1 Rolle's Theorem and the Mean Value Theorem

Theorem 1.1 (Rolle's Theorem). *Let f be a function satisfying the following properties:*

1. *f is continuous on the interval $[a, b]$*
2. *f is differentiable on the interval (a, b)*
3. *$f(a) = f(b)$*

Then there is a number c in (a, b) such that $f'(c) = 0$

Theorem 1.2 (The Mean Value Theorem). *Let f be a function satisfying the following properties:*

1. *f is continuous on the interval $[a, b]$*
2. *f is differentiable on the interval (a, b)*

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Equivalently, there is a number c in (a, b) such that

$$(b - a)f'(c) = f(b) - f(a).$$

2 Examples

Example 2.1. *Use the Intermediate Value Theorem to show the equation $1-2x = \sin x$ has at least one real solution. Then use Rolle's Theorem to show it has no more than one solution.*

Proof. Let $f(x) = 1 - 2x - \sin x$. Notice that $f(x)$ is a continuous function and that $f(0) = 1 > 0$ while $f(\pi) = 1-2\pi < 0$. The Intermediate Value Theorem guarantees there is a number, c between 0 and π such that $f(c) = 0$. Since $f(c) = 0$ we have $1 - 2c = \sin c$. Thus c is a real solution for $1 - 2x = \sin x$ showing this equation has at least one real solution.

Now suppose there are two zeros for f. That is, suppose a and b are two different real number with $f(a) = f(b) = 0$. Note that f is both continuous and differentiable for all x so by Rolle's Theorem there must be a real number c between a and b with $f'(c) = 0$. However, $f'(x) = -2 - \cos x$ cannot equal zero since $-1 \leq \cos x \leq 1$ for all x. This creates a contradiction and so the original assumption, that there are two different real zeros for f, must be false. Thus there can only be one real solution for $1 - 2x = \sin x$. □

3 Very important results that use Rolle's Theorem or the Mean Value Theorem in the proof

Theorem 3.1. *Suppose f is a function that is differentiable on the interval (a, b). Then $f'(x) = 0$ for all x in the interval (a, b)* **if and only if** *f is a constant function on (a, b).*

Theorem 3.2. *Suppose f is a function that is differentiable on the interval (a, b). Then $f'(x) > 0$ for all x in the interval (a, b), except possibly a finite number of points,* **if and only if** *f is a strictly increasing function on (a, b).*

Theorem 3.3. *Suppose f is a function that is differentiable on the interval (a, b). Then $f'(x) < 0$ for all x in the interval (a, b), except possibly a finite number of points,* **if and only if** *f is a strictly decreasing function on (a, b).*