# 4.3 THE FIRST DERIVATIVE AND THE SHAPE OF f

This section examines some of the interplay between the shape of the graph of f and the behavior of f '. If we have a graph of f , we will see what we can conclude about the values of f '. If we know values of f ', we will see what we can conclude about the graph of f .

**Definitions:** The function f is **increasing on (a,b)** if $a < x_1 < x_2 < b$ implies $f(x_1) < f(x_2)$.
The function f is **decreasing on (a,b)** if $a < x_1 < x_2 < b$ implies $f(x_1) > f(x_2)$.

f is **monotonic on (a,b)** if f is increasing on (a,b) or if f is decreasing on (a,b).

Graphically, f is **increasing** (decreasing) if, as we move from left to right along the graph of f, the height of the graph **increases** (decreases).

These same ideas make sense if we consider h(t) to be the height (in feet) of a rocket at time t seconds. We naturally say that the rocket is rising or that its height is increasing if the height h(t) increases over a period of time, as t increases.

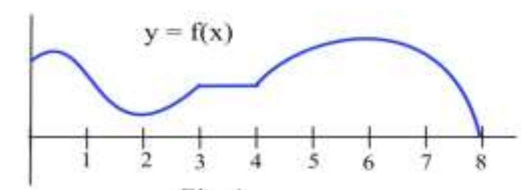

Fig. 1

**Example 1:** List the intervals on which the function given in Fig. 1 is increasing or decreasing.

Solution: f is increasing on the intervals [0,1] , [ 2,3] and [4,6]. f is decreasing on [1,2] and [6,8]. On the interval [3,4] the function is not increasing or decreasing, it is constant. It is also valid to say that f is increasing on the intervals [0.3, 0.8] and (0.2, 0.5) as well as many others, but we usually talk about the longest intervals on which f is monotonic.

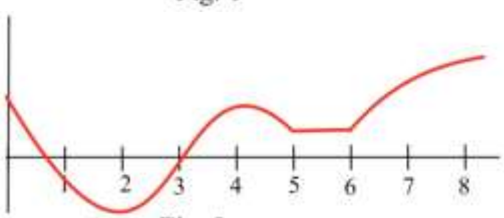

Fig. 2

**Practice 1:** List the intervals on which the function given in Fig. 2 is increasing or decreasing.

If we have an accurate graph of a function, then it is relatively easy to determine where f is monotonic, but if the function is defined by an equation, then a little more work is required. The next two theorems relate the values of the derivative of f to the monotonicity of f. The first theorem says that if we know where f is monotonic, then we also know something about the values of f '. The second theorem says that if we know about the values of f ' then we can draw conclusions about where f is monotonic.

*Source URL: http://scidiv.bellevuecollege.edu/dh/Calculus_all/Calculus_all.html*
*Saylor URL: http://www.saylor.org/courses/ma005/*

**First Shape Theorem**

For a function f which is differentiable on an interval (a,b);

(i) if f is increasing on (a,b), then $f'(x) \ge 0$ for all x in (a,b)

(ii) if f is decreasing on (a,b), then $f'(x) \le 0$ for all x in (a,b)

(iii) if f is constant on (a,b), then $f'(x) = 0$ for all x in (a,b).

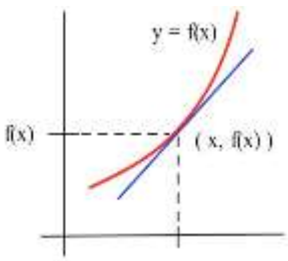

Fig. 3

Proof: Most people find a picture such as Fig. 3 to be a convincing justification of this theorem: if the graph of f increases near a point (x, f(x)), then the tangent line is also increasing, and the slope of the tangent line is positive (or perhaps zero at a few places). A more precise proof, however, requires that we use the definitions of the derivative of f and of "increasing".

(i) Assume that f is increasing on (a,b). We know that f is differentiable, so if x is any number in the interval (a,b) then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$ and this limit exists and is a finite value.

If h is any small enough **positive** number so that x + h is also in the interval (a,b), then x < x + h and f(x) < f(x + h). We know that the numerator, f(x + h) – f(x), and the denominator, h, are both positive so the limiting value, f '(x), must be positive or zero: $f'(x) \ge 0$.

(ii) Assume that f is decreasing on (a,b): The proof of this part is very similar to part (i). If x < x + h, then f(x) > f(x + h) since f is decreasing on (a,b). Then the numerator of the limit, f(x + h) – f(x), will be negative and the denominator, h, will still be positive, so the limiting value, f '(x), must be negative or zero: $f'(x) \le 0$.

(iii) The derivative of a constant is zero, so if f is constant on (a,b) then f '(x) = 0 for all x in (a,b).

The previous theorem is easy to understand, but you need to pay attention to exactly what it says and what it does not say. It is possible for a differentiable function which is increasing on an interval to have horizontal tangent lines at some places in the interval (Fig 4). It is also possible for a continuous function which is increasing on an interval to have an undefined derivative at some places in the interval (Fig. 4). Finally, it is possible for a function which is increasing on an interval to fail to be continuous at some places in the

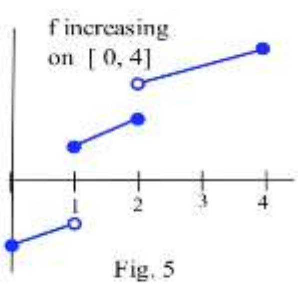

Fig. 5

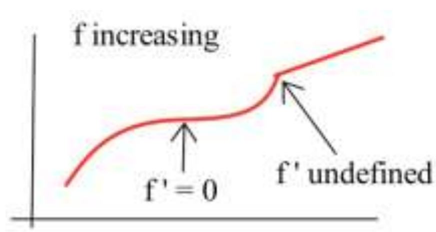

Fig. 4

interval (Fig. 5).

The First Shape Theorem has a natural interpretation in terms of the height h(t) and upward velocity h '(t) of a helicopter at time t. If the height of the helicopter is increasing ( h(t) is increasing ), then the helicopter has a positive or zero upward velocity: h '(t) ≥ 0. If the height of the helicopter is not changing, then its upward velocity is 0: h '(t) = 0.

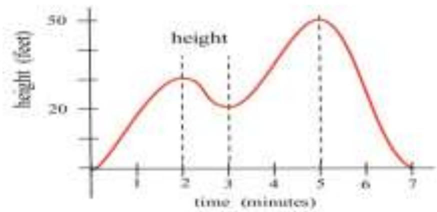

Fig. 6

**Example 2**: Fig. 6 shows the height of a helicopter during a period of time. Sketch the graph of the upward velocity of the helicopter, dh/dt.

Solution: The graph of v(t) = dh/dt is shown in Fig. 7. Notice that the h(t) has a local maximum when t = 2 and t = 5, and v(2) = 0 and v(5) = 0. Similarly, h(t) has a local minimum when t = 3, and v(3) = 0. When h is increasing, v is positive. When h is decreasing, v is negative.

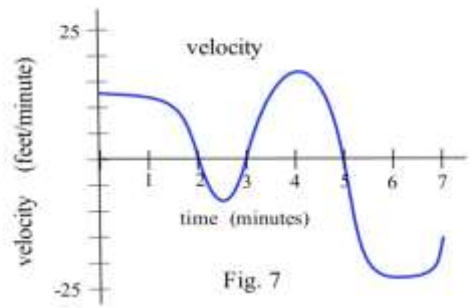

Fig. 7

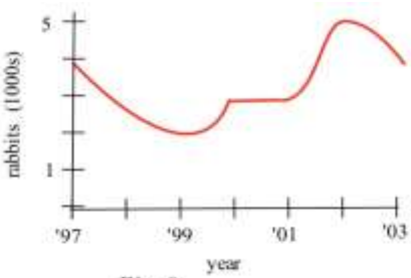

Fig. 8

**Practice 2**: Fig. 8 shows the population of rabbits on an island during 6 years. Sketch the graph of the rate of population change, dR/dt, during those years.

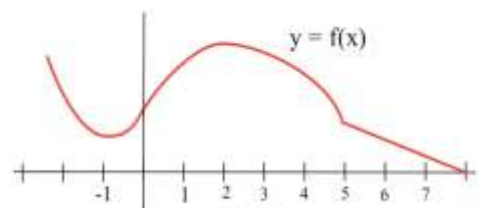

Fig. 9

**Example 3:** The graph of f is shown in Fig. 9 . Sketch the graph of f ' .

Solution: It is a good idea to look first for the points where f '(x) = 0 or where f is not differentiable, the critical points of f. These locations are usually easy to spot, and they naturally break the problem into several smaller pieces. The only numbers at which f '(x) = 0 are x = –1 and x = 2, so the only places the graph of f '(x) will cross the x–axis

are at x = –1 and x = 2 , and we can plot the point (–1,0) and (2,0) on the graph of f '. The only place that f is not differentiable is at the "corner" above x = 5, so the graph of f ' will not have a point for x = 5. The rest of the graph of f is relatively easy:

| | | |
|---|---|---|
| if $x < -1$ then | $f(x)$ is decreasing | so $f'(x)$ is negative, |
| if $-1 < x < 2$ then | $f(x)$ is increasing | so $f'(x)$ is positive, |
| if $2 < x < 5$ then | $f(x)$ is decreasing | so $f'(x)$ is negative, and |
| if $5 < x$ then | $f(x)$ is decreasing | so $f'(x)$ is negative. |

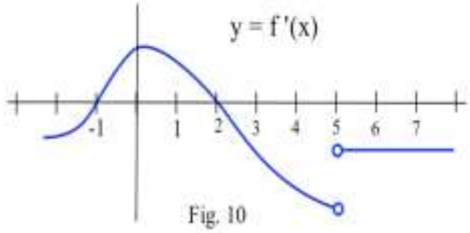

Fig. 10

The graph of $f'$ is shown in Fig. 10. $f(x)$ is continuous at $x = 5$, but $f$ is not differentiable at $x = 5$, as is indicated by the "hole" in the graph.

**Practice 3:** The graph of $f$ is shown in Fig. 11. Sketch the graph of $f'$. (The graph of f has a "corner" at $x = 5$.)

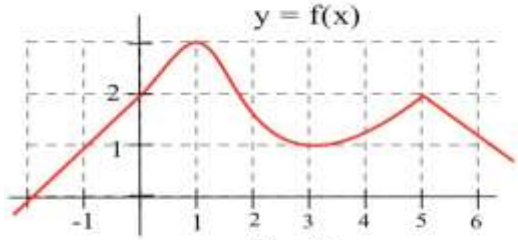

Fig. 11

The next theorem is almost the converse of the First Shape Theorem and explains the relationship between the values of the derivative and the graph of a function from a different perspective. It says that if we know something about the values of $f'$, then we can draw some conclusions about the shape of the graph of $f$.

> **Second Shape Theorem**
>
> For a function $f$ which is differentiable on an interval I;
>
> (i) if $f'(x) > 0$ for all $x$ in the interval I, then $f$ is increasing on I,
>
> (ii) if $f'(x) < 0$ for all $x$ in the interval I, then $f$ is decreasing on I,
>
> (iii) if $f'(x) = 0$ for all $x$ in the interval I, then $f$ is constant on I.

Proof: This theorem follows directly from the Mean Value Theorem, and part (c) is just a restatement of the First Corollary of the Mean Value Theorem.

(a) Assume that $f'(x) > 0$ for all $x$ in I and pick any points $a$ and $b$ in I with $a < b$. Then, by the Mean Value Theorem, there is a point $c$ between $a$ and $b$ so that $\frac{f(b) - f(a)}{b - a} = f'(c) > 0$, and we can conclude that $f(b) - f(a) > 0$ and $f(b) > f(a)$. Since $a < b$ implies that $f(a) < f(b)$, we know that $f$ is increasing on I.

(b) Assume that $f'(x) < 0$ for all $x$ in I and pick any points $a$ and $b$ in I with $a < b$. Then there is a point $c$ between $a$ and $b$ so that $\frac{f(b) - f(a)}{b - a} = f'(c) < 0$, and we can conclude that $f(b) - f(a) = (b-a) f'(c) < 0$ so $f(b) < f(a)$. Since $a < b$ implies that $f(a) > f(b)$, we know f is decreasing on I.

**Practice 4:** Rewrite the Second Shape Theorem as a statement about the height h(t) and upward velocity h '(t) of a helicopter at time t seconds.

The value of the function at a number x tells us the height of the graph of f above or below the point x on the x–axis. The value of f ' at a number x tells us whether the graph of f is increasing or decreasing (or neither) as the graph passes through the point (x, f(x) ) on the graph of f. If f(x) is positive, it is possible for f '(x) to be positive, negative, zero or undefined: the value of f(x) has absolutely nothing to do with the value of f ' . Fig. 12 illustrates some of the combinations of values for f and f ' .

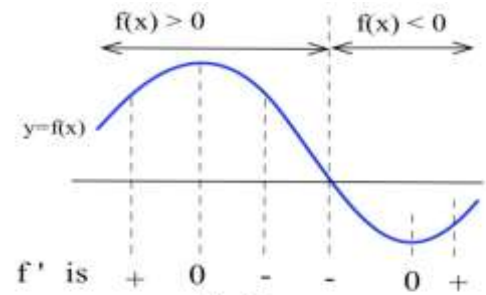

Fig. 12

**Practice 5:** Graph a continuous function which satisfies the conditions on f and f ' given below:

| x | –2 | –1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| f(x) | 1 | –1 | –2 | –1 | 0 | 2 |
| f '(x) | –1 | 0 | 1 | 2 | –1 | 1 |

The Second Shape Theorem is particularly useful if we need to graph a function f which is defined by an equation. Between any two consecutive critical numbers of f, the graph of f is monotonic (why?). If we can find all of the critical numbers of f, then the domain of f will be naturally broken into a number of pieces on which f will be monotonic.

**Example 4:** Use information about the values of f ' to help graph $f(x) = x^3 - 6x^2 + 9x + 1$ .

Solution: $f'(x) = 3x^2 - 12x + 9 = \mathbf{3(x-1)(x-3)}$ so f '(x) = 0 only when x = 1 or x = 3. f ' is a polynomial so it is always defined. The only critical numbers for f are x = 1 and x = 3, and they divide the real number line into three pieces on which f is monotonic: $(-\infty, 1)$ , (1,3) and $(3, \infty)$.

If x < 1, then f '(x) = 3(negative number)(negative number) > 0 so f is increasing.

If 1 < x < 3, then f '(x) = 3(positive number)(negative number) < 0 so f is decreasing.

If 3 < x, then f '(x) = 3(positive number)(positive number) > 0 so f is increasing.

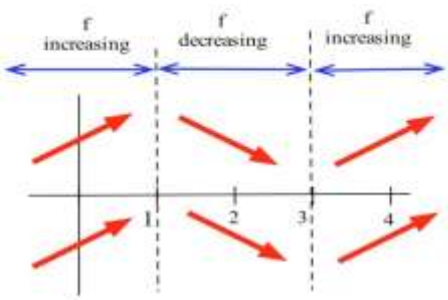


Fig. 13

Even though we don't know the value of f anywhere yet, we do know a lot about the shape of the graph of f : as we move from left to right along the x–axis, the graph of f increases until $x = 1$, then the graph decreases until $x = 3$, and then the graph increases again (Fig. 13) The graph of f makes "turns" when $x = 1$ and $x = 3$.

To plot the graph of f, we still need to evaluate f at a few values of x, but only at a very few values. $f(1) = 5$ , and (1,5) is a local maximum of f. $f(3) = 1$ , and (3,1) is a local minimum of f. The graph of f is shown in Fig. 14.

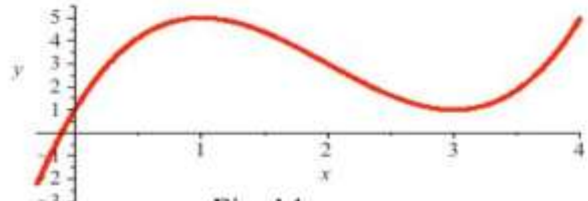


Fig. 14

**Practice 6:** Use information about the values of f ' to help graph $f(x) = x^3 - 3x^2 - 24x + 5$ .

**Example 5:** Use the graph of f ' in Fig. 15 to sketch the shape of the graph of f. Why isn't the graph of f ' enough to completely determine the graph of f ?

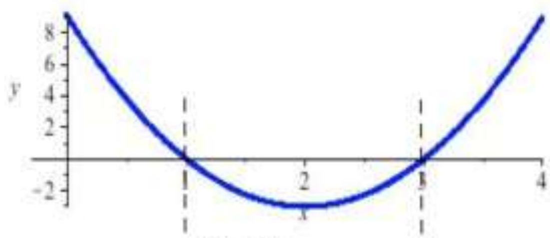


Fig. 15

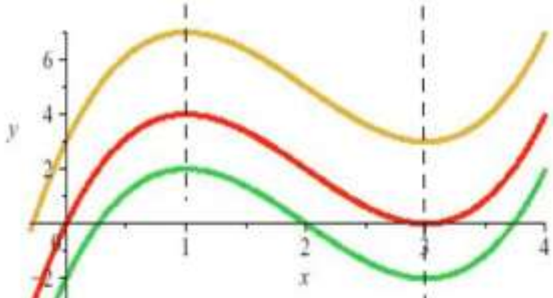


Fig. 16

Solution: Several functions which have the derivative we want are given in Fig. 16 , and each of them is a correct answer. By the Second Corollary to the Mean Value Theorem, we know there is a whole family of parallel functions which have the derivative we want, and each of these functions is a correct answer. If we had additional information about the function such as a point it went through, then only one member of the family would satisfy the extra condition and that function would be the only correct answer.

**Practice 7:** Use the graph of g ' in Fig. 17 to sketch the shape of a graph of g.

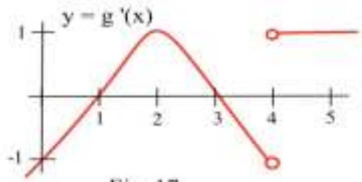

Fig. 17

**Practice 8**: A weather balloon is released from the ground and sends back its upward velocity measurements (Fig. 18). Sketch a graph of the height of the balloon. When was the balloon highest?

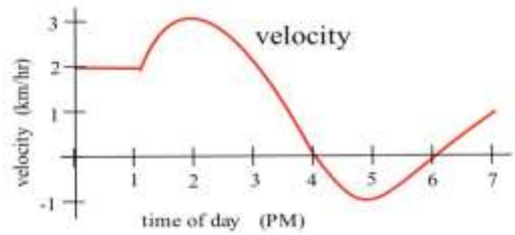

Fig. 18

## Using the Derivative to Test for Extremes

The first derivative of a function tells about the general shape of the function, and we can use that shape information to determine if an extreme point is a maximum or minimum or neither.

**First Derivative Test for Local Extremes**

Let f be a continuous function with f '(a) = 0 or f '(a) is undefined.

(i) If f '(left of a) > 0 and f '(right of a) < 0, then (a,f(a)) is a local maximum (Fig. 19a)

(ii) If f '(left of a) < 0 and f '(right of a) > 0, then (a,f(a)) is a local minimum (Fig. 19b)

(iii) If f '(left of a) > 0 and f '(right of a) > 0, then (a,f(a)) is **not** a local extreme (Fig. 19c)

(iv) If f '(left of a) < 0 and f '(right of a) < 0, then (a,f(a)) is **not** a local extreme (Fig. 19d)

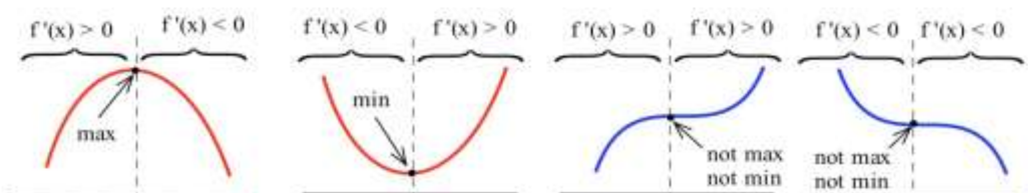

Fig. 19

**Practice 9:** Find all extremes of $f(x) = 3x^2 - 12x + 7$ and use the First Derivative Test to determine if they are maximums, minimums or neither.

A variant of the First Derivative Test can also be used to determine whether an endpoint gives a maximum or minimum for a function.

## PROBLEMS

In problems 1–3, sketch the graph of the derivative of each function.

1. Use Fig. 20.
2. Use Fig. 21.
3. Use Fig. 22

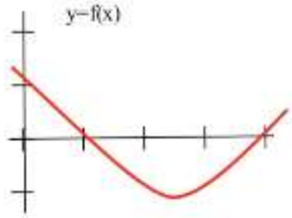

Fig. 20

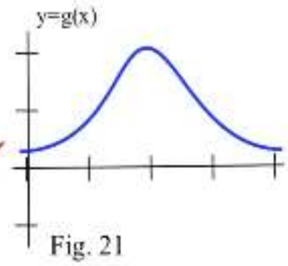

Fig. 21

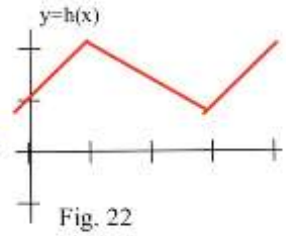

Fig. 22

In problems 4–6, the graph of the height of a helicopter is shown. Sketch the graph of the upward velocity of the helicopter.

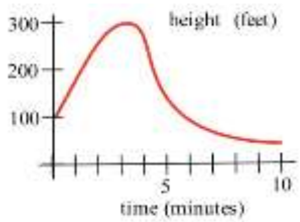

Fig. 23

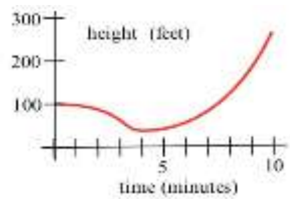

Fig. 24

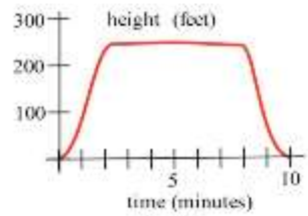

Fig. 25

4. Use Fig. 23.
5. Use Fig. 24.
6. Use Fig. 25.

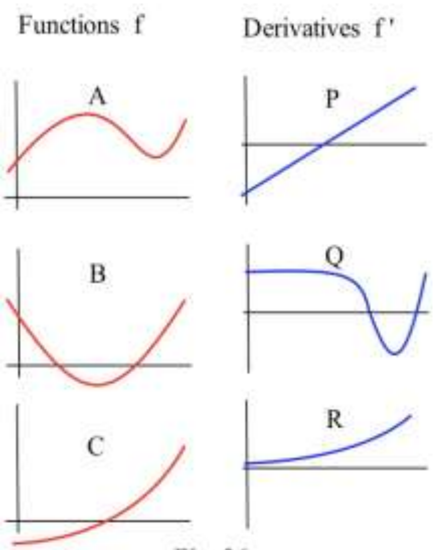

Fig. 26

7. In Fig. 26, match the graphs of the functions with those of their derivatives.

8. In Fig. 27, match the graphs showing the heights of rockets with those showing their velocities.

9. Use the Second Shape Theorem to show that $f(x) = \ln(x)$ is monotonic increasing on $(0, \infty)$.

10. Use the Second Shape Theorem to show that $g(x) = e^x$ is increasing on the entire real number line.

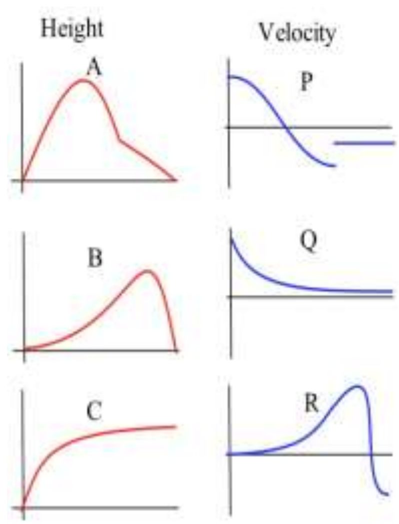

Fig. 27

11. A student is working with a complicated function f and has shown that the derivative of f is always positive. A minute later the student also claims that $f(x) = 2$ when $x = 1$ and when $x = \pi$. Without checking the student's work, how can you be certain that it contains an error?

12. Fig. 28 shows the graph of the **derivative** of a continuous function f.
   (a) List the critical numbers of f.
   (b) For what values of x does f have a local maximum?
   (c) For what values of x does f have a local minimum?

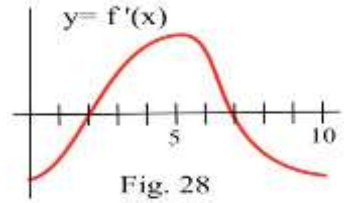

Fig. 28

13. Fig. 29 shows the graph of the **derivative** of a continuous function g.
   (a) List the critical numbers of g.
   (b) For what values of x does g have a local maximum?
   (c) For what values of x does g have a local minimum?

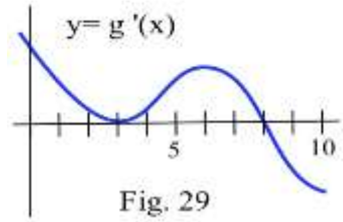

Fig. 29

In problems 14–16, the graphs of the upward velocities of several helicopters are shown. Use each graph to determine when each helicopter was at a relatively maximum and minimum height.

14. Use Fig. 30.    15. Use Fig. 31.    16. Use Fig. 32.

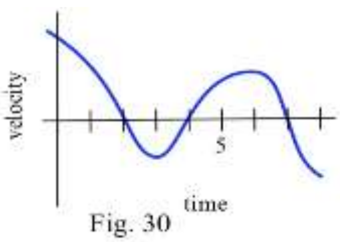

Fig. 30

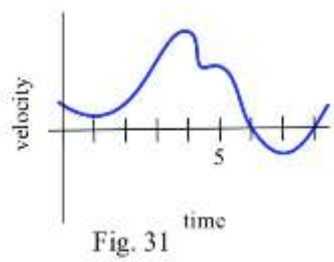

Fig. 31

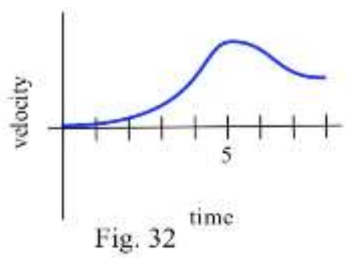

Fig. 32

In problems 17 – 22 , use information from the derivative of each function to help you graph the function. Find all local maximums and minimums of each function.

17. $f(x) = x^3 - 3x^2 - 9x - 5$

18. $g(x) = 2x^3 - 15x^2 + 6$

19. $h(x) = x^4 - 8x^2 + 3$

20. $s(t) = t + \sin(t)$

21. $r(t) = \dfrac{2}{t^2 + 1}$

22. $f(x) = \dfrac{x^2 + 3}{x}$

23. $f(x) = 2x + \cos(x)$ so $f(0) = 1$. Without graphing the function, you can be certain that f has how many **positive** roots? (zero, one, two, more than two)

24. $g(x) = 2x - \cos(x)$ so $g(0) = -1$. Without graphing the function, you can be certain that g has how many **positive** roots? (zero, one, two, more than two)

25. $h(x) = x^3 + 9x - 10$ has a root at $x = 1$. Without graphing h, show that h has no other roots.

26. Sketch the graphs of monotonic decreasing functions which have exactly (a) zero roots, (b) one root, and (c) two roots.

27. Each of the following statements is false. Give (or sketch) a counterexample for each statement.
   (a) If f is increasing on an interval I, then f '(x) > 0 for all x in I.
   (b) If f is increasing and differentiable on I, then f '(x) > 0 for all x in I.
   (c) If cars A and B always have the same speed, then they will always be the same distance apart.

28. (a) Give the equations of several different functions f which all have the same derivative f '(x) = 2.
   (b) Give the equation of the function f with derivative f '(x) = 2 which also satisfies f(1) = 5.
   (c) Give the equation of the function g with g '(x) = 2 , and the graph of g goes through ( 2, 1).

29. (a) Give the equations of several different functions h which all have the same derivative h '(x) = 2x.
   (b) Give the equation of the function f with derivative f '(x) = 2x which also satisfies f(3) = 20.
   (c) Give the equation of the function g with g '(x) = 2x , and the graph of g goes through ( 2, 7).

30. Sketch functions with the given properties to help you determine whether each statement is True or False.
   (a) If f '(7) > 0 and f '(x) > 0 for all x near 7, then f(7) is a local maximum of f on [1,7].
   (b) If g '(7) < 0 and g '(x) < 0 for all x near 7, then g(7) is a local minimum of g on [1,7].
   (c) If h '(1) > 0 and h '(x) > 0 for all x near 1, then h(1) is a local minimum of h on [1,7].
   (d) If r '(1) < 0 and r '(x) < 0 for all x near 1, then r(1) is a local maximum of r on [1,7].
   (e) If s '(7) = 0, then s(7) is a local maximum of s on [1,7].

**Section 4.3** **PRACTICE Answers**

**Practice 1**: g is increasing on [2, 4] and [6, 8].
g is decreasing on [0, 2] and [4, 5].
g is constant on [5, 6].

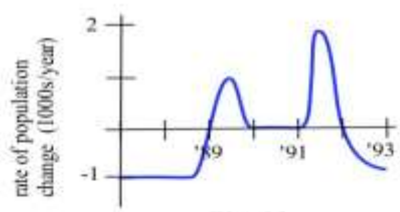


Fig. 33

**Practice 2**: The graph in Fig. 33 shows the rate of population change, dR/dt.

**Practice 3**: The graph of f ' is shown in Fig. 34. Notice how the graph of f ' is 0 where f has a maximum and minimum.

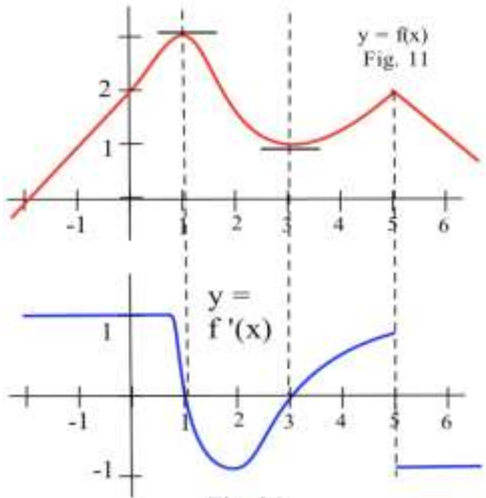

Fig. 34

**Practice 4**: The Second Shape Theorem for helicopters:

(i) If the upward velocity h ' is positive during time interval I
then the height h is increasing during time interval I.

(ii) If the upward velocity h ' is negative during time interval I
then the height h is decreasing during time interval I.

(iii) If the upward velocity h ' is zero during time interval I
then the height h is constant during time interval I.

**Practice 5**: A graph satisfying the conditions in the table is shown in Fig. 35.

| x | –2 | –1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| f(x) | 1 | –1 | –2 | –1 | 0 | 2 |
| f '(x) | –1 | 0 | 1 | 2 | –1 | 1 |

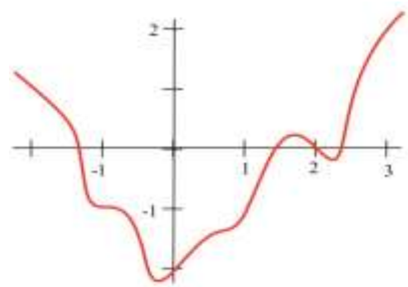
Fig. 35

**Practice 6**: $f(x) = x^3 - 3x^2 - 24x + 5$. $f'(x) = 3x^2 - 6x - 24 = 3(x - 4)(x + 2)$.

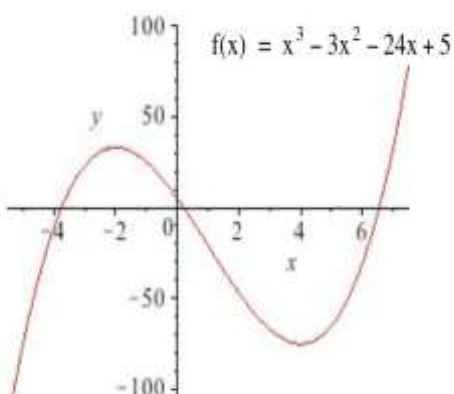

Fig. 36

$f'(x) = 0$ if $x = -2, 4$.

If $x < -2$, then $f'(x) = 3(x - 4)(x + 2) = 3(\text{negative})(\text{negative}) > 0$ so f is increasing.

If $-2 < x < 4$, then $f'(x) = 3(x - 4)(x + 2) = 3(\text{negative})(\text{positive}) < 0$ so f is decreasing.

If $x > 4$, then $f'(x) = 3(x - 4)(x + 2) = 3(\text{positive})(\text{positive}) > 0$ so f is increasing.

f has a relative maximum at x = –2 and a relative minimum at x = 4. The graph of f is shown in Fig. 36.

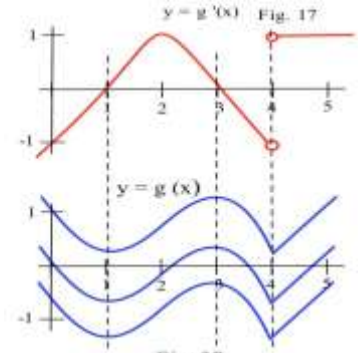


Fig. 37

**Practice 7**: Fig. 37 shows several possible graphs for g. Each of the g graphs has the correct shape to give the graph of g '. Notice that the graphs of g are "parallel," differ by a constant.

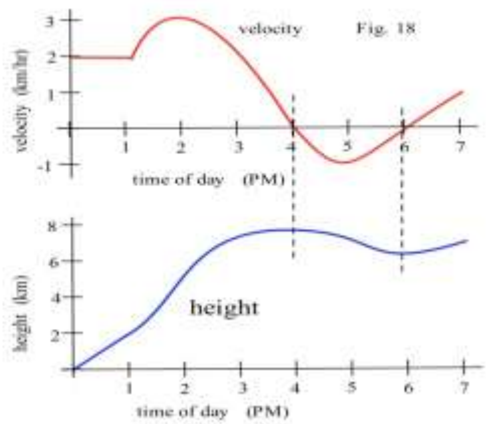


Fig. 38

**Practice 8**: Fig. 38 shows the height graph for the balloon. The balloon was highest at 4 pm and had a local minimum at 6pm.

**Practice 9**:

$f(x) = 3x^2 – 12x + 7$ so $f'(x) = 6x – 12$. $f'(x) = 0$ if $x = 2$.

If $x < 2$, then $f'(x) < 0$ and f is decreasing.

If $x > 2$, then $f'(x) > 0$ and f is increasing.

From this we can conclude that f has a minimum when $x = 2$ and has a shape similar to Fig. 19(b).

We could also notice that the graph of the quadratic $f(x) = 3x^2 – 12x + 7$ is an upward opening parabola. The graph of f is shown in Fig. 39.

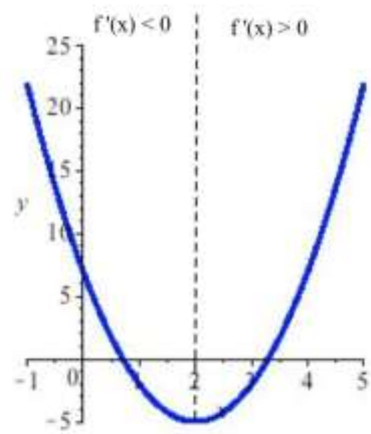


Fig. 39