3.6 Inverse Trigonometric Functions and Their Derivatives

Inverse Trigonometric Functions and Their Derivatives

You can see from Figure 1 that the sine function $y = \sin x$ is not one-to-one (use the Horizontal Line Test).

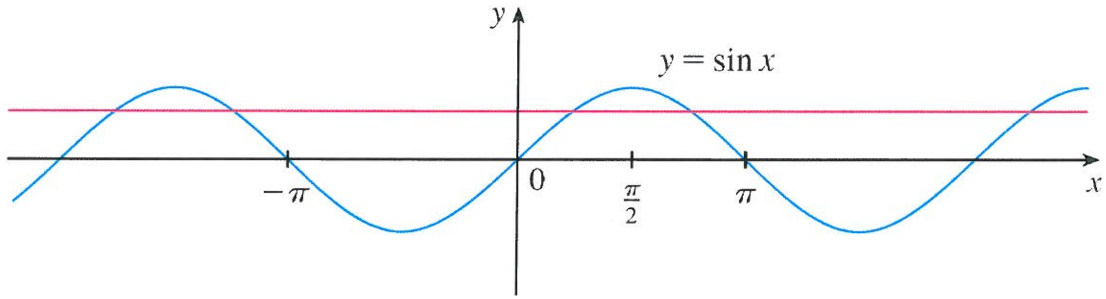

Figure 1

But the function $f(x) = \sin x$, $-\pi/2 \leq x \leq \pi/2$, *is* one-to-one (see Figure 2).

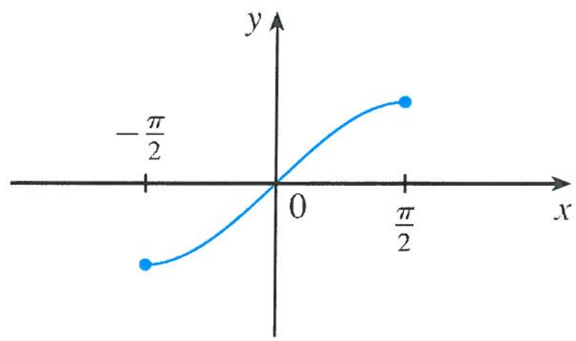

Figure 2 $y = \sin x, -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

Inverse Trigonometric Functions and Their Derivatives

The inverse function of this restricted sine function f exists and is denoted by $\sin^{-1}$ or arcsin. It is called the **inverse sine function** or the **arcsine function**.

Since the definition of an inverse function says that

$$f^{-1}(x) = y \iff f(y) = x$$

we have

$$\sin^{-1}x = y \iff \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

Thus, if $-1 \le x \le 1$, $\sin^{-1}x$ is the number between $-\pi/2$ and $\pi/2$ whose sine is x.

Example 1

Evaluate (a) $\sin^{-1}(\frac{1}{2})$ and (b) $\tan(\arcsin \frac{1}{3})$.

Solution:

(a) We have

$$\sin^{-1}\left(\tfrac{1}{2}\right) = \frac{\pi}{6}$$

because $\sin(\pi/6) = \frac{1}{2}$ and $\pi/6$ lies between $-\pi/2$ and $\pi/2$.

Example 1 – *Solution*

cont'd

(b) Let $\theta = \arcsin \frac{1}{3}$, so $\sin \theta = \frac{1}{3}$.

Then we can draw a right triangle with angle θ as in Figure 3 and deduce from the Pythagorean Theorem that the third side has length $\sqrt{9-1} = 2\sqrt{2}$.

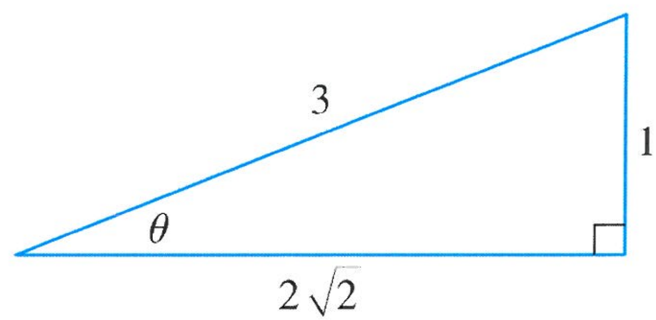

Figure 3

This enables us to read from the triangle that

$$\tan\left(\arcsin \tfrac{1}{3}\right) = \tan \theta = \frac{1}{2\sqrt{2}}$$

Inverse Trigonometric Functions and Their Derivatives

The cancellation equations for inverse functions become, in this case,

$$\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$$

$$\sin(\sin^{-1}x) = x \quad \text{for } -1 \leqslant x \leqslant 1$$

Inverse Trigonometric Functions and Their Derivatives

The inverse sine function, $\sin^{-1}$, has domain [–1, 1] and range [$-\pi/2$, $\pi/2$], and its graph, shown in Figure 4, is obtained from that of the restricted sine function (Figure 2) by reflection about the line $y = x$.

We know that the sine function f is continuous, so the inverse sine function is also continuous.

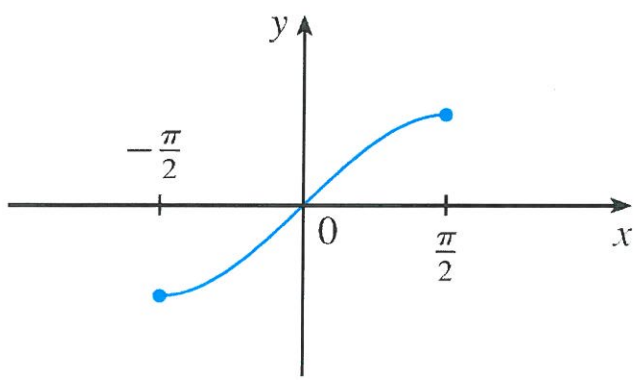

Figure 2 $y = \sin x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

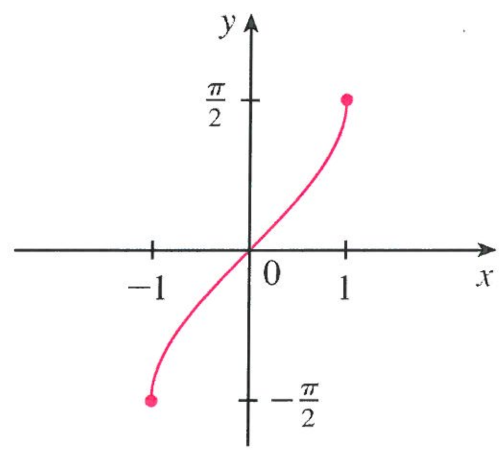

Figure 4 $y = \sin^{-1}x = \arcsin x$

Inverse Trigonometric Functions and Their Derivatives

We can use implicit differentiation to find the derivative of the inverse sine function, assuming that it is differentiable. (The differentiability is certainly plausible from its graph in Figure 4.)

Let $y = \sin^{-1}x$. Then $\sin y = x$ and $-\pi/2 \le y \le \pi/2$.

Differentiating $\sin y = x$ implicitly with respect to x, we obtain

$$\cos y \frac{dy}{dx} = 1 \quad \text{and} \quad \frac{dy}{dx} = \frac{1}{\cos y}$$

Inverse Trigonometric Functions and Their Derivatives

Now cos $y \geq 0$ since $-\pi/2 \leq y \leq \pi/2$, so

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

Therefore $$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

1 $$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1$$

Inverse Trigonometric Functions and Their Derivatives

The **inverse cosine function** is handled similarly. The restricted cosine function $f(x) = \cos x$, $0 \leq x \leq \pi$, is one-to-one (see Figure 6) and so it has an inverse function denoted by $\cos^{-1}$ or arccos.

$$\cos^{-1}x = y \iff \cos y = x \quad \text{and} \quad 0 \leqslant y \leqslant \pi$$

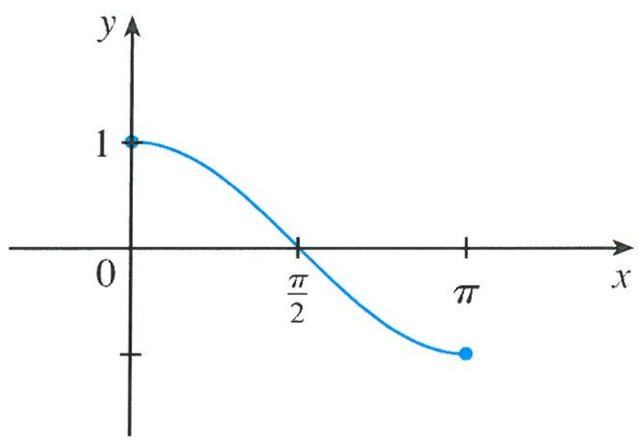

Figure 6 $y = \cos x$, $0 \leq x \leq \pi$

Inverse Trigonometric Functions and Their Derivatives

The inverse cosine function, $\cos^{-1}$, has domain [–1, 1] and range [0, π] and is a continuous function whose graph is shown in Figure 7.

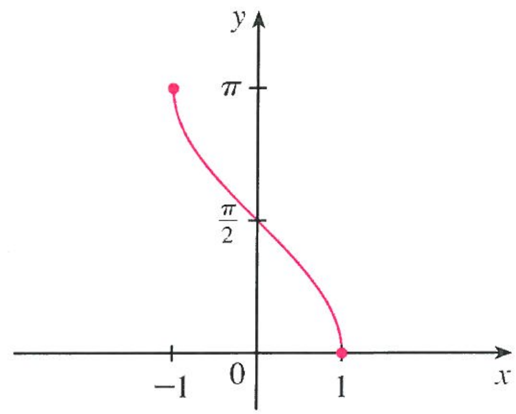

Figure 7 $y = \cos^{-1}x = \arccos x$

Its derivative is given by

2

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1$$

Inverse Trigonometric Functions and Their Derivatives

The tangent function can be made one-to-one by restricting it to the interval $(-\pi/2, \pi/2)$.

Thus the **inverse tangent function** is defined as the inverse of the function $f(x) = \tan x$, $-\pi/2 < x < \pi/2$, as shown in Figure 8.

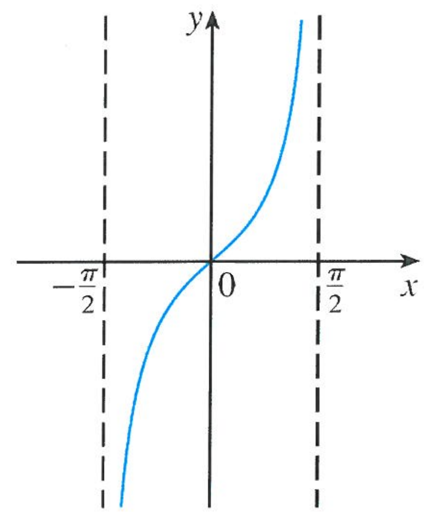

Figure 8

$y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

It is denoted by $\tan^{-1}$ or arctan.

$$\tan^{-1}x = y \iff \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Inverse Trigonometric Functions and Their Derivatives

The inverse tangent function, $\tan^{-1} = \arctan$, has domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$.

Its graph is shown in Figure 10.

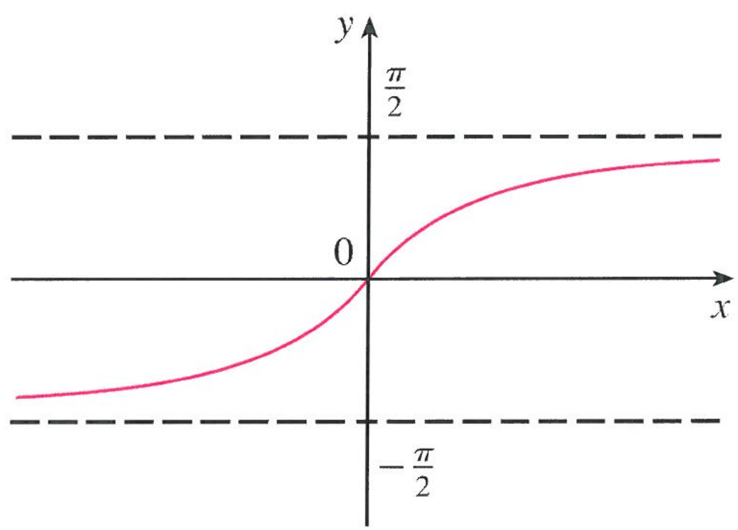

Figure 10 $y = \tan^{-1}x = \arctan x$

Inverse Trigonometric Functions and Their Derivatives

We know that

$$\lim_{x \to (\pi/2)^-} \tan x = \infty \quad \text{and} \quad \lim_{x \to -(\pi/2)^+} \tan x = -\infty$$

and so the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of tan.

Since the graph of $\tan^{-1}$ is obtained by reflecting the graph of the restricted tangent function about the line $y = x$, it follows that the lines $y = \pi/2$ and $y = -\pi/2$ are horizontal asymptotes of the graph of $\tan^{-1}$.

This fact is expressed by the following limits:

3 $$\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2} \qquad \lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$$

The formula for the derivative of the arctangent function is derived in a way that is similar to the method we used for arcsine.

Inverse Trigonometric Functions and Their Derivatives

If $y = \tan^{-1}x$, then $\tan y = x$. Differentiating this latter equation implicitly with respect to x, we have

$$\sec^2 y \frac{dy}{dx} = 1$$

and so

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

4

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2}$$

3.7 Derivatives of Logarithmic Functions

Derivatives of Logarithmic Functions

In this section we use implicit differentiation to find the derivatives of the logarithmic functions $y = \log_a x$ and, in particular, the natural logarithmic function $y = \ln x$.

1

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

Derivatives of Logarithmic Functions

If we put $a = e$ in Formula 1, then the factor $\ln a$ on the right side becomes $\ln e = 1$ and we get the formula for the derivative of the natural logarithmic function $\log_e x = \ln x$:

2

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

By comparing Formulas 1 and 2, we see one of the main reasons that natural logarithms (logarithms with base e) are used in calculus: The differentiation formula is simplest when $a = e$ because $\ln e = 1$.

Example 1

Differentiate $y = \ln(x^3 + 1)$.

Solution:

To use the Chain Rule, we let $u = x^3 + 1$.

Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= \frac{1}{u}\frac{du}{dx}$$

$$= \frac{1}{x^3 + 1}(3x^2)$$

$$= \frac{3x^2}{x^3 + 1}$$

Derivatives of Logarithmic Functions

In general, if we combine Formula 2 with the Chain Rule as in Example 1, we get

3 $$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

Example 6

Find $f'(x)$ if $f(x) = \ln |x|$.

Solution:

Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

it follows that

$$f'(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ \dfrac{1}{-x}(-1) = \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Thus $f'(x) = 1/x$ for all $x \neq 0$.

Derivatives of Logarithmic Functions

The result of Example 6 is worth remembering:

4 $$\frac{d}{dx} \ln|x| = \frac{1}{x}$$

Logarithmic Differentiation

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms.

The method used in the following example is called **logarithmic differentiation**.

Example 7 – *Logarithmic Differentiation*

Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5}$.

Solution:

We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln (x^2 + 1) - 5 \ln (3x + 2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x^2 + 1} - 5 \cdot \frac{3}{3x + 2}$$

Example 7 – *Solution*

cont'd

Solving for *dy*/*dx*, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)$$

Because we have an explicit expression for *y*, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5}\left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)$$

Logarithmic Differentiation

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

If $f(x) < 0$ for some values of x, then $\ln f(x)$ is not defined, but we can write $|y| = |f(x)|$ and use Equation 4. We illustrate this procedure by proving the general version of the Power Rule.

The Power Rule If n is any real number and $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

Logarithmic Differentiation

In general there are four cases for exponents and bases:

1. $\dfrac{d}{dx}(a^b) = 0$ (*a* and *b* are constants)

2. $\dfrac{d}{dx}[f(x)]^b = b[f(x)]^{b-1}f'(x)$

3. $\dfrac{d}{dx}[a^{g(x)}] = a^{g(x)}(\ln a)g'(x)$

4. To find $(d/dx)[f(x)]^{g(x)}$, logarithmic differentiation can be used.

3.8 Rates of Change in the Natural and Social Sciences

Rates of Change in the Natural and Social Sciences

We know that if $y = f(x)$, then the derivative dy/dx can be interpreted as the rate of change of y with respect to x.

If x changes from x_1 to x_2, then the change in x is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

Rates of Change in the Natural and Social Sciences

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the **average rate of change of *y* with respect to *x*** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line *PQ* in Figure 1.

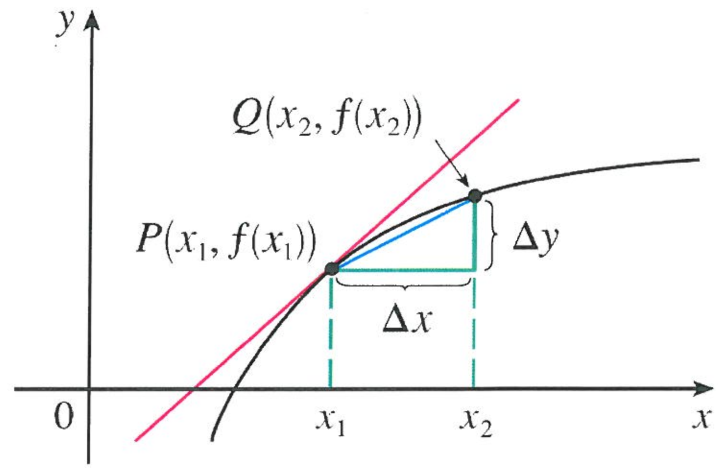

Figure 1

m_{PQ} = average rate of change
$m = f'(x_1)$ = instantaneous rate of change

Rates of Change in the Natural and Social Sciences

Its limit as $\Delta x \to 0$ is the derivative $f'(x_1)$, which can therefore be interpreted as the **instantaneous rate of change of *y* with respect to *x*** or the slope of the tangent line at $P(x_1, f(x_1))$.

Using Leibniz notation, we write the process in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Physics

Physics

If $s = f(t)$ is the position function of a particle that is moving in a straight line, then $\Delta s/\Delta t$ represents the average velocity over a time period Δt, *and* $v = ds/dt$ represents the instantaneous **velocity** (the rate of change of displacement with respect to time).

The instantaneous rate of change of velocity with respect to time is **acceleration**: $a(t) = v'(t) = s''(t)$.

Now that we know the differentiation formulas, we are able to solve problems involving the motion of objects more easily.

Example 1 – *Analyzing the Motion of a Particle*

The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 + 9t$$

where t is measured in seconds and s in meters.

(a) Find the velocity at time t.

(b) What is the velocity after 2 s? After 4 s?

(c) When is the particle at rest?

(d) When is the particle moving forward (that is, in the positive direction)?

(e) Draw a diagram to represent the motion of the particle.

(f) Find the total distance traveled by the particle during the first five seconds.

(g) Find the acceleration at time t and after 4 s.

Example 1 – *Analyzing the Motion of a Particle* cont'd

(h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 5$.

(i) When is the particle speeding up? When is it slowing down?

Solution:

(a) The velocity function is the derivative of the position function.

$$s = f(t) = t^3 - 6t^2 + 9t$$

$$v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9$$

Example 1 – *Solution*

cont'd

(b) The velocity after 2 s means the instantaneous velocity when $t = 2$, that is,

$$v(2) = \left.\frac{ds}{dt}\right|_{t=2} = 3(2)^2 - 12(2) + 9$$

$$= -3 \text{ m/s}$$

The velocity after 4 s is

$$v(4) = 3(4)^2 - 12(4) + 9$$

$$= 9 \text{ m/s}$$

Example 1 – *Solution*

cont'd

(c) The particle is at rest when $v(t) = 0$, that is,

$$3t^2 - 12t + 9 = 3(t^2 - 4t + 3)$$

$$= 3(t - 1)(t - 3)$$

$$= 0$$

and this is true when $t = 1$ or $t = 3$.

Thus the particle is at rest after 1 s and after 3 s.

Example 1 – *Solution*

cont'd

(d) The particle moves in the positive direction when $v(t) > 0$, that is,

$$3t^2 - 12t + 9 = 3(t - 1)(t - 3) > 0$$

This inequality is true when both factors are positive ($t > 3$) or when both factors are negative ($t < 1$).

Thus the particle moves in the positive direction in the time intervals $t < 1$ and $t > 3$.

It moves backward (in the negative direction) when $1 < t < 3$.

Example 1 – *Solution*

cont'd

(e) Using the information from part (d) we make a schematic sketch in Figure 2 of the motion of the particle back and forth along a line (the *s*-axis).

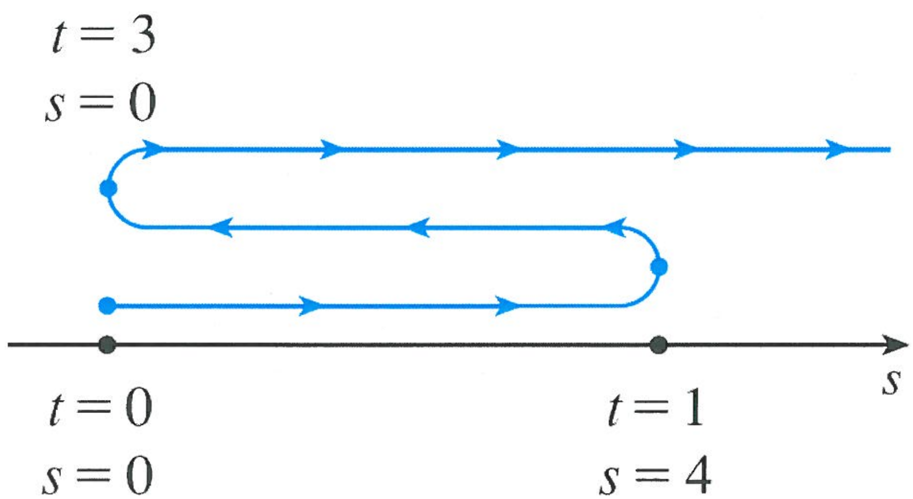

Figure 2

Example 1 – *Solution*

cont'd

(f) Because of what we learned in parts (d) and (e), we need to calculate the distances traveled during the time intervals [0, 1], [1, 3], and [3, 5] separately.

The distance traveled in the first second is

$$|f(1) - f(0)| = |4 - 0| = 4 \text{ m}$$

From $t = 1$ to $t = 3$ the distance traveled is

$$|f(3) - f(1)| = |0 - 4| = 4 \text{ m}$$

From $t = 3$ to $t = 5$ the distance traveled is

$$|f(5) - f(3)| = |20 - 0| = 20 \text{ m}$$

The total distance is $4 + 4 + 20 = 28$ m.

Example 1 – *Solution*

cont'd

(h) Figure 3 shows the graphs of *s*, *v*, and *a*.

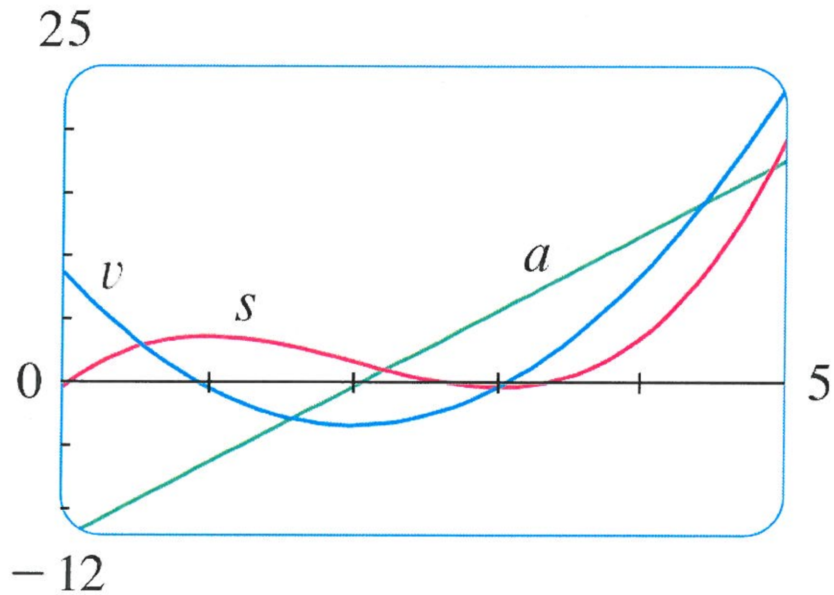

Figure 3

Example 1 – *Solution*

cont'd

(g) The acceleration is the derivative of the velocity function:

$$a(t) = \frac{d^2s}{dt^2}$$

$$= \frac{dv}{dt}$$

$$= 6t - 12$$

$$a(4) = 6(4) - 12$$

$$= 12 \text{ m/s}^2$$

Example 1 – *Solution*

cont'd

(i) The particle speeds up when the velocity is positive and increasing (v and a are both positive) and also when the velocity is negative and decreasing (v and a are both negative).

In other words, the particle speeds up when the velocity and acceleration have the same sign. (The particle is pushed in the same direction it is moving.)

From Figure 3 we see that this happens when $1 < t < 2$ and when $t > 3$.

Example 1 – *Solution*

cont'd

The particle slows down when v and a have opposite signs, that is, when $0 \le t < 1$ and when $2 < t < 3$.

Figure 4 summarizes the motion of the particle.

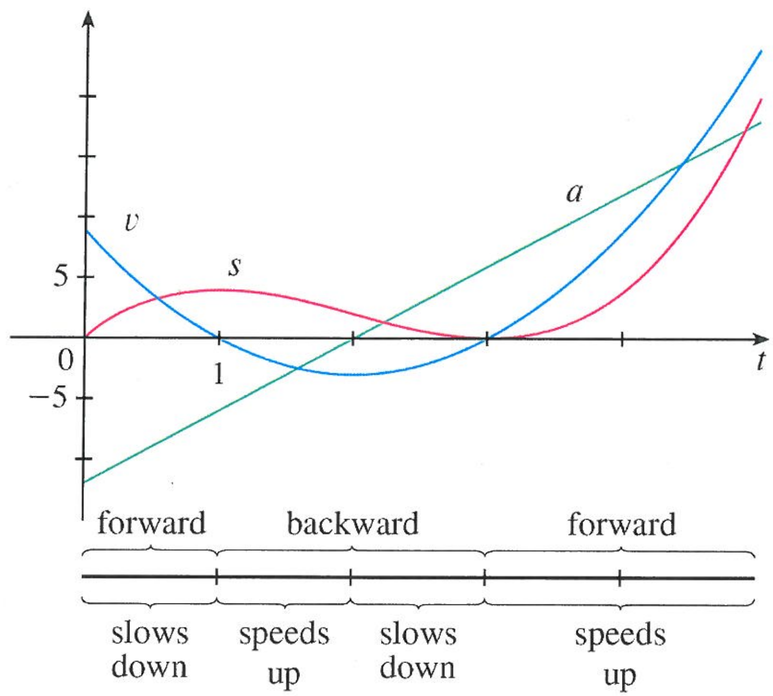

Figure 4

Biology

Example 6 – *Rate of Growth of a Population*

Let $n = f(t)$ be the number of individuals in an animal or plant population at time t.

The change in the population size between the times $t = t_1$ and $t = t_2$ is $\Delta n = f(t_2) - f(t_1)$, and so the average rate of growth during the time period $t_1 \leq t \leq t_2$ is

$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

The **instantaneous rate of growth** is obtained from this average rate of growth by letting the time period Δt approach 0:

$$\text{growth rate} = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

Example 6 – *Rate of Growth of a Population* cont'd

Strictly speaking, this is not quite accurate because the actual graph of a population function $n = f(t)$ would be a step function that is discontinuous whenever a birth or death occurs and therefore not differentiable.

However, for a large animal or plant population, we can replace the graph by a smooth approximating curve as in Figure 7.

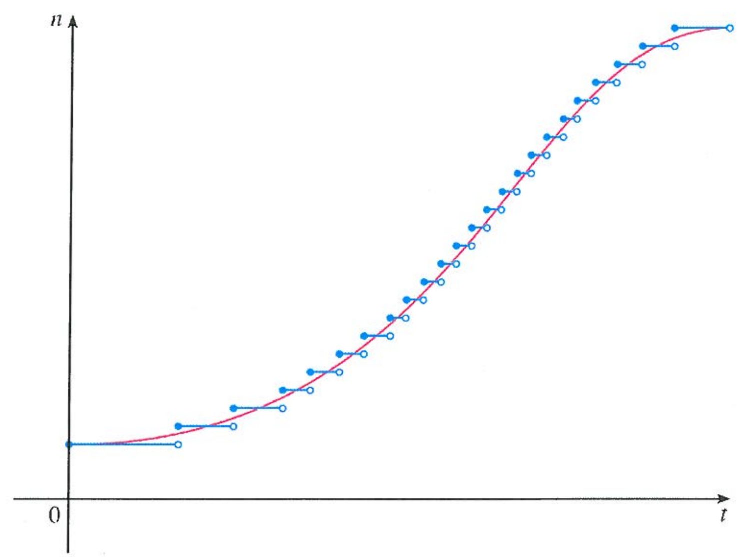

Figure 7

A smooth curve approximating a growth function

Example 6 – *Rate of Growth of a Population*

cont'd

To be more specific, consider a population of bacteria in a homogeneous nutrient medium.

Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour.

If the initial population is n_0 and the time t is measured in hours, then

$$f(1) = 2f(0) = 2n_0$$
$$f(2) = 2f(1) = 2^2n_0$$
$$f(3) = 2f(2) = 2^3n_0$$

In general,

$$f(t) = 2^t n_0$$

Example 6 – *Rate of Growth of a Population*

cont'd

The population function is $n_0 = n_0 2^t$.

We have shown that

$$\frac{d}{dx}(a^x) = a^x \ln a$$

So the rate of growth of the bacteria population at time t is

$$\frac{dn}{dt} = \frac{d}{dt}(n_0 2^t)$$

$$= n_0 2^t \ln 2$$

Example 6 – *Rate of Growth of a Population* cont'd

For example, suppose that we start with an initial population of n_0 = 100 bacteria.

Then the rate of growth after 4 hours is

$$\left.\frac{dn}{dt}\right|_{t=4} = 100 \cdot 2^4 \ln 2$$

$$= 1600 \ln 2$$

$$\approx 1109$$

This means that, after 4 hours, the bacteria population is growing at a rate of about 1109 bacteria per hour.

Economics

Example 8 – *Marginal Cost*

Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity.

The function C is called a **cost function**. If the number of items produced is increased from x_1 to x_2, then the additional cost is $\Delta C = C(x_2) - C(x_1)$, and the average rate of change of the cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}$$

Example 8 – *Marginal Cost*

cont'd

The limit of this quantity as $\Delta x \to 0$, that is, the instantaneous rate of change of cost with respect to the number of items produced, is called the **marginal cost** by economists:

$$\text{marginal cost} = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

[Since x often takes on only integer values, it may not make literal sense to let Δx approach 0, but we can always replace $C(x)$ by a smooth approximating function as in Example 6.]

Taking $\Delta x = 1$ and n large (so that Δx is small compared to n), we have

$$C'(n) \approx C(n + 1) - C(n)$$

Example 8 – *Marginal Cost*

cont'd

Thus the marginal cost of producing n units is approximately equal to the cost of producing one more unit, the $(n + 1)$st unit.

It is often appropriate to represent a total cost function by a polynomial

$$C(x) = a + bx + cx^2 + dx^3$$

where a represents the overhead cost (rent, heat, maintenance) and the other terms represent the cost of raw materials, labor, and so on. (The cost of raw materials may be proportional to x, but labor costs might depend partly on higher powers of x because of overtime costs and inefficiencies involved in large-scale operations.)

Example 8 – *Marginal Cost*

cont'd

For instance, suppose a company has estimated that the cost (in dollars) of producing x items is

$$C(x) = 10{,}000 + 5x + 0.01x^2$$

Then the marginal cost function is

$$C'(x) = 5 + 0.02x$$

The marginal cost at the production level of 500 items is

$$C'(500) = 5 + 0.02(500)$$

$$= \$15/\text{item}$$

Example 8 – *Marginal Cost*

cont'd

This gives the rate at which costs are increasing with respect to the production level when $x = 500$ and predicts the cost of the 501st item.

The actual cost of producing the 501st item is

$$C(501) - C(500) = [10{,}000 + 5(501) + 0.01(501)^2] - [10{,}000 + 5(500) + 0.01(500)^2]$$

$$= \$15.01$$

Notice that $C'(500) \approx C(501) - C(500)$.