Calculus A

§ *4.3 Monotonic Functions and the First Derivative Test*

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What We Will Look At

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Topics: In this section, we will look at two main ideas:

1. Identifying where functions are increasing and where they are decreasing.
2. The first derivative test.

Learning Objectives:

For the topics in this section, students are expected to be able to:

1. Determine where a function is increasing or decreasing.
2. Classify critical points using the first derivative test.
3. Sketch functions using the first derivative and the first derivative test.

What We Will Look At

Motivation

- In sketching the graph of a function it is useful to know where it increases and where it decreases over an interval.
- This section gives a test to determine where a function increases and where it decreases.
- We also explore one method for testing the critical points of a function to identify whether local extreme values are present.

Definition: Increasing and Decreasing Functions

1. A function f is increasing on an interval I if for any $x_1, x_2 \in I$ with $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
2. A function f is decreasing on an interval I if for any $x_1, x_2 \in I$ with $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.
3. A function f is constant on an interval I if for any $x_1, x_2 \in I$ with $x_1 < x_2 \Rightarrow f(x_1) = f(x_2)$.

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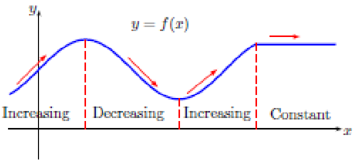
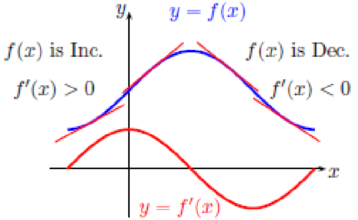
Note

A function f is increasing if , as x moves to the right, its graph moves up and is decreasing if its graph moves down and it is constant if the graph moves horizontally. See Figure





Derivatives and Increasing and Decreasing Functions

Theorem

Let f be a continuous function on the closed interval $[a, b]$, differentiable on the open interval (a, b).

- If $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on an interval $[a, b]$.
- If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on an interval $[a, b]$.

A function that is increasing (or decreasing) on an interval is **monotonic** on that interval.

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Guidelines for finding intervals on which a function f is monotonic

To find the intervals on which a function f is monotonic:

1. Find the critical numbers of f.
2. The critical numbers will divide the domain into subintervals.
3. Test the sign of f' on each subinterval you got in step 2.
4. Positive sign means f is increasing and negative sign means f is decreasing.

Monotonic Functions and The First Derivative Test

Example 1

Find the intervals on which $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ is increasing decreasing.

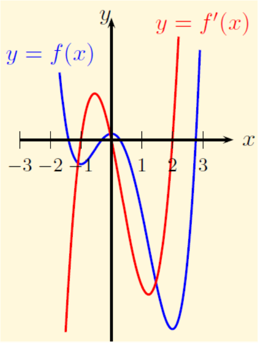
Solution: Note that f is a polynomial, then it is continuous on $\mathbb{R}$. So to find the critical numbers, set $f'(x) = 0$.

$$f'(x) = 0 \Rightarrow 12x^3 - 12x^2 - 24x = 0$$
$$\Rightarrow 12x(x^2 - x - 2) = 0$$
$$\Rightarrow 12x(x-2)(x+1) = 0$$

Hence $f'(x) = 0$ if $x = -1, 0, 2$.

So the critical numbers are $x = -1, 0, 2$. To find the intervals of increasing and decreasing we find the sign of each factor of $f'(x)$ and we get the following chart.

sign of	$-\infty$ to -1	-1 to 0	0 to 2	2 to ∞
$12x$	$-$	$-$	$+$	$+$
$x+1$	$-$	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
$f'(x)$	$-$	$+$	$-$	$+$
Concl	Dec.	Inc.	Dec.	Inc.



So f is increasing on the intervals $(-1, 0) \cup (2, \infty)$ and decreasing on $(-\infty, 1) \cup (0, 2)$. See the graph of f.

First Derivative Test for Local Extrema

Theorem

Let f be a continuous function on the closed interval $[a, b]$, and let $c \in (a, b)$ be a critical number for f. If f is differentiable on the open interval (a, b) except possibly at c.

- If $f'(x) > 0$ for all $a < x < c$, and $f'(x) < 0$ for all $c < x < b$, ($f'(x)$ changes from positive to negative at c), then $f(c)$ is a local maximum.
- If $f'(x) < 0$ for all $a < x < c$, and $f'(x) > 0$ for all $c < x < b$, ($f'(x)$ changes from negative to positive at c), then $f(c)$ is a local minimum.
- If $f'(x)$ does not change sign at c, then $f(c)$ has no local extreme.

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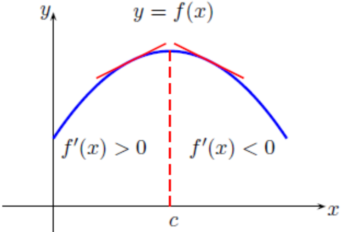
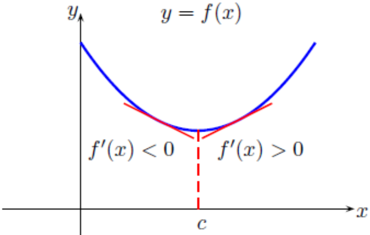


$f(c)$ is Max.



$f(c)$ is Min.

Example 1: Re-visit

Example 2

Find the local extreme for $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$.

Solution: From the discussion in Example 1, and using the same chart of signs

sign of	$-\infty$ to -1	-1 to 0	0 to 2	2 to ∞
$12x$	$-$	$-$	$+$	$+$
$x+1$	$-$	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
$f'(x)$	$-$	$+$	$-$	$+$
Concl	Dec.	Inc.	Dec.	Inc.

So f is increasing on the intervals $(-1, 0) \cup (2, \infty)$ and decreasing on $(-\infty, 1) \cup (0, 2)$. It follows from the first derivative test that f has a local maximum at $x = 0$ with value $f(0) = 1$. Also f has a local minimum at $x = -1$ and $x = 2$ with value $f(-1) = -4$ and $f(2) = -31$. See the graph of the function again.

First Derivative Test for Local Extrema

Example 3

Find the local extreme for $f(x) = \frac{x}{2} + \sin x$ in the interval $(0, 2\pi)$.

Solution: Note that f is continuous on $(0, 2\pi)$. To find the critical numbers, set $f'(x) = 1/2 + \cos x = 0$. Now,

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow 1/2 + \cos x = 0 \\ &\Leftrightarrow \cos x = -1/2 \\ &\Leftrightarrow x = 2\pi/3 \ \& \ x = 4\pi/3. \end{aligned}$$

Hence $f'(x) = 0$ if $x = 2\pi/3, \ x = 4\pi/3$ in $(0, 2\pi)$. So the critical numbers in $(0, 2\pi)$ are $x = 2\pi/3, x = 4\pi/3$. To find the intervals of increasing and decreasing we find the sign of $f'(x)$ and we get the following chart.

So f is increasing on the intervals $(0, 2\pi/3) \cup (4\pi/3, 2\pi)$ and decreasing on $(2\pi/3, 4\pi/3)$. It follows from the first derivative test that f has a local maximum at $x = 2\pi/3$ with value $f(2\pi/3) = \pi/3 + \sqrt{3}/2$. Also f has a local minimum at $x = 4\pi/3$ with value $f(4\pi/3) = 2\pi/3 - \sqrt{3}/2$.

0 $\frac{2\pi}{3}$ $\frac{4\pi}{3}$ 2π

Test Value	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
f'(Test Value)	$f'(\frac{\pi}{2}) = \frac{1}{2}$	$f'(\pi) = \frac{-1}{2}$	$f'(\frac{3\pi}{2}) = \frac{1}{2}$
sign of $f'(x)$	$+$	$-$	$+$
Concl	Inc.	Dec.	Inc.