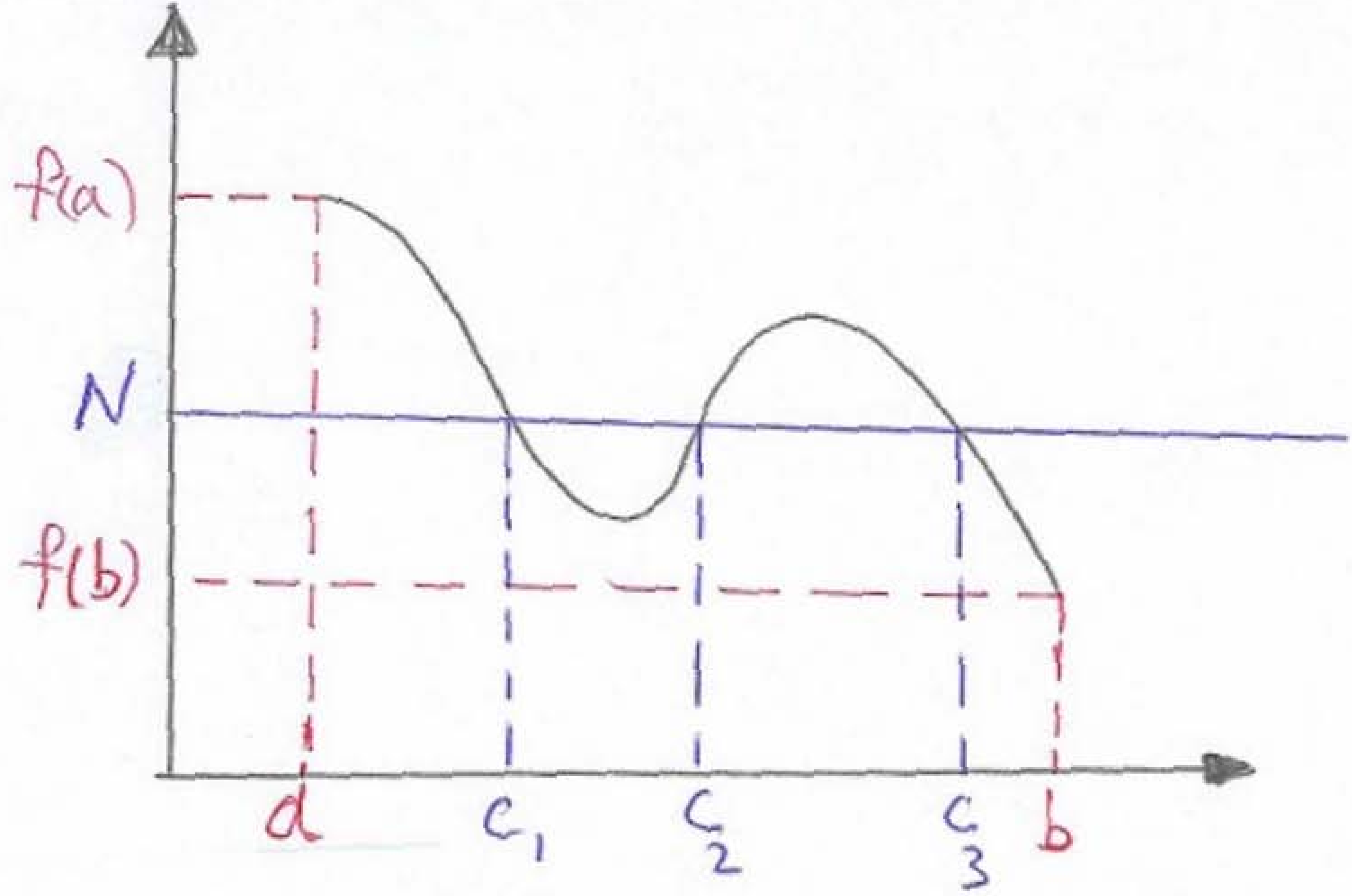
Intermediate Value Theorem

Theorem: Suppose that f is continuous on the interval $[a\,,\,b]$ (it is continuous on the path from a to b). If $f(a) \neq f(b)$ and if N is a number between $f(a)$ and $f(b)$ ($f(a) < N < f(b)$ or $f(b) < N < f(a)$), then there is number c in the open interval $a < c < b$ such that $f(c) = N$.

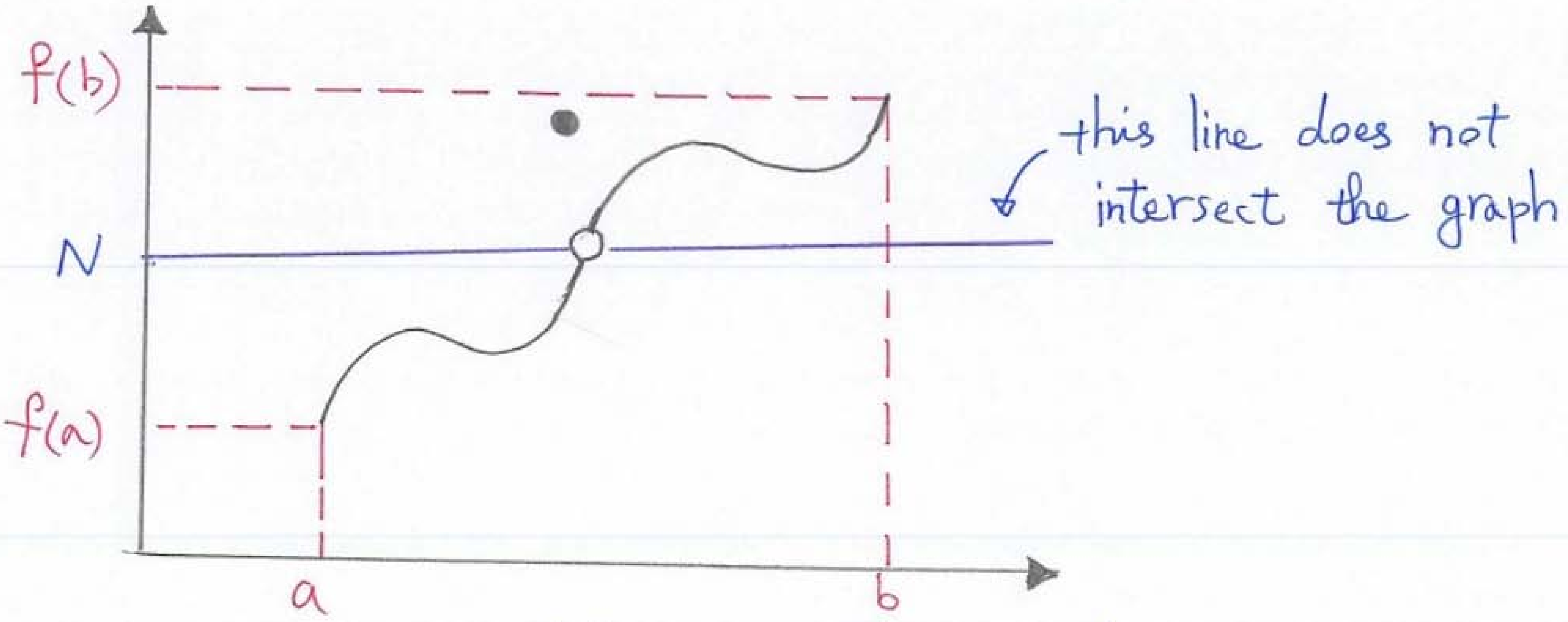
Note. This theorem says that any horizontal line between the two horizontal lines $y = f(a)$ and $y = f(b)$ intersects the graph of f somewhere between a and b.



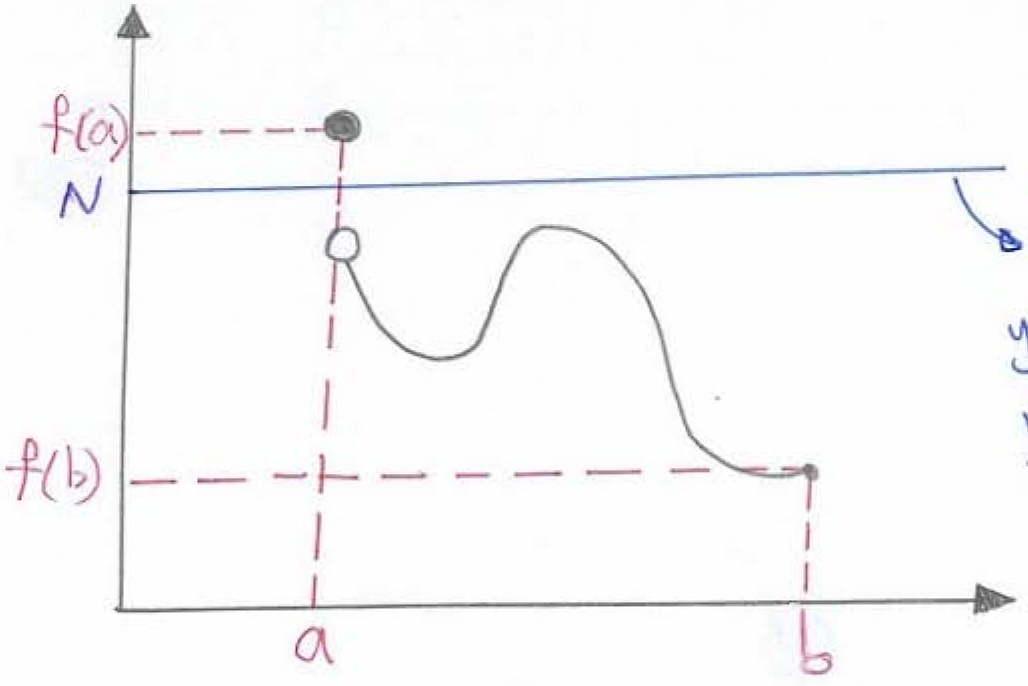
f(a)
N
f(b)
a
c1
c2
c3
b

$$\begin{cases} f(c_1) = N \\ f(c_2) = N \\ f(c_3) = N \end{cases}$$

three candidates



no candidate: discontinuity causes no candidate to exist.



this line is between $y=f(a)$ and $y=f(b)$ but it does not intersect the graph over $[a,b]$

no candidate exists because of discontinuity

Theorem 2—The First Derivative Theorem for Local Extreme Values

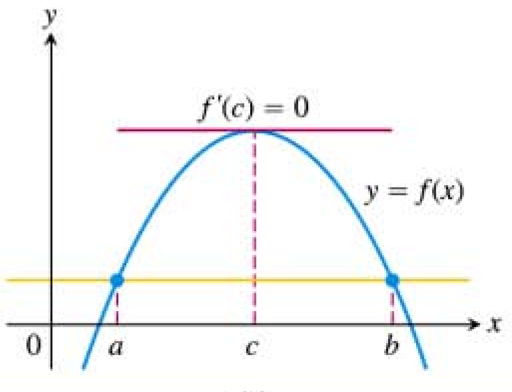
If *f* has a local maximum or minimum value at an interior point *c* of its domain, and if f' is defined at *c*, then

$$f'(c) = 0.$$

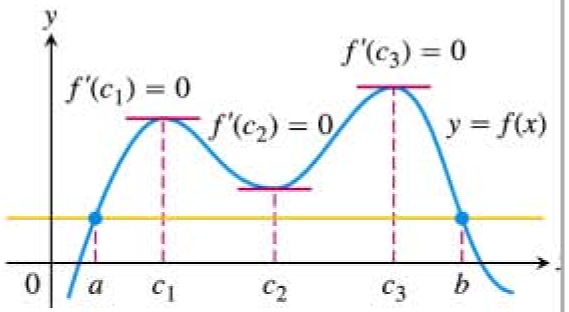
Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

$$f(a) = f(b)$$

$$a \neq b$$



(a)



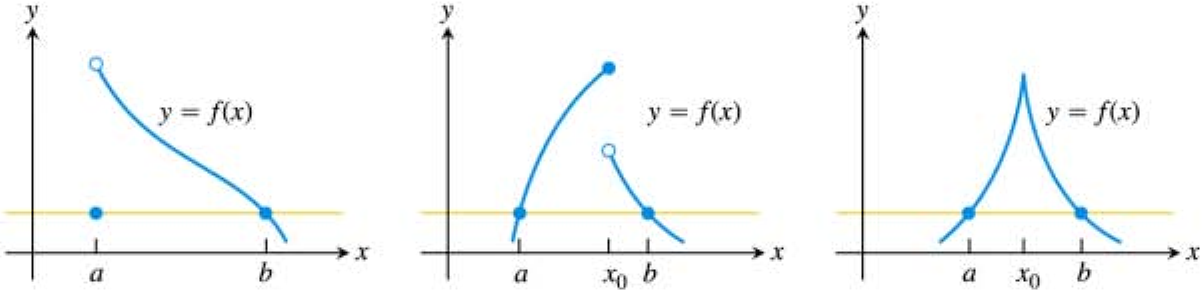
(b)

Theorem 3—Rolle's Theorem

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (*a*, *b*). If $f(a) = f(b)$, then there is at least one number *c* in (*a*, *b*) at which $f'(c) = 0$.

Figure 4.11

There may be no horizontal tangent if the hypotheses of Rolle's Theorem do not hold.



(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

Figure 4.12

The only real zero of the polynomial $y = x^3 + 3x + 1$ is the one shown here where the curve crosses the *x*-axis between −1 and 0 (Example 1).

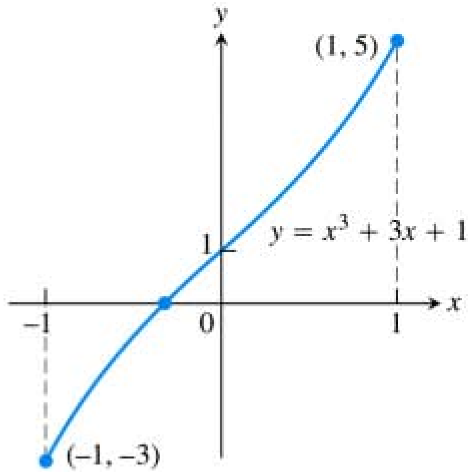
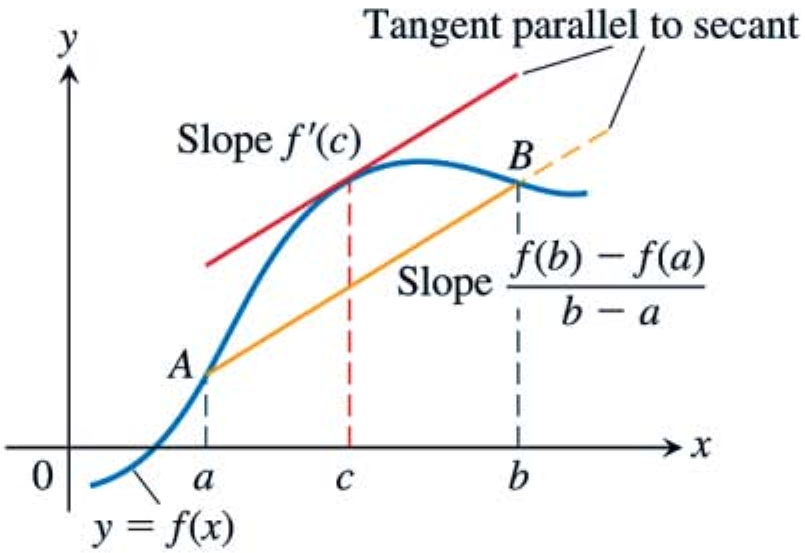


Figure 4.13

Geometrically, the Mean Value Theorem says that somewhere between *a* and *b* the curve has at least one tangent line parallel to the secant line that joins *A* and *B*.



Theorem 4—The Mean Value Theorem

Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (*a*, *b*). Then there is at least one point *c* in (*a*, *b*) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$

Figure 4.14

The graph of *f* and the secant AB over the interval $[a,b]$.

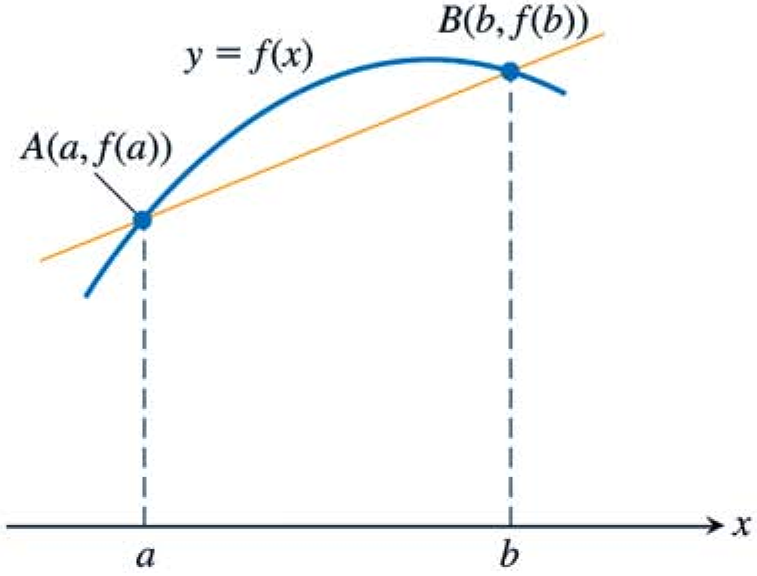


Figure 4.16

The function $f(x)=\sqrt{1-x^2}$ satisfies the hypotheses (and conclusion) of the Mean Value Theorem on $[-1,1]$ even though *f* is not differentiable at −1 and 1.

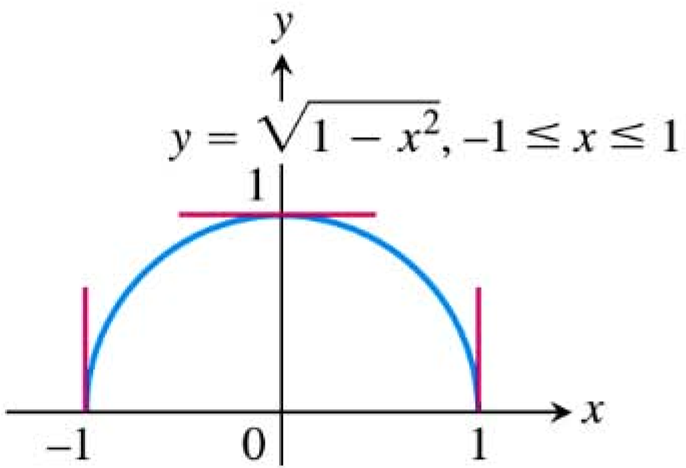


Figure 4.17

As we find in Example 2, $c = 1$ is where the tangent is parallel to the secant line.