^2 - 10\ln(x^2+1)$

SOLUTION Let $f(x) = x^2 - 10\ln(x^2+1)$. Then $f'(x) = 2x - \dfrac{20x}{x^2+1}$, and

$$f''(x) = 2 - \frac{(x^2+1)(20) - (20x)(2x)}{(x^2+1)^2} = \frac{x^4+12x^2-9}{(x^2+1)^2}.$$

There are critical points at $x = 0$ and $x = \pm 3$, and $x = \pm\sqrt{-6+3\sqrt{5}}$ are candidates for inflection points. Sign analysis shows that $f'(x)$ changes from negative to positive at $x = \pm 3$ and from positive to negative at $x = 0$. Moreover, $f''(x)$ changes from positive to negative at $x = -\sqrt{-6+3\sqrt{5}}$ and from negative to positive at $x = \sqrt{-6+3\sqrt{5}}$. Therefore, f has a local maximum at $x = 0$, local minima at $x = \pm 3$ and points of inflection at $x = \pm\sqrt{-6+3\sqrt{5}}$. Here is a graph of f with the transition points highlighted.



38. $y = e^{-x/2}\ln x$

SOLUTION Let $f(x) = e^{-x/2}\ln x$. Then

$$f'(x) = \frac{e^{-x/2}}{x} - \frac{1}{2}e^{-x/2}\ln x = e^{-x/2}\left(\frac{1}{x} - \frac{1}{2}\ln x\right)$$

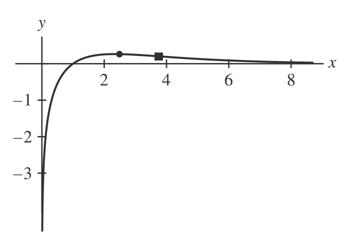
and

$$f''(x) = e^{-x/2}\left(-\frac{1}{x^2} - \frac{1}{2x}\right) - \frac{1}{2}e^{-x/2}\left(\frac{1}{x} - \frac{1}{2}\ln x\right)$$
$$= e^{-x/2}\left(-\frac{1}{x^2} - \frac{1}{x} + \frac{1}{4}\ln x\right).$$

There is a critical point at $x = 2.345751$ and a candidate point of inflection at $x = 3.792199$. Sign analysis reveals that $f'(x)$ changes from positive to negative at $x = 2.345751$ and that $f''(x)$ changes from negative to positive at $x = 3.792199$. Therefore, f has a local maximum at $x = 2.345751$ and a point of inflection at $x = 3.792199$. Moreover,

$$\lim_{x\to 0+} f(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = 0.$$

Here is a graph of f with the transition points highlighted.

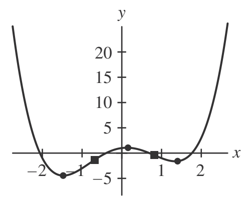


39. $y = x^4 - 4x^2 + x + 1$

SOLUTION Let $f(x) = x^4 - 4x^2 + x + 1$. Then $f'(x) = 4x^3 - 8x + 1$ and $f''(x) = 12x^2 - 8$. The critical points are $x = -1.473$, $x = 0.126$ and $x = 1.347$, while the candidates for points of inflection are $x = \pm\sqrt{\frac{2}{3}}$. Sign analysis reveals that $f'(x)$ changes from negative to positive at $x = -1.473$, from positive to negative at $x = 0.126$ and from negative to positive at $x = 1.347$. For the second derivative, $f''(x)$ changes from positive to negative at $x = -\sqrt{\frac{2}{3}}$ and from negative to positive at $x = \sqrt{\frac{2}{3}}$. Therefore, f has local minima at $x = -1.473$ and $x = 1.347$, a local maximum at $x = 0.126$ and points of inflection at $x = \pm\sqrt{\frac{2}{3}}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is a graph of f with the transition points highlighted.

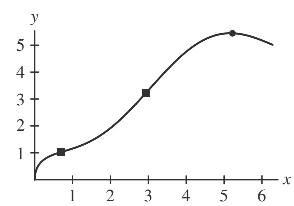


40. $y = 2\sqrt{x} - \sin x, \quad 0 \le x \le 2\pi$

SOLUTION Let $f(x) = 2\sqrt{x} - \sin x$. Then

$$f'(x) = \frac{1}{\sqrt{x}} - \cos x \quad \text{and} \quad f''(x) = -\frac{1}{2}x^{-3/2} + \sin x.$$

On $0 \le x \le 2\pi$, there is a critical point at $x = 5.167866$ and candidate points of inflection at $x = 0.790841$ and $x = 3.047468$. Sign analysis reveals that $f'(x)$ changes from positive to negative at $x = 5.167866$, while $f''(x)$ changes from negative to positive at $x = 0.790841$ and from positive to negative at $x = 3.047468$. Therefore, f has a local maximum at $x = 5.167866$ and points of inflection at $x = 0.790841$ and $x = 3.047468$. Here is a graph of f with the transition points highlighted.



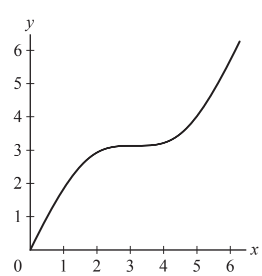
In Exercises 41–46, sketch the graph over the given interval, with all transition points indicated.

41. $y = x + \sin x, \quad [0, 2\pi]$

SOLUTION Let $f(x) = x + \sin x$. Setting $f'(x) = 1 + \cos x = 0$ yields $\cos x = -1$, so that $x = \pi$ is the lone critical point on the interval $[0, 2\pi]$. Setting $f''(x) = -\sin x = 0$ yields potential points of inflection at $x = 0, \pi, 2\pi$ on the interval $[0, 2\pi]$.

Interval	signs of f' and f''
$(0, \pi)$	$+-$
$(\pi, 2\pi)$	$++$

The graph has an inflection point at $x = \pi$, and no local maxima or minima. Here is a sketch of the graph of $f(x)$:

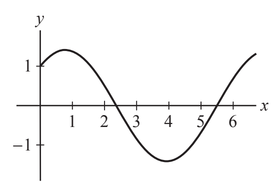


42. $y = \sin x + \cos x, \quad [0, 2\pi]$

SOLUTION Let $f(x) = \sin x + \cos x$. Setting $f'(x) = \cos x - \sin x = 0$ yields $\sin x = \cos x$, so that $\tan x = 1$, and $x = \frac{\pi}{4}, \frac{5\pi}{4}$. Setting $f''(x) = -\sin x - \cos x = 0$ yields $\sin x = -\cos x$, so that $-\tan x = 1$, and $x = \frac{3\pi}{4}, x = \frac{7\pi}{4}$.

Interval	signs of f' and f''
$(0, \frac{\pi}{4})$	$+-$
$(\frac{\pi}{4}, \frac{3\pi}{4})$	$--$
$(\frac{3\pi}{4}, \frac{5\pi}{4})$	$-+$
$(\frac{5\pi}{4}, \frac{7\pi}{4})$	$++$
$(\frac{7\pi}{4}, 2\pi)$	$+-$

The graph has a local maximum at $x = \frac{\pi}{4}$, a local minimum at $x = \frac{5\pi}{4}$, and inflection points at $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$. Here is a sketch of the graph of $f(x)$:

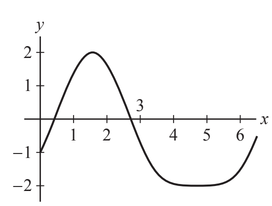


43. $y = 2\sin x - \cos^2 x, \quad [0, 2\pi]$

SOLUTION Let $f(x) = 2\sin x - \cos^2 x$. Then $f'(x) = 2\cos x - 2\cos x\,(-\sin x) = \sin 2x + 2\cos x$ and $f''(x) = 2\cos 2x - 2\sin x$. Setting $f'(x) = 0$ yields $\sin 2x = -2\cos x$, so that $2\sin x \cos x = -2\cos x$. This implies $\cos x = 0$ or $\sin x = -1$, so that $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Setting $f''(x) = 0$ yields $2\cos 2x = 2\sin x$, so that $2\sin(\frac{\pi}{2} - 2x) = 2\sin x$, or $\frac{\pi}{2} - 2x = x \pm 2n\pi$. This yields $3x = \frac{\pi}{2} + 2n\pi$, or $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{9\pi}{6} = \frac{3\pi}{2}$.

Interval	signs of f' and f''
$(0, \frac{\pi}{6})$	$++$
$(\frac{\pi}{6}, \frac{\pi}{2})$	$+-$
$(\frac{\pi}{2}, \frac{5\pi}{6})$	$--$
$(\frac{5\pi}{6}, \frac{3\pi}{2})$	$-+$
$(\frac{3\pi}{2}, 2\pi)$	$++$

The graph has a local maximum at $x = \frac{\pi}{2}$, a local minimum at $x = \frac{3\pi}{2}$, and inflection points at $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$. Here is a graph of f *without* transition points highlighted.

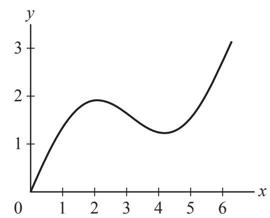


44. $y = \sin x + \frac{1}{2}x, \quad [0, 2\pi]$

SOLUTION Let $f(x) = \sin x + \frac{1}{2}x$. Setting $f'(x) = \cos x + \frac{1}{2} = 0$ yields $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$. Setting $f''(x) = -\sin x = 0$ yields potential points of inflection at $x = 0, \pi, 2\pi$.

Interval	signs of f' and f''
$\left(0, \frac{2\pi}{3}\right)$	$+-$
$\left(\frac{2\pi}{3}, \pi\right)$	$--$
$\left(\pi, \frac{4\pi}{3}\right)$	$-+$
$\left(\frac{4\pi}{3}, 2\pi\right)$	$++$

The graph has a local maximum at $x = \frac{2\pi}{3}$, a local minimum at $x = \frac{4\pi}{3}$, and an inflection point at $x = \pi$. Here is a graph of f *without* transition points highlighted.

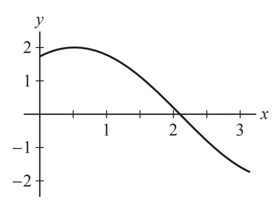


45. $y = \sin x + \sqrt{3}\cos x, \quad [0, \pi]$

SOLUTION Let $f(x) = \sin x + \sqrt{3}\cos x$. Setting $f'(x) = \cos x - \sqrt{3}\sin x = 0$ yields $\tan x = \frac{1}{\sqrt{3}}$. In the interval $[0, \pi]$, the solution is $x = \frac{\pi}{6}$. Setting $f''(x) = -\sin x - \sqrt{3}\cos x = 0$ yields $\tan x = -\sqrt{3}$. In the interval $[0, \pi]$, the lone solution is $x = \frac{2\pi}{3}$.

Interval	signs of f' and f''
$(0, \pi/6)$	$+-$
$(\pi/6, 2\pi/3)$	$--$
$(2\pi/3, \pi)$	$-+$

The graph has a local maximum at $x = \frac{\pi}{6}$ and a point of inflection at $x = \frac{2\pi}{3}$. A plot without the transition points highlighted is given below:



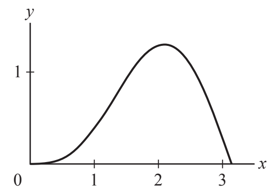
46. $y = \sin x - \frac{1}{2}\sin 2x, \quad [0, \pi]$

SOLUTION Let $f(x) = \sin x - \frac{1}{2}\sin 2x$. Setting $f'(x) = \cos x - \cos 2x = 0$ yields $\cos 2x = \cos x$. Using the double angle formula for cosine, this gives $2\cos^2 x - 1 = \cos x$ or $(2\cos x + 1)(\cos x - 1) = 0$. Solving for $x \in [0, \pi]$, we find $x = 0$ or $\frac{2\pi}{3}$.

Setting $f''(x) = -\sin x + 2\sin 2x = 0$ yields $4\sin x \cos x = \sin x$, so $\sin x = 0$ or $\cos x = \frac{1}{4}$. Hence, there are potential points of inflection at $x = 0$, $x = \pi$ and $x = \cos^{-1}\frac{1}{4} \approx 1.31812$.

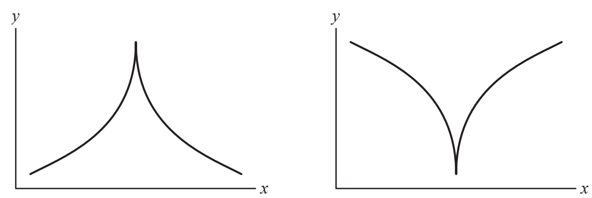
Interval	Sign of f' and f''
$\left(0, \cos^{-1}\frac{1}{4}\right)$	$++$
$\left(\cos^{-1}\frac{1}{4}, \frac{2\pi}{3}\right)$	$+-$
$\left(\frac{2\pi}{3}, \pi\right)$	$--$

The graph of $f(x)$ has a local maximum at $x = \frac{2\pi}{3}$ and a point of inflection at $x = \cos^{-1}\frac{1}{4}$.



47. Are all sign transitions possible? Explain with a sketch why the transitions $++ \to -+$ and $-- \to +-$ do not occur if the function is differentiable. (See Exercise 76 for a proof.)

SOLUTION In both cases, there is a point where f is not differentiable at the transition from increasing to decreasing or decreasing to increasing.



48. Suppose that f is twice differentiable satisfying (i) $f(0) = 1$, (ii) $f'(x) > 0$ for all $x \neq 0$, and (iii) $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$. Let $g(x) = f(x^2)$.

(a) Sketch a possible graph of $f(x)$.

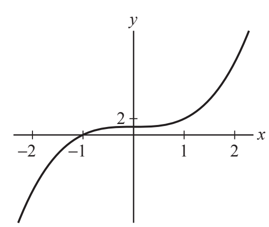
(b) Prove that $g(x)$ has no points of inflection and a unique local extreme value at $x = 0$. Sketch a possible graph of $g(x)$.

SOLUTION

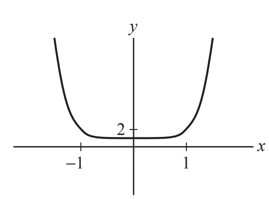
(a) To produce a possible sketch, we give the direction and concavity of the graph over every interval.

Interval	$(-\infty, 0)$	$(0, \infty)$
Direction	↗	↗
Concavity	⌢	⌣

A sketch of one possible such function appears here:



(b) Let $g(x) = f(x^2)$. Then $g'(x) = 2xf'(x^2)$. If $g'(x) = 0$, either $x = 0$ or $f'(x^2) = 0$, which implies that $x = 0$ as well. Since $f'(x^2) > 0$ for all $x \neq 0$, $g'(x) < 0$ for $x < 0$ and $g'(x) > 0$ for $x > 0$. This gives $g(x)$ a unique local extreme value at $x = 0$, a minimum. $g''(x) = 2f'(x^2) + 4x^2 f''(x^2)$. For all $x \neq 0$, $x^2 > 0$, and so $f''(x^2) > 0$ and $f'(x^2) > 0$. Thus $g''(x) > 0$, and so $g''(x)$ does not change sign, and can have no inflection points. A sketch of $g(x)$ based on the sketch we made for $f(x)$ follows: indeed, this sketch shows a unique local minimum at $x = 0$.



49. Which of the graphs in Figure 3 *cannot* be the graph of a polynomial? Explain.

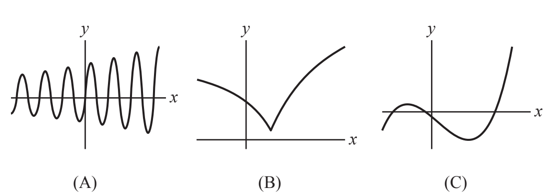
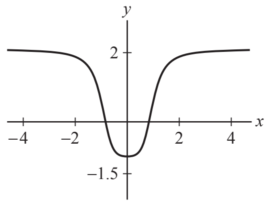


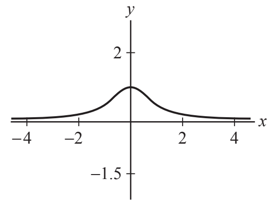
FIGURE 3

SOLUTION Polynomials are everywhere differentiable. Accordingly, graph (B) cannot be the graph of a polynomial, since the function in (B) has a cusp (sharp corner), signifying nondifferentiability at that point.

50. Which curve in Figure 4 is the graph of $f(x) = \dfrac{2x^4 - 1}{1 + x^4}$? Explain on the basis of horizontal asymptotes.



(A)



(B)

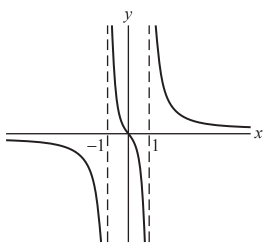
FIGURE 4

SOLUTION Since

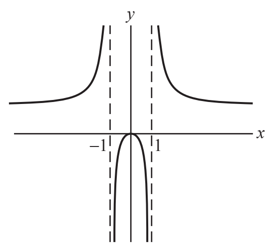
$$\lim_{x\to\pm\infty} \frac{2x^4 - 1}{1 + x^4} = \frac{2}{1} \cdot \lim_{x\to\pm\infty} 1 = 2$$

the graph has left and right horizontal asymptotes at $y = 2$, so the left curve is the graph of $f(x) = \dfrac{2x^4 - 1}{1 + x^4}$.

51. Match the graphs in Figure 5 with the two functions $y = \dfrac{3x}{x^2 - 1}$ and $y = \dfrac{3x^2}{x^2 - 1}$. Explain.



(A)



(B)

FIGURE 5

SOLUTION Since $\displaystyle\lim_{x\to\pm\infty} \frac{3x^2}{x^2 - 1} = \frac{3}{1} \cdot \lim_{x\to\pm\infty} 1 = 3$, the graph of $y = \dfrac{3x^2}{x^2 - 1}$ has a horizontal asymptote of $y = 3$; hence, the right curve is the graph of $f(x) = \dfrac{3x^2}{x^2 - 1}$. Since

$$\lim_{x\to\pm\infty} \frac{3x}{x^2 - 1} = \frac{3}{1} \cdot \lim_{x\to\pm\infty} x^{-1} = 0,$$

the graph of $y = \dfrac{3x}{x^2 - 1}$ has a horizontal asymptote of $y = 0$; hence, the left curve is the graph of $f(x) = \dfrac{3x}{x^2 - 1}$.

52. Match the functions with their graphs in Figure 6.

(a) $y = \dfrac{1}{x^2 - 1}$

(b) $y = \dfrac{x^2}{x^2 + 1}$

(c) $y = \dfrac{1}{x^2 + 1}$

(d) $y = \dfrac{x}{x^2 - 1}$

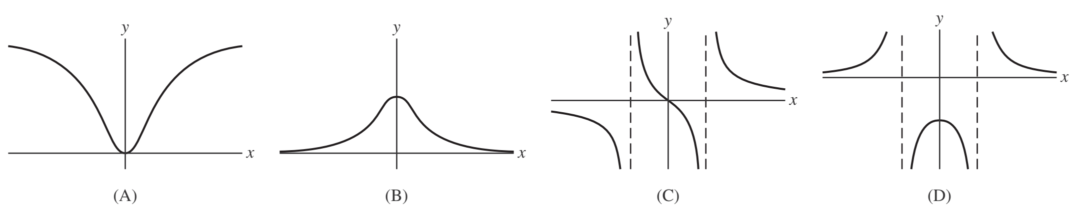


FIGURE 6

SOLUTION

(a) The graph of $\frac{1}{x^2-1}$ should have a horizontal asymptote at $y = 0$ and vertical asymptotes at $x = \pm 1$. Further, the graph should consist of positive values for $|x| > 1$ and negative values for $|x| < 1$. Hence, the graph of $\frac{1}{x^2-1}$ is (D).

(b) The graph of $\frac{x^2}{x^2+1}$ should have a horizontal asymptote at $y = 1$ and no vertical asymptotes. Hence, the graph of $\frac{x^2}{x^2+1}$ is (A).

(c) The graph of $\frac{1}{x^2+1}$ should have a horizontal asymptote at $y = 0$ and no vertical asymptotes. Hence, the graph of $\frac{1}{x^2+1}$ is (B).

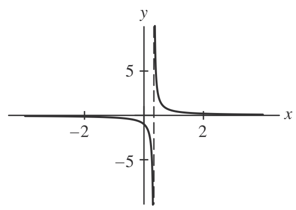
(d) The graph of $\frac{x}{x^2-1}$ should have a horizontal asymptote at $y = 0$ and vertical asymptotes at $x = \pm 1$. Further, the graph should consist of positive values for $-1 < x < 0$ and $x > 1$ and negative values for $x < 1$ and $0 < x < 1$. Hence, the graph of $\frac{x}{x^2-1}$ is (C).

In Exercises 53–70, sketch the graph of the function. Indicate the transition points and asymptotes.

53. $y = \dfrac{1}{3x-1}$

SOLUTION Let $f(x) = \dfrac{1}{3x-1}$. Then $f'(x) = \dfrac{-3}{(3x-1)^2}$, so that f is decreasing for all $x \neq \frac{1}{3}$. Moreover, $f''(x) = \dfrac{18}{(3x-1)^3}$, so that f is concave up for $x > \frac{1}{3}$ and concave down for $x < \frac{1}{3}$. Because $\lim_{x\to\pm\infty} \dfrac{1}{3x-1} = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has a vertical asymptote at $x = \frac{1}{3}$ with

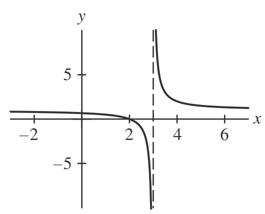
$$\lim_{x\to\frac{1}{3}-} \frac{1}{3x-1} = -\infty \quad \text{and} \quad \lim_{x\to\frac{1}{3}+} \frac{1}{3x-1} = \infty.$$



54. $y = \dfrac{x-2}{x-3}$

SOLUTION Let $f(x) = \dfrac{x-2}{x-3}$. Then $f'(x) = \dfrac{-1}{(x-3)^2}$, so that f is decreasing for all $x \neq 3$. Moreover, $f''(x) = \dfrac{2}{(x-3)^3}$, so that f is concave up for $x > 3$ and concave down for $x < 3$. Because $\lim_{x\to\pm\infty} \dfrac{x-2}{x-3} = 1$, f has a horizontal asymptote at $y = 1$. Finally, f has a vertical asymptote at $x = 3$ with

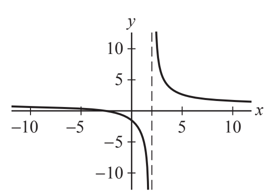
$$\lim_{x\to 3-} \frac{x-2}{x-3} = -\infty \quad \text{and} \quad \lim_{x\to 3+} \frac{x-2}{x-3} = \infty.$$



55. $y = \dfrac{x+3}{x-2}$

SOLUTION Let $f(x) = \dfrac{x+3}{x-2}$. Then $f'(x) = \dfrac{-5}{(x-2)^2}$, so that f is decreasing for all $x \neq 2$. Moreover, $f''(x) = \dfrac{10}{(x-2)^3}$, so that f is concave up for $x > 2$ and concave down for $x < 2$. Because $\lim_{x\to\pm\infty} \dfrac{x+3}{x-2} = 1$, f has a horizontal asymptote at $y = 1$. Finally, f has a vertical asymptote at $x = 2$ with

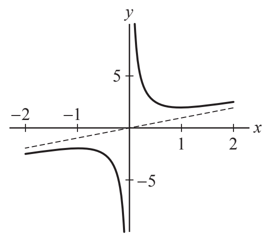
$$\lim_{x\to 2-} \frac{x+3}{x-2} = -\infty \quad \text{and} \quad \lim_{x\to 2+} \frac{x+3}{x-2} = \infty.$$



56. $y = x + \dfrac{1}{x}$

SOLUTION Let $f(x) = x + x^{-1}$. Then $f'(x) = 1 - x^{-2}$, so that f is increasing for $x < -1$ and $x > 1$ and decreasing for $-1 < x < 0$ and $0 < x < 1$. Moreover, $f''(x) = 2x^{-3}$, so that f is concave up for $x > 0$ and concave down for $x < 0$. f has no horizontal asymptote and has a vertical asymptote at $x = 0$ with

$$\lim_{x\to 0-} (x + x^{-1}) = -\infty \qquad \text{and} \qquad \lim_{x\to 0+} (x + x^{-1}) = \infty.$$



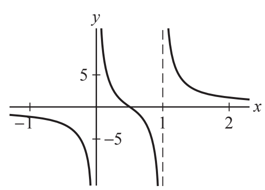
57. $y = \dfrac{1}{x} + \dfrac{1}{x-1}$

SOLUTION Let $f(x) = \dfrac{1}{x} + \dfrac{1}{x-1}$. Then $f'(x) = -\dfrac{2x^2 - 2x + 1}{x^2(x-1)^2}$, so that f is decreasing for all $x \neq 0, 1$. Moreover, $f''(x) = \dfrac{2\left(2x^3 - 3x^2 + 3x - 1\right)}{x^3(x-1)^3}$, so that f is concave up for $0 < x < \frac{1}{2}$ and $x > 1$ and concave down for $x < 0$ and $\frac{1}{2} < x < 1$. Because $\displaystyle\lim_{x\to\pm\infty}\left(\frac{1}{x} + \frac{1}{x-1}\right) = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has vertical asymptotes at $x = 0$ and $x = 1$ with

$$\lim_{x\to 0-}\left(\frac{1}{x} + \frac{1}{x-1}\right) = -\infty \qquad \text{and} \qquad \lim_{x\to 0+}\left(\frac{1}{x} + \frac{1}{x-1}\right) = \infty$$

and

$$\lim_{x\to 1-}\left(\frac{1}{x} + \frac{1}{x-1}\right) = -\infty \qquad \text{and} \qquad \lim_{x\to 1+}\left(\frac{1}{x} + \frac{1}{x-1}\right) = \infty.$$



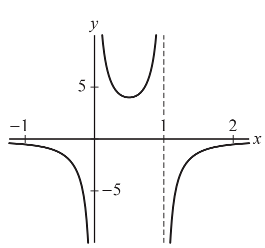
58. $y = \dfrac{1}{x} - \dfrac{1}{x-1}$

SOLUTION Let $f(x) = \dfrac{1}{x} - \dfrac{1}{x-1}$. Then $f'(x) = \dfrac{2x - 1}{x^2(x-1)^2}$, so that f is decreasing for $x < 0$ and $0 < x < \frac{1}{2}$ and increasing for $\frac{1}{2} < x < 1$ and $x > 1$. Moreover, $f''(x) = -\dfrac{2\left(3x^2 - 3x + 1\right)}{x^3(x-1)^3}$, so that f is concave up for $0 < x < 1$ and concave down for $x < 0$ and $x > 1$. Because $\displaystyle\lim_{x\to\pm\infty}\left(\frac{1}{x} - \frac{1}{x-1}\right) = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has vertical asymptotes at $x = 0$ and $x = 1$ with

$$\lim_{x\to 0-}\left(\frac{1}{x} - \frac{1}{x-1}\right) = -\infty \qquad \text{and} \qquad \lim_{x\to 0+}\left(\frac{1}{x} - \frac{1}{x-1}\right) = \infty$$

and

$$\lim_{x\to 1-}\left(\frac{1}{x} - \frac{1}{x-1}\right) = \infty \qquad \text{and} \qquad \lim_{x\to 1+}\left(\frac{1}{x} - \frac{1}{x-1}\right) = -\infty.$$



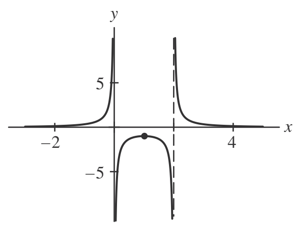
59. $y = \dfrac{1}{x(x-2)}$

SOLUTION Let $f(x) = \dfrac{1}{x(x-2)}$. Then $f'(x) = \dfrac{2(1-x)}{x^2(x-2)^2}$, so that f is increasing for $x < 0$ and $0 < x < 1$ and decreasing for $1 < x < 2$ and $x > 2$. Moreover, $f''(x) = \dfrac{2(3x^2 - 6x + 4)}{x^3(x-2)^3}$, so that f is concave up for $x < 0$ and $x > 2$ and concave down for $0 < x < 2$. Because $\lim_{x\to\pm\infty} \left(\dfrac{1}{x(x-2)}\right) = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has vertical asymptotes at $x = 0$ and $x = 2$ with

$$\lim_{x\to 0-} \left(\frac{1}{x(x-2)}\right) = +\infty \quad \text{and} \quad \lim_{x\to 0+} \left(\frac{1}{x(x-2)}\right) = -\infty$$

and

$$\lim_{x\to 2-} \left(\frac{1}{x(x-2)}\right) = -\infty \quad \text{and} \quad \lim_{x\to 2+} \left(\frac{1}{x(x-2)}\right) = \infty.$$



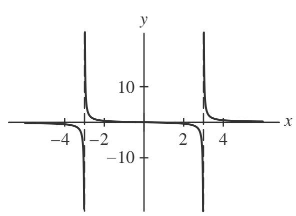
60. $y = \dfrac{x}{x^2 - 9}$

SOLUTION Let $f(x) = \dfrac{x}{x^2-9}$. Then $f'(x) = -\dfrac{x^2+9}{(x^2-9)^2}$, so that f is decreasing for all $x \neq \pm 3$. Moreover, $f''(x) = \dfrac{6x(x^2+6)}{(x^2-9)^3}$, so that f is concave down for $x < -3$ and for $0 < x < 3$ and is concave up for $-3 < x < 0$ and for $x > 3$. Because $\lim_{x\to\pm\infty} \dfrac{x}{x^2-9} = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has vertical asymptotes at $x = \pm 3$, with

$$\lim_{x\to -3-} \left(\frac{x}{x^2-9}\right) = -\infty \quad \text{and} \quad \lim_{x\to -3+} \left(\frac{x}{x^2-9}\right) = \infty$$

and

$$\lim_{x\to 3-} \left(\frac{x}{x^2-9}\right) = -\infty \quad \text{and} \quad \lim_{x\to 3+} \left(\frac{x}{x^2-9}\right) = \infty.$$



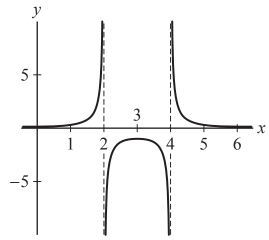
61. $y = \dfrac{1}{x^2 - 6x + 8}$

SOLUTION Let $f(x) = \dfrac{1}{x^2-6x+8} = \dfrac{1}{(x-2)(x-4)}$. Then $f'(x) = \dfrac{6-2x}{\left(x^2-6x+8\right)^2}$, so that f is increasing for $x < 2$ and for $2 < x < 3$, is decreasing for $3 < x < 4$ and for $x > 4$, and has a local maximum at $x = 3$. Moreover, $f''(x) = \dfrac{2\left(3x^2-18x+28\right)}{\left(x^2-6x+8\right)^3}$, so that f is concave up for $x < 2$ and for $x > 4$ and is concave down for $2 < x < 4$. Because $\lim_{x\to\pm\infty} \dfrac{1}{x^2-6x+8} = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has vertical asymptotes at $x = 2$ and $x = 4$, with

$$\lim_{x\to 2-} \left(\frac{1}{x^2-6x+8}\right) = \infty \quad \text{and} \quad \lim_{x\to 2+} \left(\frac{1}{x^2-6x+8}\right) = -\infty$$

and

$$\lim_{x\to 4-} \left(\frac{1}{x^2-6x+8}\right) = -\infty \quad \text{and} \quad \lim_{x\to 4+} \left(\frac{1}{x^2-6x+8}\right) = \infty.$$



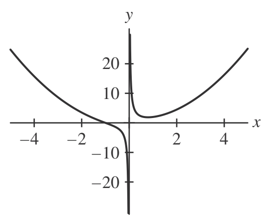
62. $y = \dfrac{x^3+1}{x}$

SOLUTION Let $f(x) = \dfrac{x^3+1}{x} = x^2 + x^{-1}$. Then $f'(x) = 2x - x^{-2}$, so that f is decreasing for $x < 0$ and for $0 < x < \sqrt[3]{1/2}$ and increasing for $x > \sqrt[3]{1/2}$. Moreover, $f''(x) = 2 + 2x^{-3}$, so f is concave up for $x < -1$ and for $x > 0$ and concave down for $-1 < x < 0$. Because

$$\lim_{x\to\pm\infty} \frac{x^3+1}{x} = \infty,$$

f has no horizontal asymptotes. Finally, f has a vertical asymptote at $x = 0$ with

$$\lim_{x\to 0-} \frac{x^3+1}{x} = -\infty \qquad \text{and} \qquad \lim_{x\to 0+} \frac{x^3+1}{x} = \infty.$$



63. $y = 1 - \dfrac{3}{x} + \dfrac{4}{x^3}$

SOLUTION Let $f(x) = 1 - \dfrac{3}{x} + \dfrac{4}{x^3}$. Then

$$f'(x) = \frac{3}{x^2} - \frac{12}{x^4} = \frac{3(x-2)(x+2)}{x^4},$$

so that f is increasing for $|x| > 2$ and decreasing for $-2 < x < 0$ and for $0 < x < 2$. Moreover,

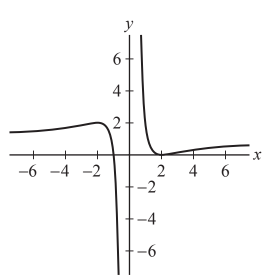
$$f''(x) = -\frac{6}{x^3} + \frac{48}{x^5} = \frac{6(8-x^2)}{x^5},$$

so that f is concave down for $-2\sqrt{2} < x < 0$ and for $x > 2\sqrt{2}$, while f is concave up for $x < -2\sqrt{2}$ and for $0 < x < 2\sqrt{2}$. Because

$$\lim_{x\to\pm\infty} \left(1 - \frac{3}{x} + \frac{4}{x^3}\right) = 1,$$

f has a horizontal asymptote at $y = 1$. Finally, f has a vertical asymptote at $x = 0$ with

$$\lim_{x\to 0-} \left(1 - \frac{3}{x} + \frac{4}{x^3}\right) = -\infty \qquad \text{and} \qquad \lim_{x\to 0+} \left(1 - \frac{3}{x} + \frac{4}{x^3}\right) = \infty.$$



64. $y = \dfrac{1}{x^2} + \dfrac{1}{(x-2)^2}$

SOLUTION Let $f(x) = \dfrac{1}{x^2} + \dfrac{1}{(x-2)^2}$. Then

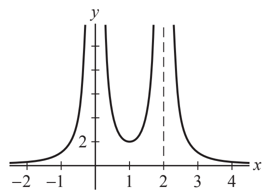
$$f'(x) = -2x^{-3} - 2\,(x-2)^{-3} = -\frac{4(x-1)(x^2-2x+4)}{x^3(x-2)^3},$$

so that f is increasing for $x < 0$ and for $1 < x < 2$, is decreasing for $0 < x < 1$ and for $x > 2$, and has a local minimum at $x = 1$. Moreover, $f''(x) = 6x^{-4} + 6\,(x-2)^{-4}$, so that f is concave up for all $x \neq 0, 2$. Because $\lim_{x\to\pm\infty}\left(\dfrac{1}{x^2} + \dfrac{1}{(x-2)^2}\right) = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has vertical asymptotes at $x = 0$ and $x = 2$ with

$$\lim_{x\to 0-}\left(\frac{1}{x^2} + \frac{1}{(x-2)^2}\right) = \infty \qquad \text{and} \qquad \lim_{x\to 0+}\left(\frac{1}{x^2} + \frac{1}{(x-2)^2}\right) = \infty$$

and

$$\lim_{x\to 2-}\left(\frac{1}{x^2} + \frac{1}{(x-2)^2}\right) = \infty \qquad \text{and} \qquad \lim_{x\to 2+}\left(\frac{1}{x^2} + \frac{1}{(x-2)^2}\right) = \infty.$$



65. $y = \dfrac{1}{x^2} - \dfrac{1}{(x-2)^2}$

SOLUTION Let $f(x) = \dfrac{1}{x^2} - \dfrac{1}{(x-2)^2}$. Then $f'(x) = -2x^{-3} + 2\,(x-2)^{-3}$, so that f is increasing for $x < 0$ and for $x > 2$ and is decreasing for $0 < x < 2$. Moreover,

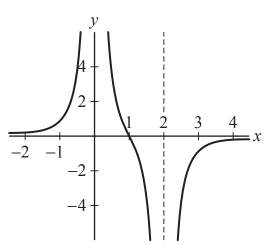
$$f''(x) = 6x^{-4} - 6\,(x-2)^{-4} = -\frac{48(x-1)(x^2-2x+2)}{x^4(x-2)^4},$$

so that f is concave up for $x < 0$ and for $0 < x < 1$, is concave down for $1 < x < 2$ and for $x > 2$, and has a point of inflection at $x = 1$. Because $\lim_{x\to\pm\infty}\left(\dfrac{1}{x^2} - \dfrac{1}{(x-2)^2}\right) = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has vertical asymptotes at $x = 0$ and $x = 2$ with

$$\lim_{x\to 0-}\left(\frac{1}{x^2} - \frac{1}{(x-2)^2}\right) = \infty \qquad \text{and} \qquad \lim_{x\to 0+}\left(\frac{1}{x^2} - \frac{1}{(x-2)^2}\right) = \infty$$

and

$$\lim_{x\to 2-}\left(\frac{1}{x^2} - \frac{1}{(x-2)^2}\right) = -\infty \qquad \text{and} \qquad \lim_{x\to 2+}\left(\frac{1}{x^2} - \frac{1}{(x-2)^2}\right) = -\infty.$$



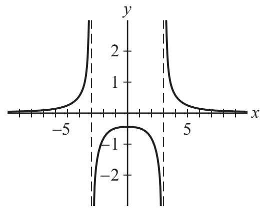
66. $y = \dfrac{4}{x^2 - 9}$

SOLUTION Let $f(x) = \dfrac{4}{x^2-9}$. Then $f'(x) = -\dfrac{8x}{\left(x^2-9\right)^2}$, so that f is increasing for $x < -3$ and for $-3 < x < 0$, is decreasing for $0 < x < 3$ and for $x > 3$, and has a local maximum at $x = 0$. Moreover, $f''(x) = \dfrac{24\left(x^2+3\right)}{\left(x^2-9\right)^3}$, so that f is concave up for $x < -3$ and for $x > 3$ and is concave down for $-3 < x < 3$. Because $\lim_{x\to\pm\infty} \frac{4}{x^2-9} = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has vertical asymptotes at $x = -3$ and $x = 3$, with

$$\lim_{x\to-3-} \left(\frac{4}{x^2-9}\right) = \infty \quad \text{and} \quad \lim_{x\to-3+} \left(\frac{4}{x^2-9}\right) = -\infty$$

and

$$\lim_{x\to3-} \left(\frac{4}{x^2-9}\right) = -\infty \quad \text{and} \quad \lim_{x\to3+} \left(\frac{4}{x^2-9}\right) = \infty.$$

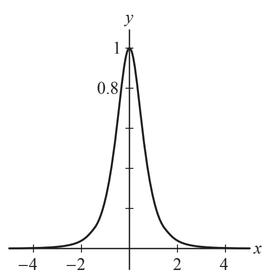


67. $y = \frac{1}{(x^2+1)^2}$

SOLUTION Let $f(x) = \frac{1}{(x^2+1)^2}$. Then $f'(x) = \frac{-4x}{(x^2+1)^3}$, so that f is increasing for $x < 0$, is decreasing for $x > 0$ and has a local maximum at $x = 0$. Moreover,

$$f''(x) = \frac{-4(x^2+1)^3 + 4x \cdot 3(x^2+1)^2 \cdot 2x}{(x^2+1)^6} = \frac{20x^2-4}{(x^2+1)^4},$$

so that f is concave up for $|x| > 1/\sqrt{5}$, is concave down for $|x| < 1/\sqrt{5}$, and has points of inflection at $x = \pm 1/\sqrt{5}$. Because $\lim_{x\to\pm\infty} \frac{1}{(x^2+1)^2} = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has no vertical asymptotes.



68. $y = \frac{x^2}{(x^2-1)(x^2+1)}$

SOLUTION Let

$$f(x) = \frac{x^2}{(x^2-1)(x^2+1)}.$$

Then

$$f'(x) = -\frac{2x(1+x^4)}{(x-1)^2(x+1)^2(x^2+1)^2},$$

so that f is increasing for $x < -1$ and for $-1 < x < 0$, is decreasing for $0 < x < 1$ and for $x > 1$, and has a local maximum at $x = 0$. Moreover,

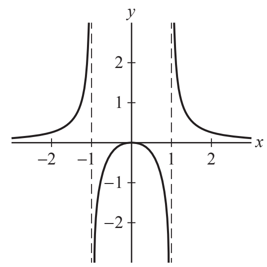
$$f''(x) = \frac{2 + 24x^4 + 6x^8}{(x-1)^3(x+1)^3(x^2+1)^3},$$

so that f is concave up for $|x| > 1$ and concave down for $|x| < 1$. Because $\lim_{x\to\pm\infty} \frac{x^2}{(x^2-1)(x^2+1)} = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has vertical asymptotes at $x = -1$ and $x = 1$, with

$$\lim_{x\to-1-} \frac{x^2}{(x^2-1)(x^2+1)} = \infty \quad \text{and} \quad \lim_{x\to-1+} \frac{x^2}{(x^2-1)(x^2+1)} = -\infty$$

and

$$\lim_{x\to 1-} \frac{x^2}{(x^2-1)(x^2+1)} = -\infty \qquad \text{and} \qquad \lim_{x\to 1+} \frac{x^2}{(x^2-1)(x^2+1)} = \infty.$$



69. $y = \dfrac{1}{\sqrt{x^2+1}}$

SOLUTION Let $f(x) = \frac{1}{\sqrt{x^2+1}}$. Then

$$f'(x) = -\frac{x}{\sqrt{(x^2+1)^3}} = -x(x^2+1)^{-3/2},$$

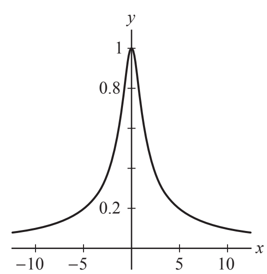
so that f is increasing for $x < 0$ and decreasing for $x > 0$. Moreover,

$$f''(x) = -\frac{3}{2}x(x^2+1)^{-5/2}(-2x) - (x^2+1)^{-3/2} = (2x^2-1)(x^2+1)^{-5/2},$$

so that f is concave down for $|x| < \frac{\sqrt{2}}{2}$ and concave up for $|x| > \frac{\sqrt{2}}{2}$. Because

$$\lim_{x\to\pm\infty} \frac{1}{\sqrt{x^2+1}} = 0,$$

f has a horizontal asymptote at $y = 0$. Finally, f has no vertical asymptotes.



70. $y = \dfrac{x}{\sqrt{x^2+1}}$

SOLUTION Let

$$f(x) = \frac{x}{\sqrt{x^2+1}}.$$

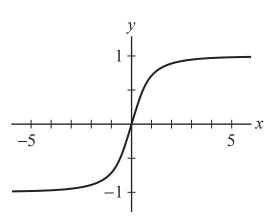
Then

$$f'(x) = (x^2+1)^{-3/2} \quad \text{and} \quad f''(x) = \frac{-3x}{(x^2+1)^{5/2}}.$$

Thus, f is increasing for all x, is concave up for $x < 0$, is concave down for $x > 0$, and has a point of inflection at $x = 0$. Because

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = 1 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = -1,$$

f has horizontal asymptotes of $y = -1$ and $y = 1$. There are no vertical asymptotes.



Further Insights and Challenges

In Exercises 71–75, we explore functions whose graphs approach a nonhorizontal line as $x \to \infty$. A line $y = ax + b$ is called a ***slant asymptote*** *if*

$$\lim_{x\to\infty}(f(x) - (ax + b)) = 0$$

or

$$\lim_{x\to-\infty}(f(x) - (ax + b)) = 0$$

71. Let $f(x) = \dfrac{x^2}{x-1}$ (Figure 7). Verify the following:

(a) $f(0)$ is a local max and $f(2)$ a local min.

(b) f is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.

(c) $\lim_{x\to 1-} f(x) = -\infty$ and $\lim_{x\to 1+} f(x) = \infty$.

(d) $y = x + 1$ is a slant asymptote of $f(x)$ as $x \to \pm\infty$.

(e) The slant asymptote lies above the graph of $f(x)$ for $x < 1$ and below the graph for $x > 1$.

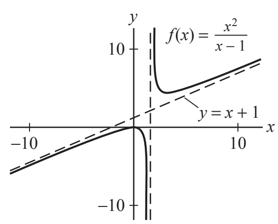


FIGURE 7

SOLUTION Let $f(x) = \dfrac{x^2}{x-1}$. Then $f'(x) = \dfrac{x(x-2)}{(x-1)^2}$ and $f''(x) = \dfrac{2}{(x-1)^3}$.

(a) Critical points of $f'(x)$ occur at $x = 0$ and $x = 2$. $x = 1$ is not a critical point because it is not in the domain of f. Sign analyses reveal that $x = 2$ is a local minimum of f and $x = 0$ is a local maximum.

(b) Sign analysis of $f''(x)$ reveals that $f''(x) < 0$ on $(-\infty, 1)$ and $f''(x) > 0$ on $(1, \infty)$.

(c)

$$\lim_{x\to 1-} f(x) = -1 \lim_{x\to 1-} \frac{1}{1-x} = -\infty \quad \text{and} \quad \lim_{x\to 1+} f(x) = 1 \lim_{x\to 1+} \frac{1}{x-1} = \infty.$$

(d) Note that using polynomial division, $f(x) = \dfrac{x^2}{x-1} = x + 1 + \dfrac{1}{x-1}$. Then $\lim_{x\to\pm\infty}(f(x) - (x+1)) = \lim_{x\to\pm\infty} x + 1 + \dfrac{1}{x-1} - (x+1) = \lim_{x\to\pm\infty} \dfrac{1}{x-1} = 0.$

(e) For $x > 1$, $f(x) - (x+1) = \dfrac{1}{x-1} > 0$, so $f(x)$ approaches $x + 1$ from above. Similarly, for $x < 1$, $f(x) - (x+1) = \dfrac{1}{x-1} < 0$, so $f(x)$ approaches $x + 1$ from below.

72. If $f(x) = P(x)/Q(x)$, where P and Q are polynomials of degrees $m + 1$ and m, then by long division, we can write

$$f(x) = (ax + b) + P_1(x)/Q(x)$$

where P_1 is a polynomial of degree $< m$. Show that $y = ax + b$ is the slant asymptote of $f(x)$. Use this procedure to find the slant asymptotes of the following functions:

(a) $y = \dfrac{x^2}{x+2}$

(b) $y = \dfrac{x^3 + x}{x^2 + x + 1}$

SOLUTION Since $\deg(P_1) < \deg(Q)$,

$$\lim_{x\to\pm\infty} \frac{P_1(x)}{Q(x)} = 0.$$

Thus

$$\lim_{x\to\pm\infty}(f(x) - (ax + b)) = 0$$

and $y = ax + b$ is a slant asymptote of f.

(a) $\dfrac{x^2}{x+2} = x - 2 + \dfrac{4}{x+2}$; hence $y = x - 2$ is a slant asymptote of $\dfrac{x^2}{x+2}$.

(b) $\dfrac{x^3+x}{x^2+x+1} = (x-1) + \dfrac{x+1}{x^2-1}$; hence, $y = x - 1$ is a slant asymptote of $\dfrac{x^3+x}{x^2+x+1}$.

73. Sketch the graph of

$$f(x) = \frac{x^2}{x+1}.$$

Proceed as in the previous exercise to find the slant asymptote.

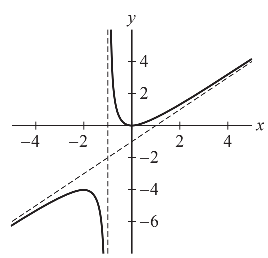
SOLUTION Let $f(x) = \dfrac{x^2}{x+1}$. Then $f'(x) = \dfrac{x(x+2)}{(x+1)^2}$ and $f''(x) = \dfrac{2}{(x+1)^3}$. Thus, f is increasing for $x < -2$ and for $x > 0$, is decreasing for $-2 < x < -1$ and for $-1 < x < 0$, has a local minimum at $x = 0$, has a local maximum at $x = -2$, is concave down on $(-\infty, -1)$ and concave up on $(-1, \infty)$. Limit analyses give a vertical asymptote at $x = -1$, with

$$\lim_{x\to -1-} \frac{x^2}{x+1} = -\infty \qquad \text{and} \qquad \lim_{x\to -1+} \frac{x^2}{x+1} = \infty.$$

By polynomial division, $f(x) = x - 1 + \dfrac{1}{x+1}$ and

$$\lim_{x\to\pm\infty} \left(x - 1 + \frac{1}{x+1} - (x-1) \right) = 0,$$

which implies that the slant asymptote is $y = x - 1$. Notice that f approaches the slant asymptote as in exercise 71.

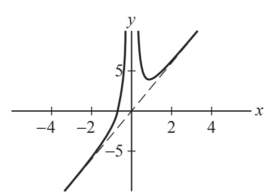


74. Show that $y = 3x$ is a slant asymptote for $f(x) = 3x + x^{-2}$. Determine whether $f(x)$ approaches the slant asymptote from above or below and make a sketch of the graph.

SOLUTION Let $f(x) = 3x + x^{-2}$. Then

$$\lim_{x\to\pm\infty} (f(x) - 3x) = \lim_{x\to\pm\infty} (3x + x^{-2} - 3x) = \lim_{x\to\pm\infty} x^{-2} = 0$$

which implies that $3x$ is the slant asymptote of $f(x)$. Since $f(x) - 3x = x^{-2} > 0$ as $x \to \pm\infty$, $f(x)$ approaches the slant asymptote from above in both directions. Moreover, $f'(x) = 3 - 2x^{-3}$ and $f''(x) = 6x^{-4}$. Sign analyses reveal a local minimum at $x = \left(\frac{3}{2}\right)^{-1/3} \approx 0.87358$ and that f is concave up for all $x \neq 0$. Limit analyses give a vertical asymptote at $x = 0$.

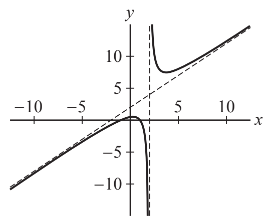


75. Sketch the graph of $f(x) = \dfrac{1-x^2}{2-x}$.

SOLUTION Let $f(x) = \dfrac{1-x^2}{2-x}$. Using polynomial division, $f(x) = x + 2 + \dfrac{3}{x-2}$. Then

$$\lim_{x\to\pm\infty} (f(x) - (x+2)) = \lim_{x\to\pm\infty} \left((x+2) + \frac{3}{x-2} - (x+2) \right) = \lim_{x\to\pm\infty} \frac{3}{x-2} = \frac{3}{1} \cdot \lim_{x\to\pm\infty} x^{-1} = 0$$

which implies that $y = x + 2$ is the slant asymptote of $f(x)$. Since $f(x) - (x+2) = \dfrac{3}{x-2} > 0$ for $x > 2$, $f(x)$ approaches the slant asymptote from above for $x > 2$; similarly, $\dfrac{3}{x-2} < 0$ for $x < 2$ so $f(x)$ approaches the slant asymptote from below for $x < 2$. Moreover, $f'(x) = \dfrac{x^2 - 4x + 1}{(2-x)^2}$ and $f''(x) = \dfrac{-6}{(2-x)^3}$. Sign analyses reveal a local minimum at $x = 2 + \sqrt{3}$, a local maximum at $x = 2 - \sqrt{3}$ and that f is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$. Limit analyses give a vertical asymptote at $x = 2$.



76. Assume that $f'(x)$ and $f''(x)$ exist for all x and let c be a critical point of $f(x)$. Show that $f(x)$ cannot make a transition from ++ to −+ at $x = c$. *Hint:* Apply the MVT to $f'(x)$.

SOLUTION Let $f(x)$ be a function such that $f''(x) > 0$ for all x and such that it transitions from ++ to −+ at a critical point c where $f'(c)$ is defined. That is, $f'(c) = 0$, $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$. Let $g(x) = f'(x)$. The previous statements indicate that $g(c) = 0$, $g(x_0) > 0$ for some $x_0 < c$, and $g(x_1) < 0$ for some $x_1 > c$. By the Mean Value Theorem,

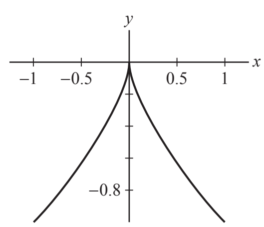
$$\frac{g(x_1) - g(x_0)}{x_1 - x_0} = g'(c_0),$$

for some c_0 between x_0 and x_1. Because $x_1 > c > x_0$ and $g(x_1) < 0 < g(x_0)$,

$$\frac{g(x_1) - g(x_0)}{x_1 - x_0} < 0.$$

But, on the other hand $g'(c_0) = f''(c_0) > 0$, so there is a contradiction. This means that our assumption of the existence of such a function $f(x)$ must be in error, so no function can transition from ++ to −+.

If we drop the requirement that $f'(c)$ exist, such a function can be found. The following is a graph of $f(x) = -x^{2/3}$. $f''(x) > 0$ wherever $f''(x)$ is defined, and $f'(x)$ transitions from positive to negative at $x = 0$.



77. Assume that $f''(x)$ exists and $f''(x) > 0$ for all x. Show that $f(x)$ cannot be negative for all x. *Hint:* Show that $f'(b) \neq 0$ for some b and use the result of Exercise 64 in Section 4.4.

SOLUTION Let $f(x)$ be a function such that $f''(x)$ exists and $f''(x) > 0$ for all x. Since $f''(x) > 0$, there is at least one point $x = b$ such that $f'(b) \neq 0$. If not, $f'(x) = 0$ for all x, so $f''(x) = 0$. By the result of Exercise 64 in Section 4.4, $f(x) \geq f(b) + f'(b)(x - b)$. Now, if $f'(b) > 0$, we find that $f(b) + f'(b)(x - b) > 0$ whenever

$$x > \frac{bf'(b) - f(b)}{f'(b)},$$

a condition that must be met for some x sufficiently large. For such x, $f(x) > f(b) + f'(b)(x - b) > 0$. On the other hand, if $f'(b) < 0$, we find that $f(b) + f'(b)(x - b) > 0$ whenever

$$x < \frac{bf'(b) - f(b)}{f'(b)}.$$

For such an x, $f(x) > f(b) + f'(b)(x - b) > 0$.