Theorem 2—The First Derivative Theorem for Local Extreme Values

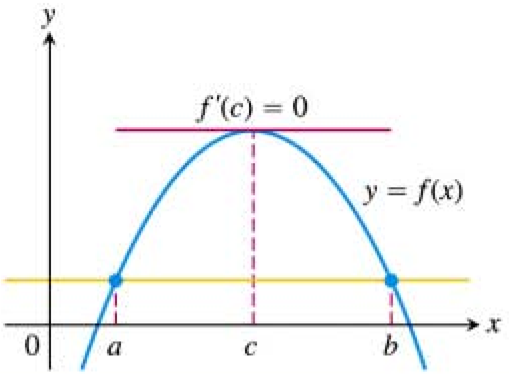
If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

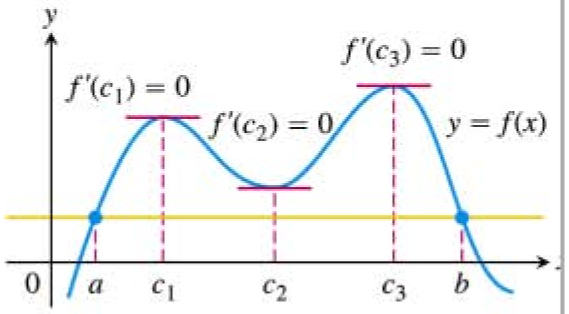
Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

$f(a) = f(b)$

$a \neq b$



(a)



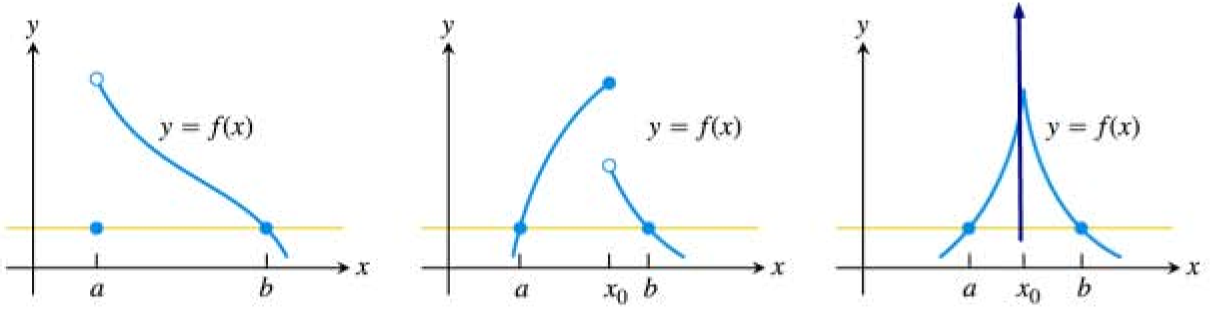
(b)

Theorem 3—Rolle's Theorem

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a,b]$ and differentiable at every point of its interior (*a*, *b*). If $f(a) = f(b)$, then there is at least one number *c* in (*a*, *b*) at which $f'(c) = 0$.

Figure 4.11

There may be no horizontal tangent if the hypotheses of Rolle's Theorem do not hold.



(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

Figure 4.12

The only real zero of the polynomial $y = x^3 + 3x + 1$ is the one shown here where the curve crosses the *x*-axis between −1 and 0 (Example 1).

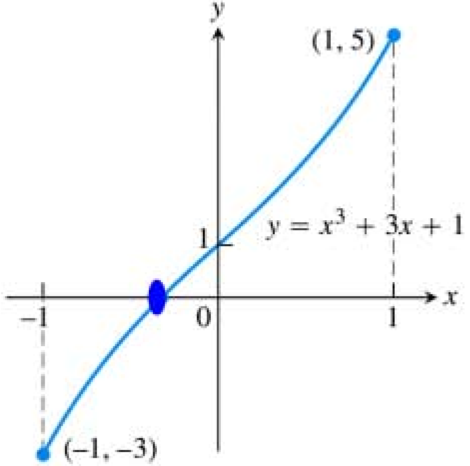
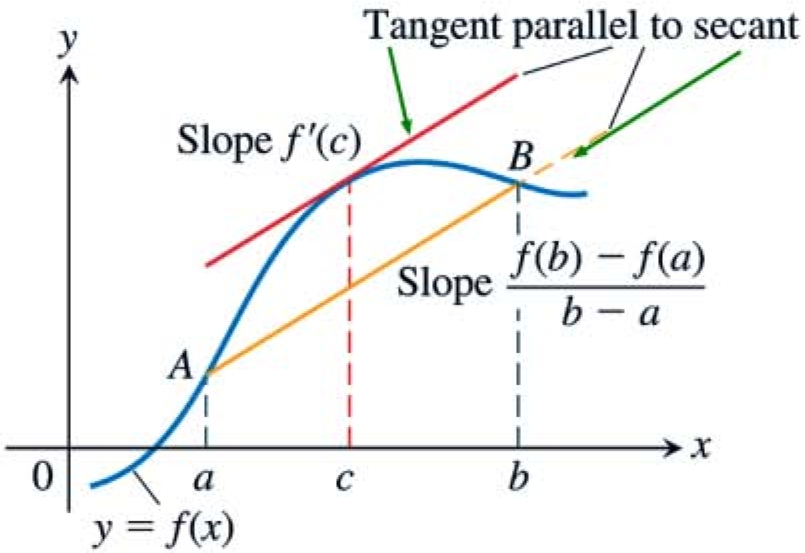


Figure 4.13

Geometrically, the Mean Value Theorem says that somewhere between *a* and *b* the curve has at least one tangent line parallel to the secant line that joins *A* and *B*.



Extreme Theory

Theorem 4—The Mean Value Theorem

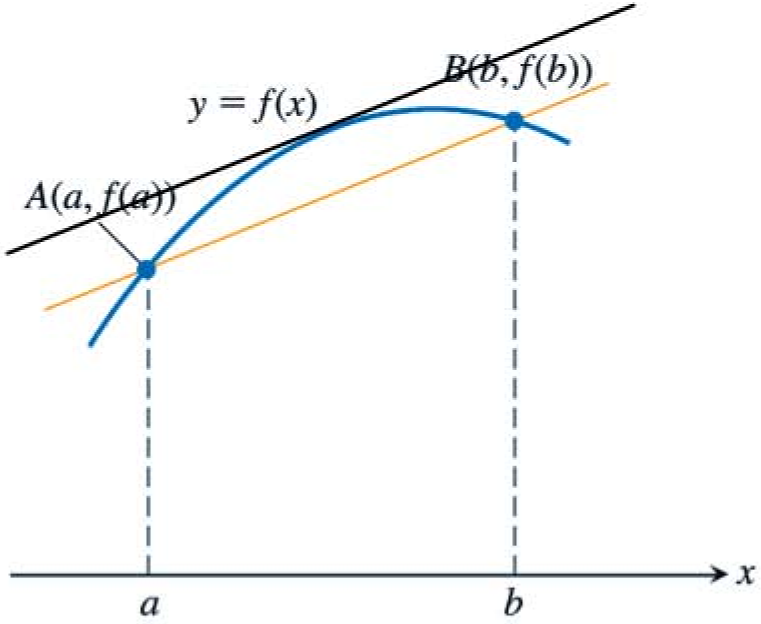
Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (*a*, *b*). Then there is at least one point *c* in (*a*, *b*) at which

Fig

$$\frac{f(b) - f(a)}{b - a} = f'(c). \qquad (1)$$

The graph of *f* and the secant AB over the interval

$[a, b]$.



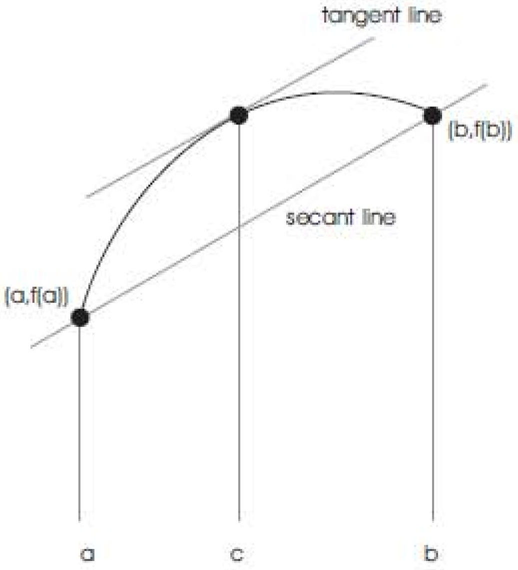
The Mean Value Theorem

A **secant line** is a line drawn through two points on a curve.
The Mean Value Theorem relates the slope of a secant line to the slope of a tangent line.

Theorem. **(The Mean Value Theorem)** If f is continuous on $a \le x \le b$ and differentiable on $a < x < b$, there is a number c in $a < x < b$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

I won't give a proof here, but the picture below shows why this makes sense. I've drawn a secant line through the points $(a, f(a))$ and $(b, f(b))$. The Mean Value Theorem says that somewhere in between a and b, there is a point c on the curve where the tangent line has the same slope as the secant line.



Lines with the same slope are parallel. To find a point where the tangent line is parallel to the secant line, take the secant line and "slide" it (without changing its slope) until it's tangent to the curve.

If you experiment with some curves, you'll find that it's always possible to do this (provided that the curve is continuous and differentiable as stipulated in the theorem).

Example. For the function $f(x) = x^3 + 3x^2$ on the interval $-5 \le x \le 1$, find a number (or numbers) satisfying the conclusion of the Mean Value Theorem.

Since f is a polynomial, f is continuous on $-5 \le x \le 1$ and differentiable on $-5 < x < 1$. Moreover,

$$\frac{f(1) - f(-5)}{1 - (-5)} = \frac{4 - (-50)}{1 - (-5)} = 9.$$

Hence, there is a number c — maybe more than one — between -5 and 1 such that $f'(c) = 9$. I'll try to find one.

$f'(x) = 3x^2 + 6x$, so $f'(c) = 3c^2 + 6c$. Set $f'(c)$ equal to 9 and solve for c:

$$\begin{aligned} 3c^2 + 6c &= 9 \\ c^2 + 2c &= 3 \\ c^2 + 2c - 3 &= 0 \\ (c+3)(c-1) &= 0 \\ c = -3 \quad \text{or} &\quad c = 1 \end{aligned}$$