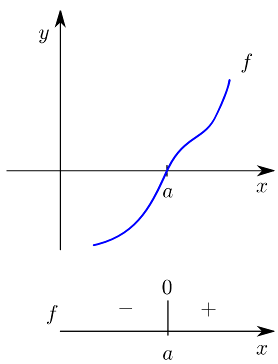
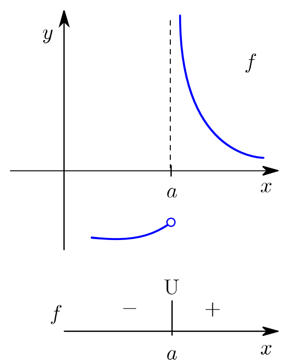
33 Graph sketching

33.1 Constructing number lines

A continuous function f can change signs only at a number a for which $f(a)$ is zero (left diagram) or undefined (right diagram):





(The function on the right is indeed continuous. See 11, in particular Example 11.2.1.) Therefore, the numbers that make f zero or undefined break the number line for f into intervals and on each interval the sign of f is consistent, that is, either always $+$ or always $-$. In order to decide which of these signs occurs for a particular interval, we need only evaluate f at *any* x in the interval and look at the sign of the result.

33.1.1 Example Complete the number line for the function

$$f(x) = \frac{x^3 - 6x^2 + 9x}{x^2 + 2x - 3}$$

showing where it is zero or undefined as well as the signs on the resulting intervals.

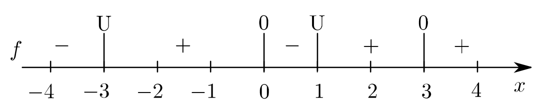
Solution We factor both numerator and denominator in order to find where the function is zero and where it is undefined:

$$f(x) = \frac{x(x-3)^2}{(x+3)(x-1)}.$$

The function is zero when $x = 0, 3$ and it is undefined when $x = -3, 1$. These x values divide the number line into intervals. According to the discussion above, in order to find the sign of the function on an interval we can just evaluate f at any convenient number in the interval and look at the sign. The original expression for the function could be used for this, but it is much easier to use the factored form. In fact, since it is just the sign we are after, we need only record the sign of each factor:

$$\begin{aligned} f(-4) &= \frac{(-)(+)}{(-)(-)} = - \\ f(-1) &= \frac{(-)(+)}{(+)(-)} = + \\ f(\tfrac{1}{2}) &= \frac{(+)(+)}{(+)(-)} = - \\ f(2) &= \frac{(+)(+)}{(+)(+)} = + \\ f(4) &= \frac{(+)(+)}{(+)(+)} = + \end{aligned}$$

The number line is



□

33.1.2 Example Complete the three number lines for the function $f(x) = 2 + 3x - x^3 = (x+1)^2(2-x)$ and use them to sketch the graph of f.

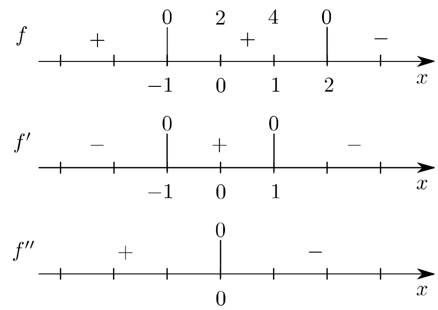
Solution The function f is zero when $x = -1, 2$ and it is never undefined. The derivative is

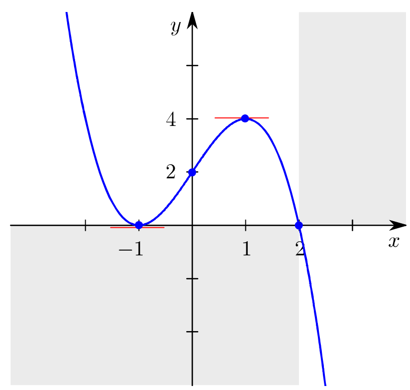
$$f'(x) = 3 - 3x^2 = 3(1+x)(1-x),$$

which is zero when $x = -1, 1$ and is never undefined. The second derivative is

$$f''(x) = -6x,$$

which is zero when $x = 0$ and is never undefined. The signs on the number lines are now completed as in the preceding example. On the number line for f, we record the heights of the graph corresponding to the x values that make the derivative and the second derivative zero since these are places where the graph has significant features.





□

33.2 Graph features: inflection point, local extreme

Looking at the graph in the preceding example (33.1.2), we see that the point $(0, 2)$ is a place where the concavity changes (from up to down). It is an example of an inflection point.

> INFLECTION POINT. Let f be a function that is continuous at the number a. The point $(a, f(a))$ is an **inflection point** if the concavity of the graph changes at a (either from up to down, or from down to up).

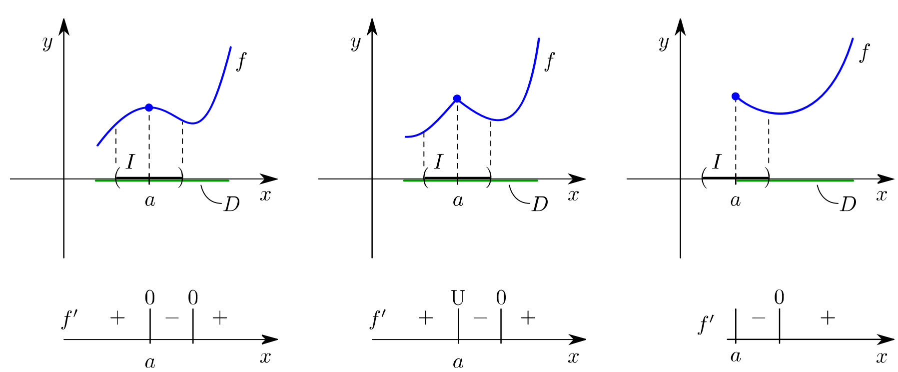
The function f in the preceding example (33.1.2) is said to have a local maximum of 4 at $x = 1$, meaning roughly that, in the vicinity of $x = 1$, the graph of f reaches a high point right at $x = 1$ and this height is 4. Since the height of the graph eventually exceeds the height at 1 (as one moves to the left), the function does not have a maximum at 1, only a *local* maximum. Similarly, f has a local minimum of 0 at $x = -1$.

The precise definitions are as follows:

Local extremes. Let f be a function with domain D and let a be an element of D.

- f has a **local maximum** at a if there exists an open interval I containing a such that $f(a)$ is a maximum of f on the set $I \cap D$.
- f has a **local minimum** at a if there exists an open interval I containing a such that $f(a)$ is a minimum of f on the set $I \cap D$.

In each of the following cases, f has a local maximum at a since there is an open interval I satisfying the requirements of the definition. The diagrams also illustrate the three ways a local maximum can occur: where the derivative is zero, where the derivative is undefined, at an endpoint.



The number line for f' is shown below each diagram. It illustrates the following way of detecting a local extreme without actually looking at the graph.

First derivative test. Let f be a function that is continuous at the number a.

- If the sign of f' changes from $+$ to $-$ at a, then f has a local maximum at a.
- If the sign of f' changes from $-$ to $+$ at a, then f has a local minimum at a.
- If the sign of f' does not change at a, then f has neither a local maximum nor a local minimum at a.

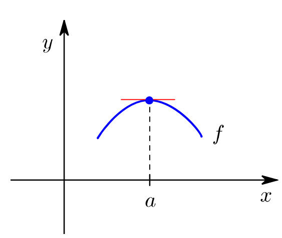
This test can be modified in a natural way to detect a local extreme occurring at an endpoint of the domain of f. For instance, if a is a left endpoint, and the sign of f' to the right of a is $-$, then f has a local maximum at a (see third diagram above).

Here is another way to detect a local extreme without looking at the graph:

Second derivative test. Let f be a differentiable function.

- If $f'(a) = 0$ and $f''(a) < 0$, then f has a local maximum at a.
- If $f'(a) = 0$ and $f''(a) > 0$, then f has a local minimum at a.

Although one does not require the graph of f in order to use this test, a good way to remember what the test says is to visualize what the conditions say about the shape of the graph. Taking the first statement, the condition $f'(a) = 0$ says that f has a horizontal tangent at a; the condition $f''(a) < 0$ says that the graph of f is concave down at a. In such a situation, the graph of f in the vicinity of a looks like this



so f has a local maximum at a as claimed. A similar analysis can be made for the second statement.

33.2.1 Example Let $f(x) = x^5 - 5x + 7$.

(a) Find all values of x at which $f'(x) = 0$.

(b) Use the first derivative test to decide whether a local minimum or a local maximum occurs at each of the numbers found in part (a).

(c) Do the same using the second derivative test.

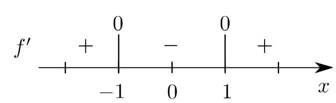
Solution

(a) The derivative is

$$f'(x) = 5x^4 - 5 = 5(x^4 - 1) = 5(x^2 - 1)(x^2 + 1) = 5(x + 1)(x - 1)(x^2 + 1),$$

which is zero when $x = -1, 1$.

(b) We use the f' number line to organize the information:



The first derivative test says that f has a local maximum at -1 and a local minimum at 1.

(c) The second derivative is

$$f''(x) = 20x^3.$$

Since $f'(-1) = 0$ and $f''(-1) = -20 < 0$, the second derivative test says that f has a local maximum at -1. Similarly, since $f'(1) = 0$ and $f''(1) = 20 > 0$, the second derivative test says that f has a local minimum at 1.

□

33.3 Asymptote

33.3.1 Example Complete the three number lines for the function

$$f(x) = \frac{3x + 6}{2x - 2}$$

and use them to sketch the graph of f.

Solution The function f is zero when $x = -2$ and it is undefined when $x = 1$. The derivative is

$$f'(x) = \frac{(2x-2)(3) - (3x+6)(2)}{(2x-2)^2} = \frac{-18}{(2x-2)^2} = -18(2x-2)^{-2},$$

which is never zero and is undefined when $x = 1$. The second derivative is

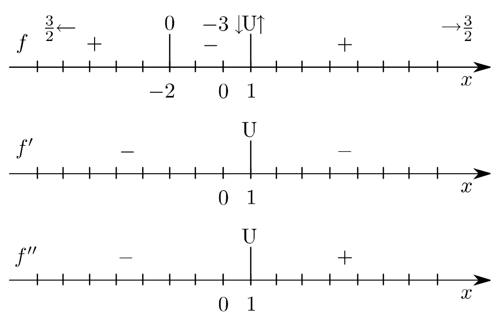
$$f''(x) = 36(2x-2)^{-3}(2) = \frac{72}{(2x-2)^3},$$

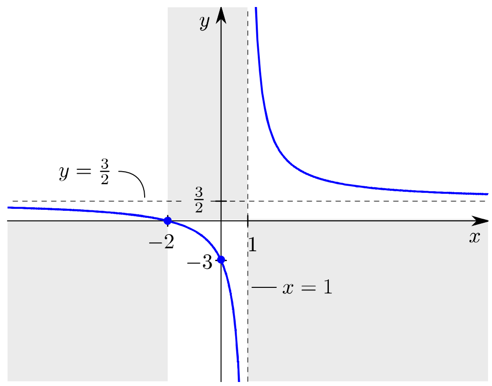
which is never zero and is undefined when $x = 1$.

A reasonable guess as to the shape of the graph can now be made by just using the signs on the three number lines. However, the height of the graph as x moves off to the right and also as x moves off to the left is in question, as is the height of the graph as x nears 1 from either side. We compute limits in order to settle these questions:

$$\begin{aligned} &\lim_{x\to\pm\infty} \frac{3x+6}{2x-2} \left(\frac{\pm\infty}{\pm\infty}\right) \overset{\text{l'H}}{=} \lim_{x\to\pm\infty} \frac{3}{2} = \frac{3}{2}, \\ &\lim_{x\to 1^+} \frac{3x+6}{2x-2} = \infty \quad \left(\frac{\text{about 9}}{\text{small pos.}}\right), \\ &\lim_{x\to 1^-} \frac{3x+6}{2x-2} = -\infty \quad \left(\frac{\text{about 9}}{\text{small neg.}}\right). \end{aligned}$$

These findings are recorded on the number line for f.





□

The dotted lines in the preceding example are called asymptotes. Roughly speaking, an asymptote is a line that the graph of a function gets ever closer to as the graph moves away from the origin. Here are the precise definitions for two special cases:

Horizontal/Vertical asymptote. Let f be a function.

- The horizontal line $y = b$ is a **horizontal asymptote** for the graph of f if

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

- The vertical line $x = a$ is a **vertical asymptote** for the graph of f if

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty.$$

In words,

- $y = b$ is a horizontal asymptote for the graph of f if the height of the graph approaches b as x moves off to the right or off to the left.
- $x = a$ is a vertical asymptote for the graph of f if the height of the graph goes to either ∞ or $-\infty$ as x gets close to a from either the right or the left.

33.4 Method for graph sketching

Now that the main topics have been introduced we can formulate a method for producing an accurate sketch of the graph of a function f:

METHOD FOR GRAPH SKETCHING.

- Determine number lines for f, f', and f'', showing where each is zero (0) or undefined (U) and signs (+ or −) on resulting intervals.
- On number line for f, record significant points (e.g., where horizontal tangents occur, inflection points), record limits at infinity, and record limits from each side where a U occurs.
- Sketch graph: Use signs for f to block out regions where graph cannot go; plot significant points; indicate horizontal tangent lines; complete sketch guided by f' number line (+ rising, − falling) and f'' number line (+ concave up, − concave down).

33.4.1 Example Use the method outlined above to sketch the graph of

$$f(x) = \frac{x^2 - x - 2}{x^2}.$$

Solution The function has factored form

$$f(x) = \frac{(x+1)(x-2)}{x^2},$$

so it is zero when $x = -1, 2$ and undefined when $x = 0$. The derivative is

$$f'(x) = \frac{x^2(2x-1) - (x^2 - x - 2)(2x)}{x^4} = \frac{x^2 + 4x}{x^4} = \frac{x+4}{x^3},$$

which is zero when $x = -4$ and undefined when $x = 0$. The second derivative is

$$f''(x) = \frac{x^3(1) - (x+4)(3x^2)}{x^6} = \frac{-2x^3 - 12x^2}{x^6} = \frac{-2(x+6)}{x^4},$$

which is zero when $x = -6$ and undefined when $x = 0$.

Since

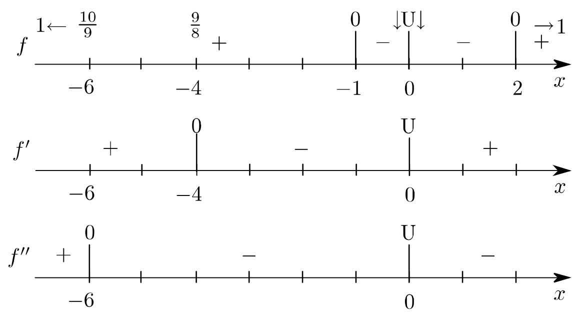
$$\lim_{x\to\pm\infty} \frac{x^2 - x - 2}{x^2} \quad \left(\frac{\infty}{\infty}\right) = \lim_{x\to\pm\infty} 1 - \frac{1}{x} - \frac{2}{x^2} = 1,$$

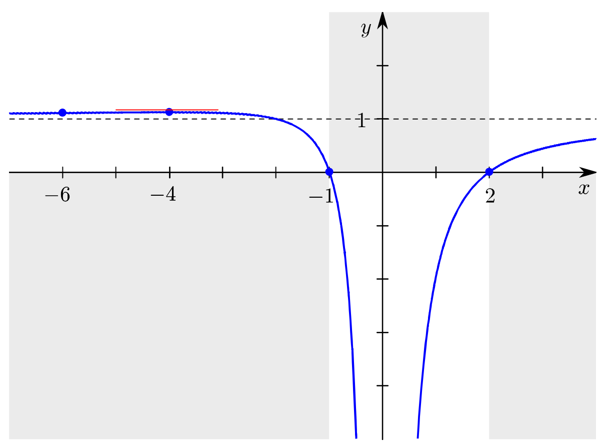
$y = 1$ is a horizontal asymptote; the height of the graph approaches 1 as x moves away from the origin in either direction.

Since

$$\lim_{x\to 0^{\pm}} \frac{x^2 - x - 2}{x^2} = -\infty \qquad \left(\frac{\text{about } -2}{\text{small pos.}}\right)$$

$x = 0$ is a vertical asymptote; the height of the graph goes to $-\infty$ as x gets close to 0 from either side.





(It is difficult to see the concavity change from up to down at $x = -6$ since the graph is nearly straight there, but it makes sense that there needs to be a change in concavity in order for the graph to approach the line $y = 1$ asymptotically on the one hand and achieve a horizontal tangent at -4 on the other hand.) □

33.4.2 Example Use the method outlined above to sketch the graph of $f(x) = x \ln x$.

Solution Due to the factor $\ln x$, the domain of f is $(0, \infty)$. Since x cannot be zero, the only way f can be zero is for $\ln x$ to be zero, and this happens when $x = 1$. The derivative is

$$f'(x) = (1)\ln x + x(1/x) = \ln x + 1,$$

which is zero when $\ln x = -1$, that is, when $x = e^{-1} = 1/e$. The second derivative is

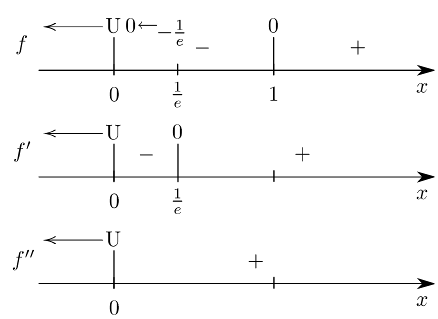
$$f''(x) = \frac{1}{x},$$

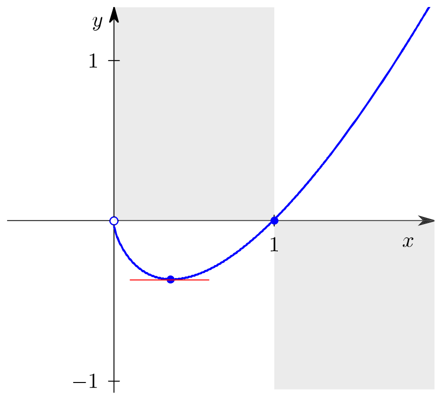
which is never zero and never undefined (0 is not in the domain of the function f).

The number lines show that the graph of f is rising and concave up as x moves to the right, so there is no chance of a horizontal asymptote. We have

$$\begin{aligned}\lim_{x\to 0^+} x\ln x \ (0\cdot -\infty) &= \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} \left(\frac{-\infty}{\infty}\right)\\ &\overset{\text{l'H}}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0,\end{aligned}$$

so the height of the graph approaches 0 as x approaches 0 from the right.





□

33.4.3 Example Use the method outlined above to sketch the graph of

$$f(x) = \frac{\cos x}{2 + \sin x}.$$

Solution Since both $\cos x$ and $\sin x$ repeat every 2π units, the graph of f also repeats every 2π units. We will sketch the graph just on the interval $[0, 2\pi]$.

The function is zero when $\cos x$ is zero, that is, when $x = \pi/2, 3\pi/2$. The function is never undefined (the least $\sin x$ can be is -1, so the denominator is never zero). The derivative is

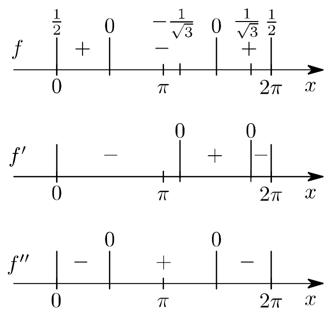
$$f'(x) = \frac{(2 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(2 + \sin x)^2} = \frac{-2\sin x - 1}{(2 + \sin x)^2},$$

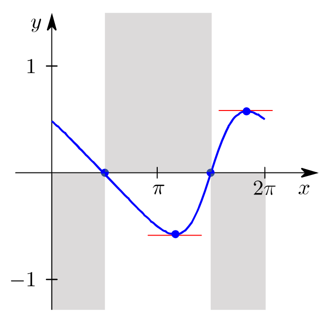
where we have used the identity $\sin^2 x + \cos^2 x = 1$ to simplify. The derivative is zero when $\sin x = -1/2$, that is, when $x = 7\pi/6, 11\pi/6$. The derivative is never undefined. The second derivative is

$$\begin{aligned} f''(x) &= \frac{(2 + \sin x)^2(-2\cos x) - (-2\sin x - 1)2(2 + \sin x)\cos x}{(2 + \sin x)^4} \\ &= \frac{(2 + \sin x)(-4\cos x - 2\sin x \cos x + 4\sin x \cos x + 2\cos x)}{(2 + \sin x)^4} \\ &= \frac{2\cos x(\sin x - 1)}{(2 + \sin x)^3}. \end{aligned}$$

(To simplify, we factored $2 + \sin x$ from both terms in the numerator and expanded what remained, ultimately canceling this factor with the same in the denominator.) The second derivative is zero when $\cos x$ is zero, that is, when $x = \pi/2, 3\pi/2$. It is also zero when $\sin x = 1$, but this is when $x = \pi/2$, which we already have. The second derivative is never undefined.

Due to the periodicity of the graph, there are no horizontal asymptotes.





□

33 – Exercises