Lecture 13: Examples of Extreme Value Problems

13.1 Example: Continuous function on a closed interval

Example Suppose a farmer wishes to enclose a rectangular field using 1000 yards of fencing in such a way that the area of the field is maximized. Let x and y be the dimensions of the field and let A be the area of the field. Then

$$A = xy.$$

Moreover,

$$1000 = 2x + 2y,$$

so

$$y = 500 - x.$$

Hence

$$A = x(500 - x) = 500x - x^2.$$

We want to find the maximum value of A on the interval $[0, 500]$. Now

$$\frac{dA}{dx} = 500 - 2x,$$

so

$$\frac{dA}{dx} = 0$$

when

$$x = 250.$$

Evaluating, we have

$$\begin{aligned} A\Big|_{x=0} &= 0, \\ A\Big|_{x=250} &= (250)(250) = 62,500, \\ A\Big|_{x=500} &= 0. \end{aligned}$$

So A has a maximum value of 62,500 square yards when $x = 250$ yards and $y = 500 - 250 = 250$ yards.

Example We will find the area of the largest rectangle that can be inscribed in a semi-circle of radius r. That is, consider rectangles inscribed in the region bounded by the x-axis and the graph of $y = \sqrt{r^2 - x^2}$. Let A be the area of a rectangle inscribed in this region with its lower left-hand corner at the point $(x, 0)$ and its upper left-hand corner at (x, y) on the graph of $y = \sqrt{r^2 - x^2}$. Then

$$A = 2xy.$$

Now $y = \sqrt{r^2 - x^2}$, so we have

$$A = 2x\sqrt{r^2 - x^2}.$$

We want to find the maximum value of A on the interval $[0, r]$. Now

$$\frac{dA}{dx} = -\frac{2x^2}{\sqrt{r^2 - x^2}} + 2\sqrt{r^2 - x^2},$$

so

$$\frac{dA}{dx} = 0$$

when

$$2\sqrt{r^2 - x^2} = \frac{2x^2}{\sqrt{r^2 - x^2}},$$

that is, when

$$r^2 - x^2 = x^2.$$

Hence

$$x^2 = \frac{r^2}{2},$$

so

$$x = \frac{r}{\sqrt{2}}.$$

Now

$$A\Big|_{x=0} = 0,$$

$$A\Big|_{x=\frac{r}{\sqrt{2}}} = \frac{2r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}} = r^2,$$

and

$$A\Big|_{x=r} = 0.$$

Hence A has a maximum value of r^2 when $x = \frac{r}{\sqrt{2}}$. Note that for this value of x, we have

$$y = \sqrt{r^2 - \frac{r^2}{2}} = \frac{r}{\sqrt{2}}.$$

13.2 Open intervals

The following case arises frequently when considering the extreme values of a continuous function on an open interval. Suppose f and f' are continuous on an open interval (a, b) and c is the only critical number of f in (a, b). Then (1) if f has a local minimum at c, then the absolute minimum of f occurs at c, and (2) if f has a local maximum at c, then the absolute maximum of f occurs at c.

Example Suppose a company wishes to manufacture a can in the shape of a right circular cylinder which will hold a volume V but have minimal surface area. Let h be the height of the can, r the radius of the base of the can, and S the surface area of the can. Then

$$S = 2\pi rh + 2\pi r^2.$$

Moreover,

$$V = \pi r^2 h,$$

so

$$h = \frac{V}{\pi r^2}.$$

Hence

$$S = 2\pi r\left(\frac{V}{\pi r^2}\right) + 2\pi r^2 = \frac{2V}{r} + 2\pi r^2.$$

We want to find the minimum value of S on the interval $(0, \infty)$. Now

$$\frac{dS}{dr} = 4\pi r - \frac{2V}{r^2},$$

so

$$\frac{dS}{dr} = 0$$

when

$$r = \sqrt[3]{\frac{V}{2\pi}}.$$

Now

$$\frac{d^2S}{dr^2} = 4\pi + \frac{4V}{r^3},$$

so

$$\left.\frac{d^2S}{dr^2}\right|_{r=\sqrt[3]{\frac{V}{2\pi}}} = 4\pi + 8\pi = 12\pi > 0.$$

Thus S has a local minimum at $r = \sqrt[3]{\frac{V}{2\pi}}$ and so, since this is the only critical number in $(0, \infty)$, S has an absolute minimum at $r = \sqrt[3]{\frac{V}{2\pi}}$. Finally, when $r = \sqrt[3]{\frac{V}{2\pi}}$, we have

$$h = \frac{V}{\pi\left(\frac{V}{2\pi}\right)^{\frac{2}{3}}} = \frac{V^{\frac{1}{3}}2^{\frac{2}{3}}}{\pi^{\frac{1}{3}}} = 2\sqrt[3]{\frac{V}{2\pi}} = 2r.$$

Example Suppose we wish to find the point on the line $y = 2x+1$ which is closest to the point $(2, 1)$. That is, we wish to minimize the distance between $(2, 1)$ and the points on the line, or, what is equivalent, minimize the square of the distance between $(2, 1)$ and points on the line. Now the square of the distance between $(2, 1)$ and a point $(x, y) = (x, 2x + 1)$ which lies on $y = 2x + 1$ is given by

$$f(x) = (x - 2)^2 + ((2x + 1) - 1)^2 = x^2 - 4x + 4 + 4x^2 = 5x^2 - 4x + 4.$$

Hence we want to find the minimum value of f on the interval $(-\infty, \infty)$. Now

$$f'(x) = 10x - 4,$$

so $f'(x) = 0$ when $x = \frac{2}{5}$. Since $f''(x) = 10$, $f''(\frac{2}{5}) = 10 > 0$, and f has a local minimum at $x = \frac{2}{5}$. Since f has only one critical number in $(-\infty, \infty)$, f has, in fact, an absolute minimum at $x = \frac{2}{5}$. Now when $x = \frac{2}{5}$,

$$y = 2(\frac{2}{5}) + 1 = \frac{9}{5},$$

so the point $(\frac{2}{5}, \frac{9}{5})$ is the point on the line $y = 2x + 1$ which is closest to $(2, 1)$.