Chap 4.4: Concavity And Curve Sketching

COROLLARY 3 Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

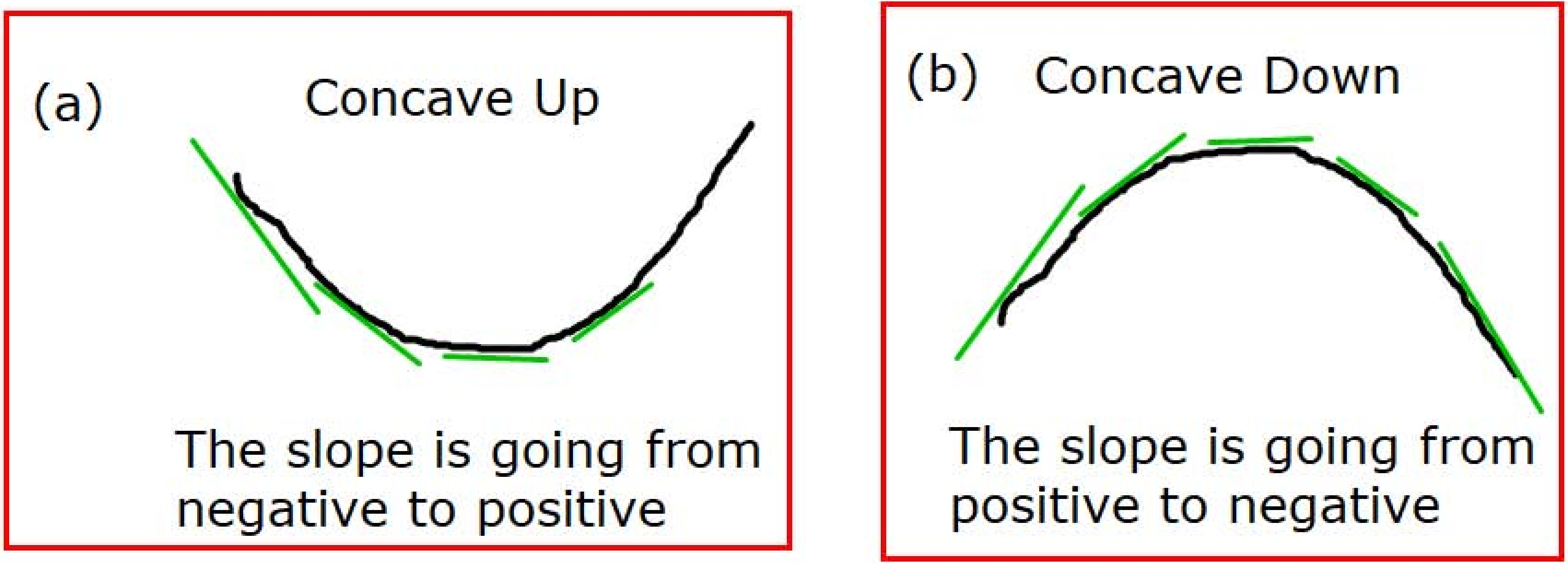
If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.

If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

DEFINITION The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I;

(b) **concave down** on an open interval I if f' is decreasing on I.


(a)
Concave Up
The slope is going from negative to positive
(b)
Concave Down
The slope is going from positive to negative

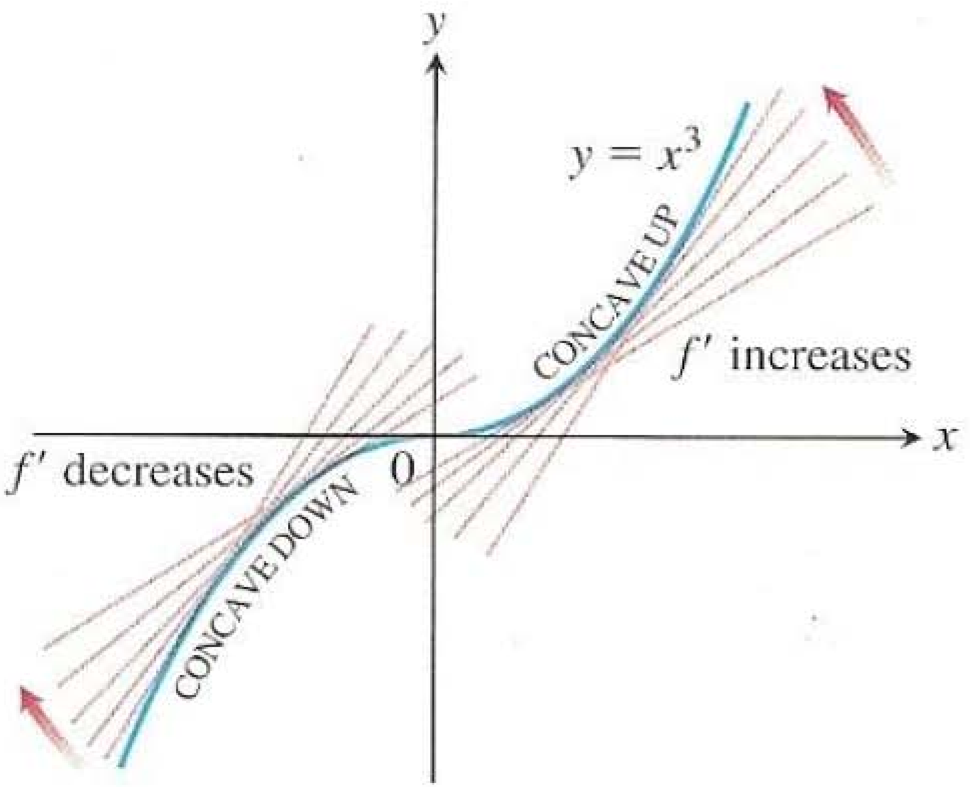


FIGURE 4.24 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

Apply the Mean Value Theorem

Part 1.) Since f'' exists, f' is continuous in $[a, b]$ and differentiable in (a, b).

By the Mean Value Theorem, there exists a c, where $a < c < b$, such that

$$f''(c) = \frac{f'(b) - f'(a)}{b - a}$$

Equivalent Fractions

$$\frac{f''(c)}{1} = \frac{f'(b) - f'(a)}{b - a}$$

we are given that $f''(c) > 0$

we have

$$\frac{f'(b)-f'(a)}{(b-a)} > 0$$

$$f'(b)-f'(a) > 0\cdot(b-a)$$

$$f'(b) - f'(a) > 0$$

$$f'(b) > f'(a)$$

$f'(x)$ is increasing in $[a, b]$

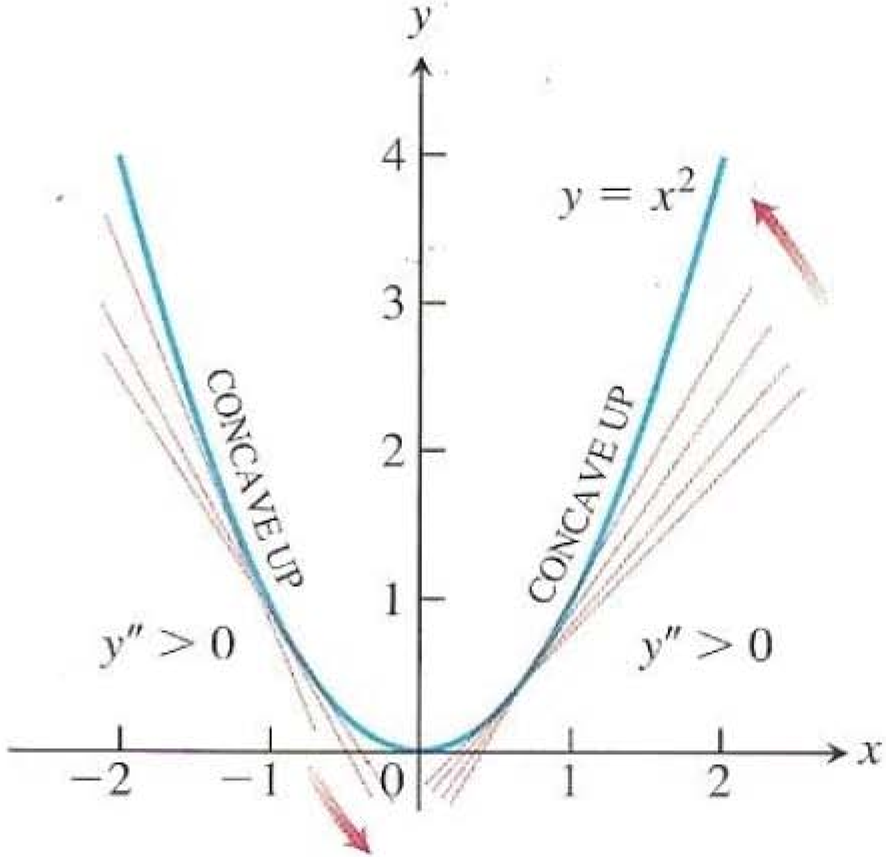


FIGURE 4.25 The graph of $f(x) = x^2$ is concave up on every interval (Example 1b).

DEFINITION A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection.**

At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

EXAMPLE 3 Determine the concavity and find the inflection points of the function

$$f(x) = x^3 - 3x^2 + 2.$$

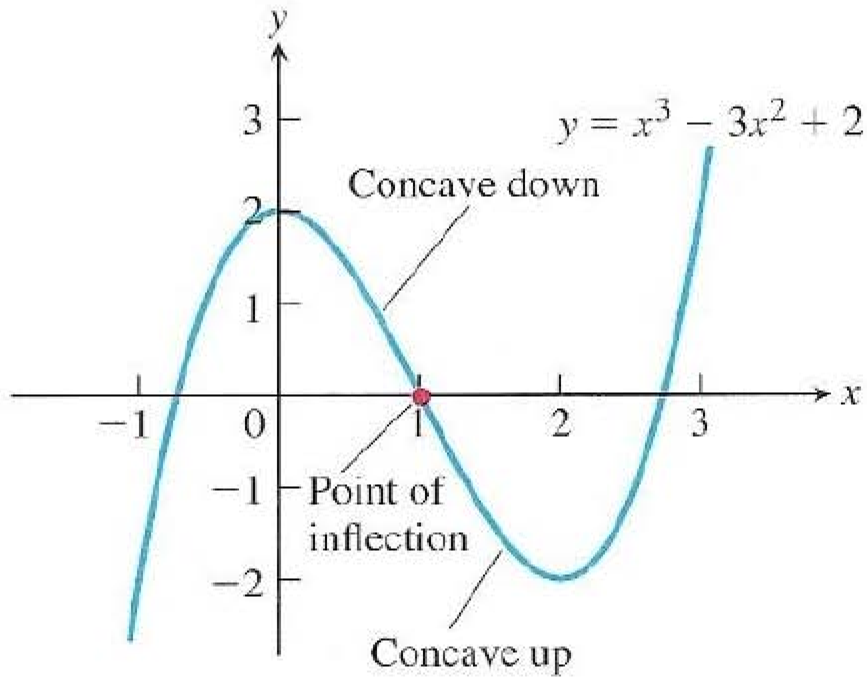


FIGURE 4.27 The concavity of the graph of f changes from concave down to concave up at the inflection point.

EXAMPLE 4 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin because $f'(x) = (5/3)x^{2/3} = 0$ when $x = 0$. However, the second derivative

$$f''(x) = \frac{d}{dx}\left(\frac{5}{3}x^{2/3}\right) = \frac{10}{9}x^{-1/3}$$

fails to exist at $x = 0$. Nevertheless, $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so the second derivative changes sign at $x = 0$ and there is a point of inflection at the origin. The graph is shown in Figure 4.28. ■

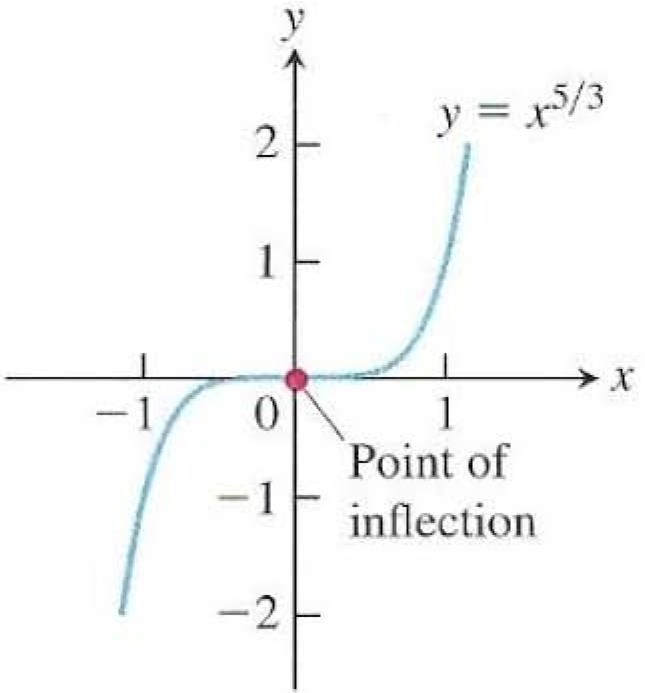
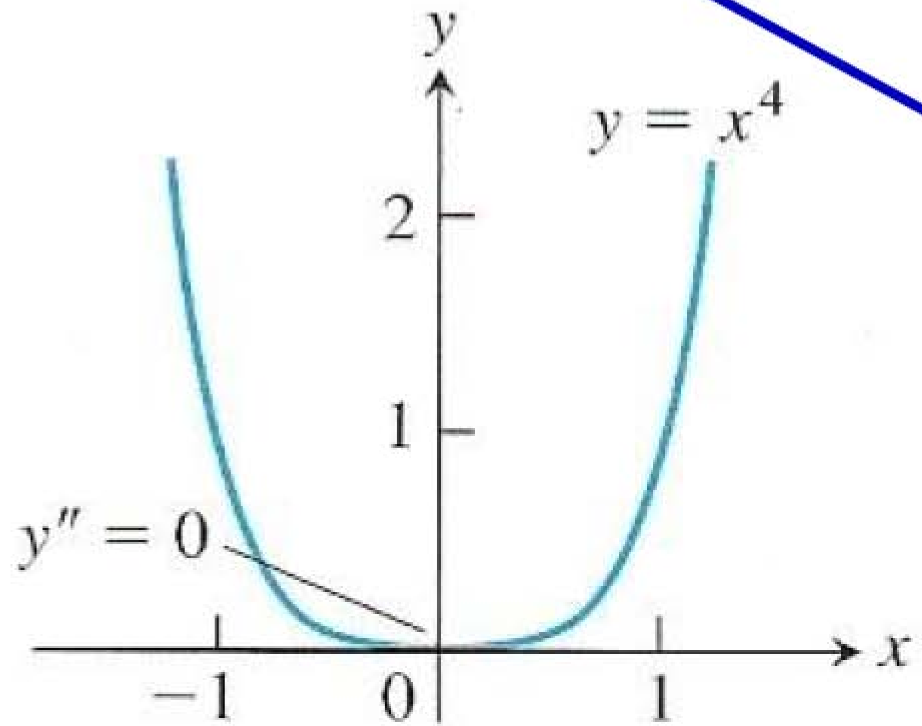


FIGURE 4.28 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin where the concavity changes, although f'' does not exist at $x = 0$ (Example 4).

EXAMPLE 5 The curve $y = x^4$ has no inflection point at $x = 0$ (Figure 4.29). Even though the second derivative $y'' = 12x^2$ is zero there, it does not change sign. The curve is concave up everywhere. ■



The values of y'' are always positive

FIGURE 4.29 The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there (Example 5).

EXAMPLE 6 The graph of $y = x^{1/3}$ has a point of inflection at the origin because the second derivative is positive for $x < 0$ and negative for $x > 0$:

$$y'' = \frac{d^2}{dx^2}\left(x^{1/3}\right) = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}.$$

However, both $y' = x^{-2/3}/3$ and y'' fail to exist at $x = 0$, and there is a vertical tangent there. See Figure 4.30.

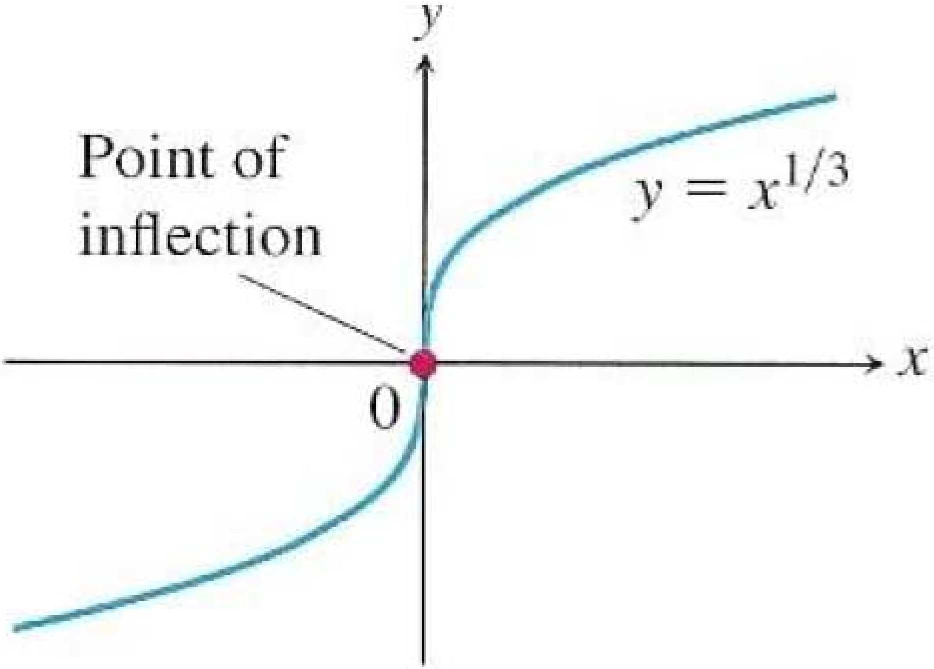


FIGURE 4.30 A point of inflection where y' and y'' fail to exist (Example 6).

THEOREM 5—Second Derivative Test for Local Extrema

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.