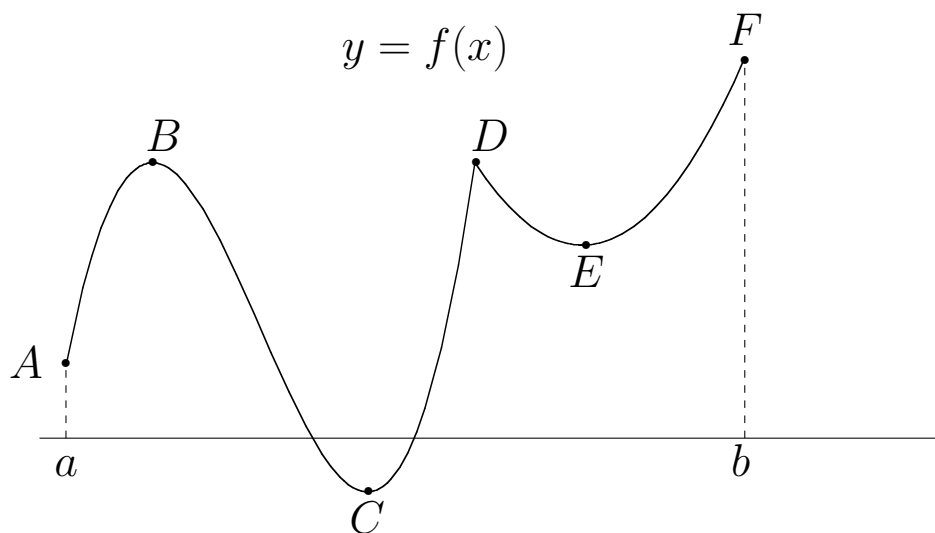# Finding Extreme Values.   Lecture 17

An important practical problem for which differentiation can often provide quick and easy answers is that of finding the **extreme values**, that is **maximum** and **minimum** values of a function. To set the stage consider the following graph of a function $y = f(x)$ defined on the closed interval $a \leq x \leq b$

The points A, B, C, D, E, and F are the **extrema** (singular **extreme points**) of the function. In particular,

- f has a **relative minimum** at A, C and E (i.e., f at that point is less than or equal to all values of $f(x)$ in some interval about that point);

- f has a **relative maximum** at B, D, and F (i.e., f at that point is greater than or equal to all values of $f(x)$ in some interval about that point);

- f has an **absolute minimum** at C (i.e., at that point f is less than or equal to all values of f on the interval);

- f has an **absolute maximum** at F (i.e., at that point f is greater than or equal to all values of f on the interval).

The extrema in this example typify virtually all of the extrema that we shall encounter in this course. For continuous functions extrema occur at only a limited class of points and the five extrema above illustrate each class. For a function $y = f(x)$ a point in its graph is

- A **critical point** if either

  It is a **stationary point**, that is, its derivative $f'(x)$ is zero there;

  It is a **singular point**, that is, its derivative does not exist there;

- It is an **end point**, that is, some interval on one side of the point is not in the domain of f.

For example, for the above function, the points B, C and E are stationary and D is singular, so these are the critical points of the function. The points A and F are the end-points.

And here is the key fact about extreme points:

> **The extreme points of a continuous function occur only at critical points and end-points.**

This pretty clearly makes the task of finding all extreme points a much easier task.

Example 1. Let's find all extreme points of $f(x) = 12x - x^3$ on the interval $-3 \le x \le 5$. We begin our search by finding all critical points and that begins with the derivative:
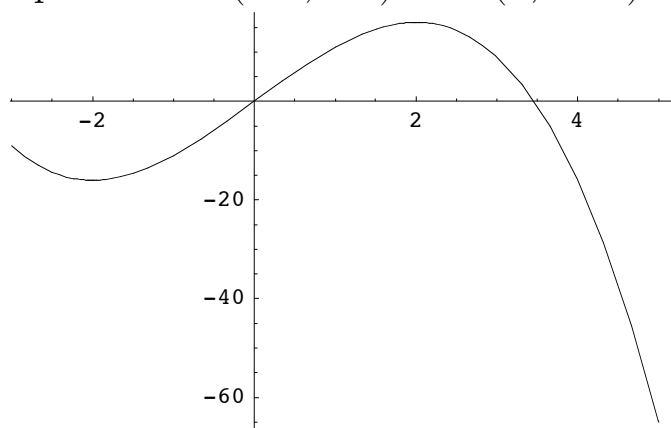
$$f'(x) = \frac{d}{dx}(12x - x^3) = 12 - 3x^2.$$

Since this is defined for all values of x on the interval, there are no singular points. But

$$f'(x) = 12 - 3x^2 = 0 \iff x^2 = 4 \iff x = \pm 2.$$

So there are only two critical points: $(-2, -16)$ and $(2, 16)$.

Next, there are only two end-points at $x = -3$ and $x = 5$. That is the end-points are $(-3, -9)$ and $(5, -65)$.

Since these four points are the only possible extreme points, we need only compare them to see that

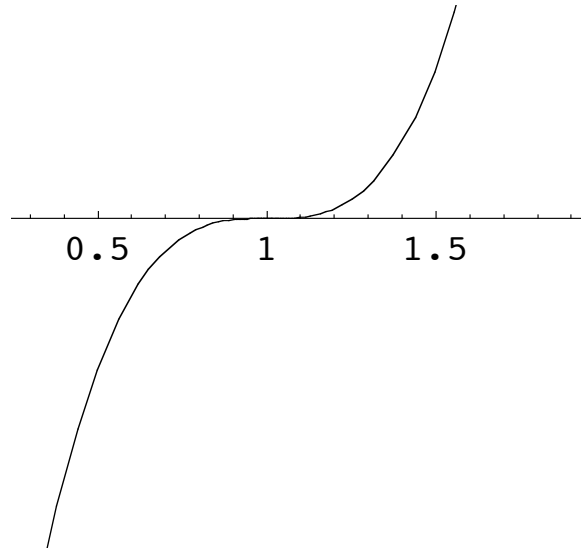$$f(-3) \ge f(-2), \quad f(-2) \le f(2), \quad \text{and} \quad f(2) \ge f(5),$$

so

- $(-2, -16)$ and $(5, -65)$ are relative minima;

- $(-3, -9)$ and $(2, 16)$ are relative maxima.

Finally just comparing the relative extrema, we see that $(5, -65)$ is the absolute minimum and $(2, 16)$ is the absolute maximum.

Example 2. Here is a curious one. Let's find the extrema of

$$f(x) = (x-1)^3.$$

This has no end-points so we need worry only about critical points. But the derivative is

$$f'(x) = 3(x-1)^2.$$

So there are no singular points and the only stationary point occurs when

$$f'(x) = 0 \iff x - 1 = 0 \iff x = 1.$$

But notice that $f(x) < f(1)$ for $x < 1$ and $f(x) > 1$ for $x > 1$. So $(1, 0)$ **is neither a relative maximum nor a relative minimum!!** The lesson to be learned here is that

> **Critical points need not be relative minima or relative maxima.**

Example 3. Let's try this one together: Find the extreme values of the function

$$f(x) = x^2 e^{-x}.$$
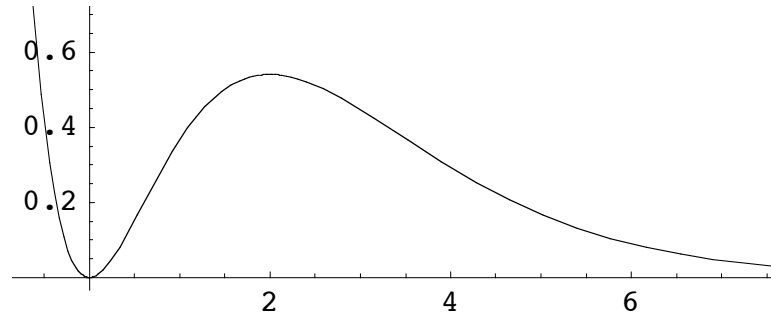
Again we begin with the derivative. We could use logarithmic differentiation. But let's just do it directly with the product rule:

So this function has an absolute minimum but no absolute maximum. It's easy to find one with neither absolute extrema. But there is one very important condition that guarantees both an absolute minimum and an absolute maximum.

> **A continuous function $f(x)$ on a closed and bounded interval $[a, b]$ has both an absolute minimum and an absolute maximum on the interval.**

Let's see some of this optimization in action:

Example 4. Find the maximum and minimum values of $F = 2xy$ given that $x^2 + 2y = 12$ and $-3 \le x \le 3$. So here

$F = 2xy$ is the **objective function**;

$x^2 + 2y = 12$ and $-3 \le x \le 3$ are the **constraints**.

We can reduce this to a straightforward optimization problem simply by solving the constraints and substituting in the objective function to express F in x alone. That is,

$$F = x(2y) = x(12 - x^2) = 12x - x^3 \quad \text{on} \quad -3 \le x \le 3.$$

$$[(2, 16),(-2, -16).]$$

Example 5. A hot-dog vendor wants to minimize the *average* cost of selling his dogs. After a careful analysis he determines that his actual daily cost is

$$C(x) = 16 + 0.5x + 0.01x^2$$

dollars when selling x dogs. What level of sales will minimize his *average* cost? And what will that minimum cost be?

$$[\overline{C} = \$1.15 \text{ when } x = 40.]$$

Example 6. You want to fence a rectangular piece of land and can afford to pay at most $1200 for the fencing. Along an adjoining road it will cost $12 per foot while the other three sides will cost only $8 per foot. What are the dimensions of the largest piece of land you can fence this way? $[30' \times 37.5']$