Examples 8.3 – Rolle's Theorem and the Mean Value Theorem

1. Show that $f(x) = \frac{1}{2}x - \sqrt{x}$ satisfies the hypothesis of Rolle's Theorem on [0, 4], and find all values of c in (0, 4) that satisfy the conclusion of the theorem.

   Solution: Based on out previous work, f is continuous on its domain, which includes [0, 4], and differentiable on (0, 4). In addition, $f(0) = f(4) = 0$ so the hypothesis is satisfied. Now we want to find all values of c in (0, 4) such that $f'(c) = 0$. Since $f'(x) = \frac{1}{2} - \frac{1}{2\sqrt{x}}$, we have $f'(c) = \frac{1}{2} - \frac{1}{2\sqrt{c}} = 0$. It follows that

$$\begin{aligned} \tfrac{1}{2} &= \tfrac{1}{2\sqrt{c}} \\ \sqrt{c} &= 1 \\ c &= 1 \end{aligned}$$

   Therefore, $c = 1$ is the only value in (0, 4) that satisfies the conclusion of the theorem.

2. Show that $f(x) = \sqrt{25 - x^2}$ satisfies the hypothesis of the Mean Value Theorem on [–5, 3], and find all values of c in (–5, 3) that satisfy the conclusion of the theorem.

   Solution: Note that the domain of $f(x) = \sqrt{25 - x^2}$ is [–5, 5]. Based on our previous work, we already know that f is continuous on [–5, 3] and differentiable on in (–5, 3). Now we must find all values of c in (–5, 3) such that $f'(c) = \frac{f(3) - f(-5)}{3 - (-5)} = \frac{4}{8} = \frac{1}{2}$. By the chain rule, $f'(x) = \frac{-2x}{2\sqrt{25 - x^2}} = \frac{-x}{\sqrt{25 - x^2}}$, hence $f'(c) = \frac{-c}{\sqrt{25 - c^2}} = \frac{1}{2}$. It follows that

$$\begin{aligned} -2c &= \sqrt{25 - c^2} \\ 4c^2 &= 25 - c^2 \\ c^2 &= 5 \\ c &= \pm\sqrt{5} \end{aligned}$$

   Squaring an equation and then applying a square root may introduce extraneous solutions. Note that c must be negative if it is to satisfy the equation $\frac{-c}{\sqrt{25 - c^2}} = \frac{1}{2}$. Therefore, $c = -\sqrt{5}$ is the only value that satisfies the conclusion of the Mean Value Theorem in (–5, 3).