# Monotonicity and Concavity

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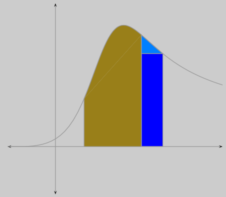
## Abstract

This work presents the traditional material of calculus I with some of the material from a traditional precalculus course interwoven throughout the discussion. Precalculus topics are discussed at or soon before the time they are needed, in order to facilitate the learning of the calculus material. Miniature animated demonstrations and interactive quizzes will be available to help the reader deepen his or her understanding of the material under discussion.



This project is funded in part by the Mellon Foundation through the Mellon Furman-Wofford Program. Many of the illustrations were designed by Furman undergraduate student Brian Wagner using the PSTricks package. Thanks go to my wife Suzan, and my two daughters, Hannah and Darby for patience while this project was being produced.

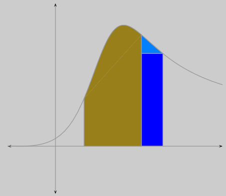





Mark R. Woodard

# Contents







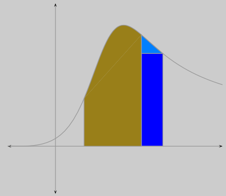

Title Page
Contents
Go Back
Close
Quit

# 1. Monotonicity

We all have an intuitive idea about what it means for a function to "go up" or increase (or to do the opposite: "go down" or decrease.) Let's formalize this idea with a definition:

> We say that a function is *increasing on an interval* $I$ if for all $x_1, x_2 \in I$ with $x_1 < x_2$ we have $f(x_1) < f(x_2)$. We say that $f(x)$ is *decreasing on an interval* $I$ if for all $x_1, x_2 \in I$ with $x_1 < x_2$ we have $f(x_1) > f(x_2)$

For example, $f(x) = x^2$ is increasing on the interval $[0, \infty)$ and decreasing on the interval $(-\infty, 0]$. A function is said to be *monotonic on an interval* $I$ if it is strictly increasing or strictly decreasing on $I$. In attempting to understand the geometry of the graph of a particular function $f(x)$, it is desirable to find the intervals on which the function is monotonic. Of course, it is most interesting to find the largest such intervals. There is a clear and almost obvious relationship between the derivative of a function and the monotonicity. This can be seen by drawing a number of strictly increasing functions, and drawing tangent lines on any of them at any point, and thinking about what is always true about the slopes of the tangent lines. You should be able to informally convince yourself pretty quickly that they all have positive slopes. Thus there seems to be a relationship between an increasing function and a positive derivative. Likewise, there seems to be a relationship between a decreasing function and a negative derivative. The following seems plausible:





Title Page
Contents
Go Back
Close
Quit

**Monotonicity Theorem**

If $f(x)$ is so that $f'(x) > 0$ for all $x$ in the interval $I$, then $f(x)$ is increasing on $I$. If $f'(x) < 0$ for all $x$ in $I$, then $f(x)$ is decreasing on $I$.

The proof of the Monotonicity Theorem uses the Mean Value Theorem. The idea is that if $f'(x) > 0$ on $I$, and if $x_1 < x_2$ on $I$, then there is a point $c$ between $x_1$ and $x_2$ with $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c)$. But if $f'(c) > 0$, we must have $f(x_2) - f(x_1) > 0$, since the denominator $x_2 - x_1 > 0$. So $f(x_1) < f(x_2)$, so $f(x)$ is increasing. A similar proof works for $f(x)$ decreasing when $f'(x) < 0$ on $I$. This result can be used to find the intervals of monotonicity for a given function: by finding the largest intervals on which the derivative of $f(x)$ is positive, we are also finding the largest intervals on which $f(x)$ is increasing. A similar statement can be made replacing the word "increasing" by "decreasing" and the word "positive" by "negative."

Exercise 1. Find the largest intervals of monotonicity for $f(x) = \frac{x}{x^2+2}$.

One upshot of the Monotonicity Theorem is that we can now see how to use the first derivative to help us locate relative maximum and relative minimums. This is because if $f(x)$ is continuous on the interval $(a, b)$, and is increasing on an interval $(a, c)$ and decreasing on $(c, b)$ where $c$ is between $a$ and $b$, then there must clearly be at least a relative maximum at $x = c$. A similar statement can be made about relative minimums if $f(x)$ changes



Title Page
Contents
◀◀ ▶▶
◀ ▶
Go Back
Close
Quit
Page 5 of 9

from decreasing to increasing in the middle of some interval on which $f$ is continuous. These ideas are collectively know as

**The First Derivative Test**

Suppose that $f(x)$ is continuous on $(a,b)$ and differentiable on $(a,b)$ except perhaps at a point $c$ with $a < c < b$. Suppose also that either $f'(c) = 0$ or $f'(c)$ doesn't exist. Then

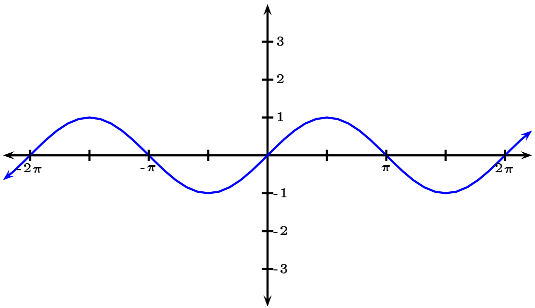
1. If $f'(x) < 0$ on $(a,c)$ and $f'(x) > 0$ on $(c,b)$, then there is a relative minimum value at $x = c$.
2. If $f'(x) > 0$ on $(a,c)$ and $f'(x) < 0$ on $(c,b)$, then there is a relative maximum value at $x = c$.
3. If the sign of $f'(x)$ is the same on both $(a,c)$ and $(c,b)$, then there is no relative extremum at $x = c$.

Exercise 2. Use the analysis done previously to find the relative extrema for $f(x) = \frac{x}{x^2+2}$.


Title Page
Contents
◀◀ ▶▶
◀ ▶
Go Back
Close
Quit

## 2. Concavity

We've had such good luck relating the sign of the derivative of a function to information about the graph of the function that one can't help but wonder whether or not the sign of the second derivative would yield any nice information about the graph of the function. Consider the following graph of $y = \sin(x)$.

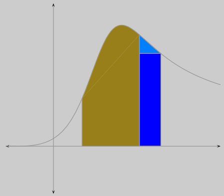



Note that between $-\pi$ and 0, the slopes of the tangent lines are increasing as we go from left to right. (The lines start out having negative slopes, and then the slopes increase to zero, and then the lines continue to get steeper.) Since the derivative of $y = \sin(x)$ tells us the slope of the tangent line at a given point, we can see that the derivative of the derivative of $y$ must be positive here. Thus the second derivative is positive. In this situation, we say that $y$ is *concave up* on the interval $(-\pi, 0)$. On the other hand, between 0 and $\pi$ we see that the slopes of the tangent lines are decreasing as we move from left to right, so the derivative of the derivative must be negative here, so $y'' < 0$ on $(0, \pi)$. Summarizing:

We say that $f(x)$ is *concave up* on an interval $I$ iff $f''(x) > 0$ on $I$. We say that $f(x)$ is *concave down* on $I$ iff $f''(x) < 0$ on $I$. If $c$ is a point in the domain of $f(x)$, and if $f(x)$ has one type of concavity on $(a, c)$ and the other type on $(c, b)$, then we say that the point $(c, f(c))$ is a *point of inflection* of $f(x)$.

EXERCISE 1. Continue studying $f(x) = \frac{x}{x^2+2}$ by analyzing its concavity.

It turns out that for stationary points, we can use the second derivative to help analyze whether a given point is a relative maximum or a relative minimum. This is because if $f(x)$ has a relative minimum and a horizontal tangent line at $x = c$, then it must be concave up there. (Try drawing an example and see for yourself!) On the other hand if it has a relative maximum at $x = c$ and a horizontal tangent line, it must be concave down there. Thus we have:



**The Second Derivative Test**

Suppose that $f(x)$ is so that $f''(x)$ exists in some interval containing $c$, and $f'(c) = 0$.

1. If $f''(c) < 0$ then there is a relative maximum at $(c, f(c))$.
2. If $f''(c) > 0$ then there is a relative minimum at $(c, f(c))$.
3. If $f''(c) = 0$, we make no conclusion.

Exercise 2. Use the analysis done previously to find the relative extrema for $f(x) = \frac{x}{x^2+2}$ using the second derivative test.

Stewart: Section 4.3: 1 - 37 odd.



Title Page
Contents
Go Back
Close
Quit

# Solutions to Exercises

**Exercise 1.**

Find the largest intervals of monotonicity for $f(x) = \frac{x}{x^2+2}$.

First we need to compute and simplify $f'(x)$. We have

$$\begin{aligned} f'(x) &= \frac{(x^2+2)(1) - (x)(2x)}{(x^2+2)^2} && \text{Quotient Rule} \\ &= \frac{2 - x^2}{(x^2+2)^2} && \text{Simplifying} \end{aligned}$$

Note that the numerator of $f'(x)$ is zero when $x = \pm\sqrt{2}$, and the denominator is never zero. Thus, $f'(x)$ is either strictly positive or strictly negative on the intervals $(-\infty, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$, and $(\sqrt{2}, \infty)$. Using test values, we see that $f'(x)$ is positive on $(-\sqrt{2}, \sqrt{2})$ and negative on the other two intervals. Thus $f(x)$ is increasing on $(-\sqrt{2}, \sqrt{2})$ and decreasing on $(-\infty, -\sqrt{2})$ and on $(\sqrt{2}, \infty)$. Exercise 1

**Exercise 2.**

Use the analysis done previously to find the relative extrema for $f(x) = \frac{x}{x^2+2}$.

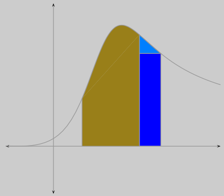
We already found that there were stationary points at $x = \pm\sqrt{2}$, and that $f(x)$ is increasing on $(-\sqrt{2}, \sqrt{2})$ and decreasing on $(-\infty, -\sqrt{2})$ and on $(\sqrt{2}, \infty)$. Using the first derivative test, we know see that there is a local maximum at $x = \sqrt{2}$ and a local minimum at $x = -\sqrt{2}$. The local maximum value is $\frac{\sqrt{2}}{4}$ while the local minimum value is $\frac{-\sqrt{2}}{4}$. Exercise 2

**Exercise 1.**

Continue studying $f(x) = \frac{x}{x^2+2}$ by analyzing its concavity. Recall that $f'(x) = \frac{2-x^2}{(x^2+2)^2}$. So

$$\begin{aligned}
f''(x) &= \frac{(x^2+2)^2(-2x) - (2-x^2)(2)(x^2+2)(2x)}{(x^2+2)^4} && \text{Quotient Rule} \\
&= \frac{(x^2+2)((x^2+2)(-2x) - (4x)(2-x^2))}{(x^2+2)^4} && \text{Factoring the Numerator} \\
&= \frac{(x^2+2)((-2x^3+-4x) - (8x-4x^3))}{(x^2+2)^4} && \text{Distributing} \\
&= \frac{(-2x^3+-4x) - (8x-4x^3)}{(x^2+2)^3} && \text{Cancelling Factors} \\
&= \frac{(2x^3-12x)}{(x^2+2)^3} && \text{Combining Like Terms} \\
&= \frac{(2x)(x^2-6)}{(x^2+2)^3} && \text{Factoring the Numerator}
\end{aligned}$$

Now an analysis of this second derivative for places where the sign might change shows that $x = 0$ and $x = \pm\sqrt{6}$ are possibilites, since that is where the numerator (and thus $f''(x)$) is zero. Also note that there are no places where the denominator is zero. The potential intervals of concavity are thus $(-\infty, -\sqrt{6})$, $(-\sqrt{6}, 0)$, $(0, \sqrt{6})$, and $(\sqrt{6}, \infty)$. Using test values we

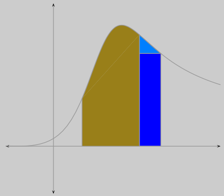




Title Page
Contents
Go Back
Close
Quit

see that the second derivative is negative on $(-\infty, -\sqrt{6})$ and on $(0, \sqrt{6})$, and is positive on $(-\sqrt{6}, 0)$ and on $(\sqrt{6}, \infty)$. Thus $f(x)$ is concave down on $(-\infty, -\sqrt{6})$ and on $(0, \sqrt{6})$, and is concave up on $(-\sqrt{6}, 0)$ and on $(\sqrt{6}, \infty)$. There are points of inflection at $(0, 0)$, $(-\sqrt{6}, \frac{-\sqrt{6}}{8})$, and $(\sqrt{6}, \frac{\sqrt{6}}{8})$.

Exercise 1





Title Page
Contents
◀◀ ▶▶
◀ ▶
Go Back
Close
Quit
Page 13 of 9

**Exercise 2.**

Use the analysis done previously to find the relative extrema for $f(x) = \frac{x}{x^2+2}$.

We already found that there were stationary points at $x = \pm\sqrt{2}$, and we found $f''(x) = \frac{(2x)(x^2-6)}{(x^2+2)^3}$. Since $f''(-\sqrt{2}) = \frac{-2\sqrt{2}(-4)}{4^3} > 0$ there must be a relative minimum at $x = -\sqrt{2}$. Since $f''(\sqrt{2}) = \frac{2\sqrt{2}(-4)}{4^3} < 0$, there must be a relative maximum at $x = \sqrt{2}$. Note that this coincides with what we obtained using the first derivative test. Exercise 2



Title Page
Contents
Go Back
Close
Quit
Page 14 of 9