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## Section 5.3 Inverse Functions

- **Verify that one function is the inverse function of another function.**
- **Determine whether a function has an inverse function.**
- **Find the derivative of an inverse function.**

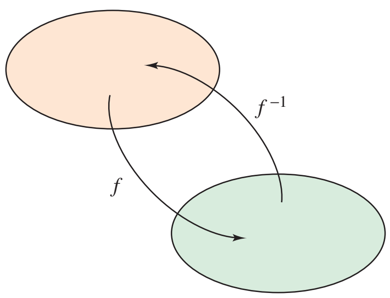
### Inverse Functions

Recall from Section P.3 that a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x + 3$ from $A = \{1, 2, 3, 4\}$ to $B = \{4, 5, 6, 7\}$ can be written as

$$f: \{(1, 4), (2, 5), (3, 6), (4, 7)\}.$$

By interchanging the first and second coordinates of each ordered pair, you can form the **inverse function** of $f$. This function is denoted by $f^{-1}$. It is a function from $B$ to $A$, and can be written as

$$f^{-1}: \{(4, 1), (5, 2), (6, 3), (7, 4)\}.$$




Domain of $f =$ range of $f^{-1}$
Domain of $f^{-1} =$ range of $f$
**Figure 5.10**

Note that the domain of $f$ is equal to the range of $f^{-1}$, and vice versa, as shown in Figure 5.10. The functions $f$ and $f^{-1}$ have the effect of "undoing" each other. That is, when you form the composition of $f$ with $f^{-1}$ or the composition of $f^{-1}$ with $f$, you obtain the identity function.

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x$$

**EXPLORATION**

***Finding Inverse Functions*** Explain how to "undo" each of the following functions. Then use your explanation to write the inverse function of $f$.

**a.** $f(x) = x - 5$

**b.** $f(x) = 6x$

**c.** $f(x) = \dfrac{x}{2}$

**d.** $f(x) = 3x + 2$

**e.** $f(x) = x^3$

**f.** $f(x) = 4(x - 2)$

Use a graphing utility to graph each function and its inverse function in the same "square" viewing window. What observation can you make about each pair of graphs?

**Definition of Inverse Function**

A function $g$ is the **inverse function** of the function $f$ if

$$f(g(x)) = x \quad \text{for each } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \quad \text{for each } x \text{ in the domain of } f.$$

The function $g$ is denoted by $f^{-1}$ (read "$f$ inverse").

NOTE Although the notation used to denote an inverse function resembles *exponential notation*, it is a different use of $-1$ as a superscript. That is, in general, $f^{-1}(x) \neq 1/f(x)$.

Here are some important observations about inverse functions.

1. If $g$ is the inverse function of $f$, then $f$ is the inverse function of $g$.
2. The domain of $f^{-1}$ is equal to the range of $f$, and the range of $f^{-1}$ is equal to the domain of $f$.
3. A function need not have an inverse function, but if it does, the inverse function is unique (see Exercise 99).

You can think of $f^{-1}$ as undoing what has been done by $f$. For example, subtraction can be used to undo addition, and division can be used to undo multiplication. Use the definition of an inverse function to check the following.

$$f(x) = x + c \quad \text{and} \quad f^{-1}(x) = x - c \quad \text{are inverse functions of each other.}$$

$$f(x) = cx \quad \text{and} \quad f^{-1}(x) = \frac{x}{c},\ c \neq 0, \quad \text{are inverse functions of each other.}$$

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## EXAMPLE 1 Verifying Inverse Functions

Show that the functions are inverse functions of each other.

$$f(x) = 2x^3 - 1 \quad \text{and} \quad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

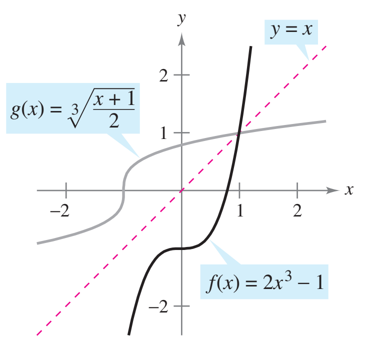
**Solution** Because the domains and ranges of both $f$ and $g$ consist of all real numbers, you can conclude that both composite functions exist for all $x$. The composition of $f$ with $g$ is given by

$$\begin{aligned} f(g(x)) &= 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 \\ &= 2\left(\frac{x+1}{2}\right) - 1 \\ &= x + 1 - 1 \\ &= x. \end{aligned}$$

The composition of $g$ with $f$ is given by

$$\begin{aligned} g(f(x)) &= \sqrt[3]{\frac{(2x^3 - 1) + 1}{2}} \\ &= \sqrt[3]{\frac{2x^3}{2}} \\ &= \sqrt[3]{x^3} \\ &= x. \end{aligned}$$

Because $f(g(x)) = x$ and $g(f(x)) = x$, you can conclude that $f$ and $g$ are inverse functions of each other (see Figure 5.11).




$f$ and $g$ are inverse functions of each other.
**Figure 5.11**

**STUDY TIP** In Example 1, try comparing the functions $f$ and $g$ verbally.

For $f$: First cube $x$, then multiply by 2, then subtract 1.

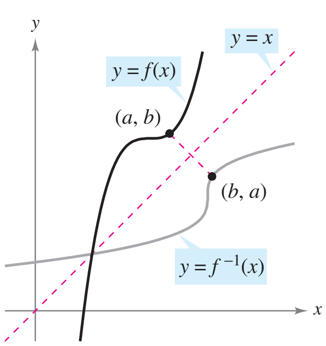
For $g$: First add 1, then divide by 2, then take the cube root.

Do you see the "undoing pattern"?

In Figure 5.11, the graphs of $f$ and $g = f^{-1}$ appear to be mirror images of each other with respect to the line $y = x$. The graph of $f^{-1}$ is a **reflection** of the graph of $f$ in the line $y = x$. This idea is generalized in the following theorem.

### THEOREM 5.6 Reflective Property of Inverse Functions

The graph of $f$ contains the point $(a, b)$ if and only if the graph of $f^{-1}$ contains the point $(b, a)$.




The graph of $f^{-1}$ is a reflection of the graph of $f$ in the line $y = x$.
**Figure 5.12**

**Proof** If $(a, b)$ is on the graph of $f$, then $f(a) = b$ and you can write

$$f^{-1}(b) = f^{-1}(f(a)) = a.$$

So, $(b, a)$ is on the graph of $f^{-1}$, as shown in Figure 5.12. A similar argument will prove the theorem in the other direction.

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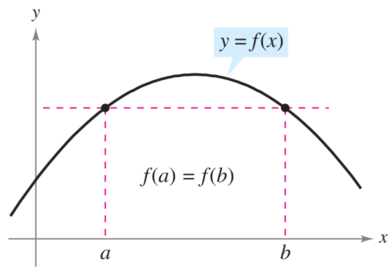



If a horizontal line intersects the graph of $f$ twice, then $f$ is not one-to-one.
**Figure 5.13**

## Existence of an Inverse Function

Not every function has an inverse function, and Theorem 5.6 suggests a graphical test for those that do—the **Horizontal Line Test** for an inverse function. This test states that a function $f$ has an inverse function if and only if every horizontal line intersects the graph of $f$ at most once (see Figure 5.13). The following theorem formally states why the horizontal line test is valid. (Recall from Section 3.3 that a function is *strictly monotonic* if it is either increasing on its entire domain or decreasing on its entire domain.)

**THEOREM 5.7 The Existence of an Inverse Function**

1. A function has an inverse function if and only if it is one-to-one.
2. If $f$ is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.

**Proof** To prove the second part of the theorem, recall from Section P.3 that $f$ is one-to-one if for $x_1$ and $x_2$ in its domain

$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2.$$

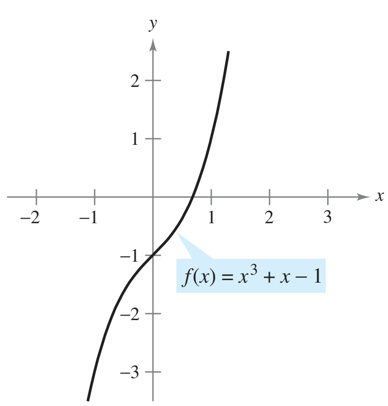
The *contrapositive* of this implication is logically equivalent and states that

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2).$$

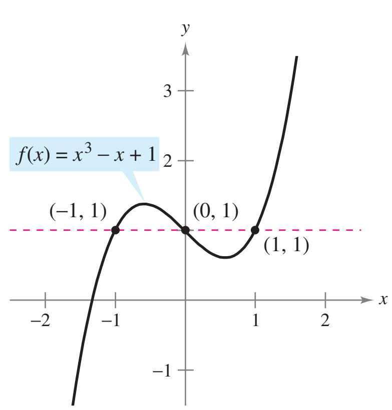
Now, choose $x_1$ and $x_2$ in the domain of $f$. If $x_1 \neq x_2$, then, because $f$ is strictly monotonic, it follows that either

$$f(x_1) < f(x_2) \qquad \text{or} \qquad f(x_1) > f(x_2).$$

In either case, $f(x_1) \neq f(x_2)$. So, $f$ is one-to-one on the interval. The proof of the first part of the theorem is left as an exercise (see Exercise 100).




**(a)** Because $f$ is increasing over its entire domain, it has an inverse function.




**(b)** Because $f$ is not one-to-one, it does not have an inverse function.
**Figure 5.14**

### *EXAMPLE 2* The Existence of an Inverse Function

Which of the functions has an inverse function?

**a.** $f(x) = x^3 + x - 1$ **b.** $f(x) = x^3 - x + 1$

**Solution**

**a.** From the graph of $f$ shown in Figure 5.14(a), it appears that $f$ is increasing over its entire domain. To verify this, note that the derivative, $f'(x) = 3x^2 + 1$, is positive for all real values of $x$. So, $f$ is strictly monotonic and it must have an inverse function.

**b.** From the graph of $f$ shown in Figure 5.14(b), you can see that the function does not pass the horizontal line test. In other words, it is not one-to-one. For instance, $f$ has the same value when $x = -1$, 0, and 1.

$$f(-1) = f(1) = f(0) = 1 \qquad \text{Not one-to-one}$$

So, by Theorem 5.7, $f$ does not have an inverse function.

NOTE Often it is easier to prove that a function *has* an inverse function than to find the inverse function. For instance, it would be difficult algebraically to find the inverse function of the function in Example 2(a).

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The following guidelines suggest a procedure for finding an inverse function.

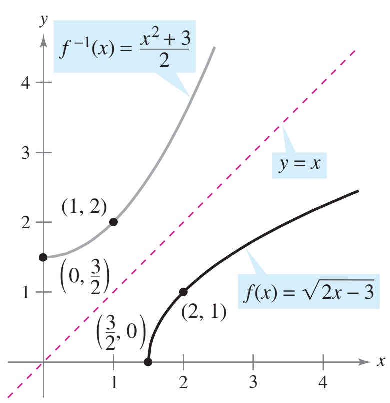
**Guidelines for Finding an Inverse Function**

1. Use Theorem 5.7 to determine whether the function given by $y = f(x)$ has an inverse function.
2. Solve for $x$ as a function of $y$: $x = g(y) = f^{-1}(y)$.
3. Interchange $x$ and $y$. The resulting equation is $y = f^{-1}(x)$.
4. Define the domain of $f^{-1}$ to be the range of $f$.
5. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

## EXAMPLE 3 Finding an Inverse Function

Find the inverse function of

$$f(x) = \sqrt{2x - 3}.$$




The domain of $f^{-1}$, $[0, \infty)$, is the range of $f$.
**Figure 5.15**

**Solution** The function has an inverse function because it is increasing on its entire domain (see Figure 5.15). To find an equation for the inverse function, let $y = f(x)$ and solve for $x$ in terms of $y$.

$$\sqrt{2x-3} = y \qquad \text{Let } y = f(x).$$
$$2x - 3 = y^2 \qquad \text{Square each side.}$$
$$x = \frac{y^2+3}{2} \qquad \text{Solve for } x.$$
$$y = \frac{x^2+3}{2} \qquad \text{Interchange } x \text{ and } y.$$
$$f^{-1}(x) = \frac{x^2+3}{2} \qquad \text{Replace } y \text{ by } f^{-1}(x).$$

The domain of $f^{-1}$ is the range of $f$, which is $[0, \infty)$. You can verify this result as shown.

$$f(f^{-1}(x)) = \sqrt{2\left(\frac{x^2+3}{2}\right) - 3} = \sqrt{x^2} = x, \quad x \geq 0$$
$$f^{-1}(f(x)) = \frac{\left(\sqrt{2x-3}\right)^2 + 3}{2} = \frac{2x - 3 + 3}{2} = x, \quad x \geq \frac{3}{2}$$

NOTE Remember that any letter can be used to represent the independent variable. So,

$$f^{-1}(y) = \frac{y^2+3}{2}$$
$$f^{-1}(x) = \frac{x^2+3}{2}$$
$$f^{-1}(s) = \frac{s^2+3}{2}$$

all represent the same function.

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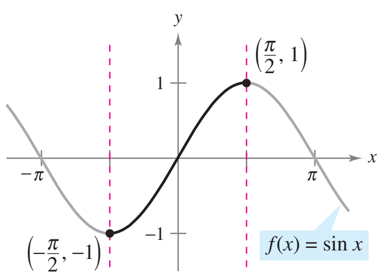
Theorem 5.7 is useful in the following type of problem. Suppose you are given a function that is *not* one-to-one on its domain. By restricting the domain to an interval on which the function is strictly monotonic, you can conclude that the new function *is* one-to-one on the restricted domain.

### EXAMPLE 4 Testing Whether a Function Is One-to-One

Show that the sine function

$$f(x) = \sin x$$

is not one-to-one on the entire real line. Then show that $[-\pi/2, \pi/2]$ is the largest interval, centered at the origin, for which $f$ is strictly monotonic.




$f$ is one-to-one on the interval $[-\pi/2, \pi/2]$.
**Figure 5.16**

**Solution** It is clear that $f$ is not one-to-one, because many different $x$-values yield the same $y$-value. For instance,

$$\sin(0) = 0 = \sin(\pi).$$

Moreover, $f$ is increasing on the open interval $(-\pi/2, \pi/2)$, because its derivative

$$f'(x) = \cos x$$

is positive there. Finally, because the left and right endpoints correspond to relative extrema of the sine function, you can conclude that $f$ is increasing on the closed interval $[-\pi/2, \pi/2]$ *and* that in any larger interval the function is not strictly monotonic (see Figure 5.16).

## Derivative of an Inverse Function

The next two theorems discuss the derivative of an inverse function. The reasonableness of Theorem 5.8 follows from the reflective property of inverse functions as shown in Figure 5.12. Proofs of the two theorems are given in Appendix A.

**THEOREM 5.8 Continuity and Differentiability of Inverse Functions**

Let $f$ be a function whose domain is an interval $I$. If $f$ has an inverse function, then the following statements are true.

1. If $f$ is continuous on its domain, then $f^{-1}$ is continuous on its domain.
2. If $f$ is increasing on its domain, then $f^{-1}$ is increasing on its domain.
3. If $f$ is decreasing on its domain, then $f^{-1}$ is decreasing on its domain.
4. If $f$ is differentiable at $c$ and $f'(c) \neq 0$, then $f^{-1}$ is differentiable at $f(c)$.

**THEOREM 5.9 The Derivative of an Inverse Function**

Let $f$ be a function that is differentiable on an interval $I$. If $f$ has an inverse function $g$, then $g$ is differentiable at any $x$ for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

**EXPLORATION**

Graph the inverse functions

$$f(x) = x^3$$

and

$$g(x) = x^{1/3}.$$

Calculate the slope of $f$ at $(1, 1)$, $(2, 8)$, and $(3, 27)$, and the slope of $g$ at $(1, 1)$, $(8, 2)$, and $(27, 3)$. What do you observe? What happens at $(0, 0)$?

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### EXAMPLE 5 Evaluating the Derivative of an Inverse Function

Let $f(x) = \frac{1}{4}x^3 + x - 1$.

**a.** What is the value of $f^{-1}(x)$ when $x = 3$?

**b.** What is the value of $(f^{-1})'(x)$ when $x = 3$?

**Solution** Notice that $f$ is one-to-one and therefore has an inverse function.

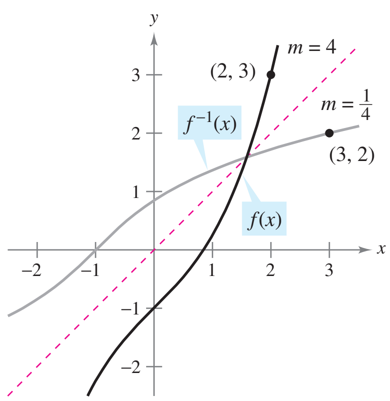
**a.** Because $f(x) = 3$ when $x = 2$, you know that $f^{-1}(3) = 2$.

**b.** Because the function $f$ is differentiable and has an inverse function, you can apply Theorem 5.9 (with $g = f^{-1}$) to write

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(2)}.$$

Moreover, using $f'(x) = \frac{3}{4}x^2 + 1$, you can conclude that

$$(f^{-1})'(3) = \frac{1}{f'(2)} = \frac{1}{\frac{3}{4}(2^2) + 1} = \frac{1}{4}.$$




The graphs of the inverse functions $f$ and $f^{-1}$ have reciprocal slopes at points $(a, b)$ and $(b, a)$.
**Figure 5.17**

In Example 5, note that at the point $(2, 3)$ the slope of the graph of $f$ is 4 and at the point $(3, 2)$ the slope of the graph of $f^{-1}$ is $\frac{1}{4}$ (see Figure 5.17). This reciprocal relationship (which follows from Theorem 5.9) can be written as shown below. If $y = g(x) = f^{-1}(x)$, then $f(y) = x$ and $f'(y) = \frac{dx}{dy}$. Theorem 5.9 says that

$$g'(x) = \frac{dy}{dx} = \frac{1}{f'(g(x))} = \frac{1}{f'(y)} = \frac{1}{(dx/dy)}.$$

So, $$\frac{dy}{dx} = \frac{1}{dx/dy}.$$

### EXAMPLE 6 Graphs of Inverse Functions Have Reciprocal Slopes

Let $f(x) = x^2$ (for $x \geq 0$) and let $f^{-1}(x) = \sqrt{x}$. Show that the slopes of the graphs of $f$ and $f^{-1}$ are reciprocals at each of the following points.

**a.** $(2, 4)$ and $(4, 2)$ **b.** $(3, 9)$ and $(9, 3)$

**Solution** The derivatives of $f$ and $f^{-1}$ are given by

$$f'(x) = 2x \quad \text{and} \quad (f^{-1})'(x) = \frac{1}{2\sqrt{x}}.$$

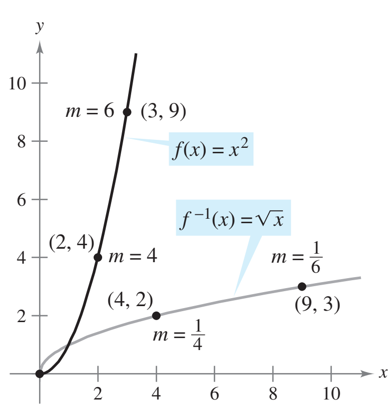
**a.** At $(2, 4)$, the slope of the graph of $f$ is $f'(2) = 2(2) = 4$. At $(4, 2)$, the slope of the graph of $f^{-1}$ is

$$(f^{-1})'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2(2)} = \frac{1}{4}.$$

**b.** At $(3, 9)$, the slope of the graph of $f$ is $f'(3) = 2(3) = 6$. At $(9, 3)$, the slope of the graph of $f^{-1}$ is

$$(f^{-1})'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}.$$

So, in both cases, the slopes are reciprocals, as shown in Figure 5.18.




At $(0, 0)$, the derivative of $f$ is 0, and the derivative of $f^{-1}$ does not exist.
**Figure 5.18**

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## Exercises for Section 5.3

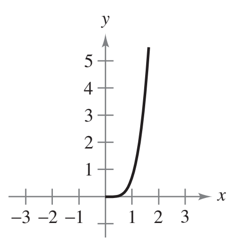
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**In Exercises 1–8, show that $f$ and $g$ are inverse functions (a) analytically and (b) graphically.**

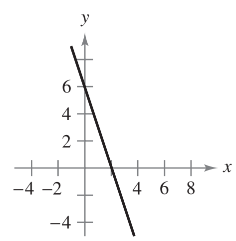
1. $f(x) = 5x + 1$, $g(x) = (x - 1)/5$
2. $f(x) = 3 - 4x$, $g(x) = (3 - x)/4$
3. $f(x) = x^3$, $g(x) = \sqrt[3]{x}$
4. $f(x) = 1 - x^3$, $g(x) = \sqrt[3]{1 - x}$
5. $f(x) = \sqrt{x - 4}$, $g(x) = x^2 + 4$, $x \geq 0$
6. $f(x) = 16 - x^2$, $x \geq 0$, $g(x) = \sqrt{16 - x}$
7. $f(x) = 1/x$, $g(x) = 1/x$
8. $f(x) = \dfrac{1}{1 + x}$, $x \geq 0$, $g(x) = \dfrac{1 - x}{x}$, $0 < x \leq 1$

**In Exercises 9–12, match the graph of the function with the graph of its inverse function. [The graphs of the inverse functions are labeled (a), (b), (c), and (d).]**

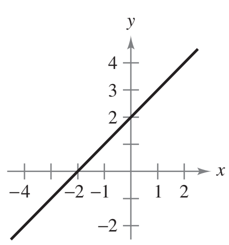
(a)



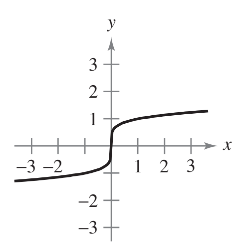
(b)



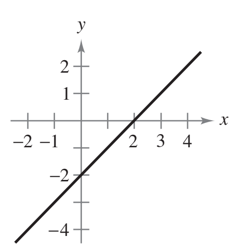
(c)



(d)



9.



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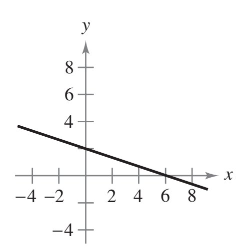

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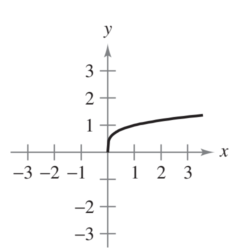
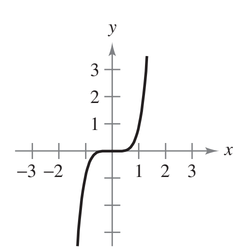
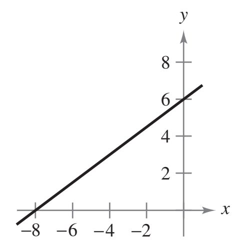


12.

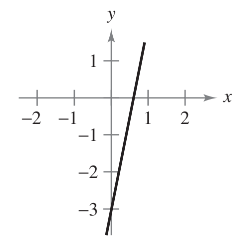


**In Exercises 13–16, use the Horizontal Line Test to determine whether the function is one-to-one on its entire domain and therefore has an inverse function. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

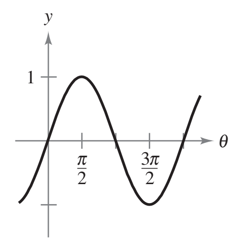
13. $f(x) = \frac{3}{4}x + 6$



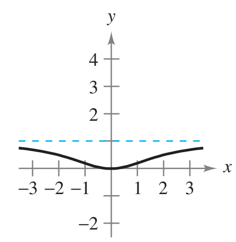
14. $f(x) = 5x - 3$



15. $f(\theta) = \sin \theta$



16. $f(x) = \dfrac{x^2}{x^2 + 4}$



**In Exercises 17–22, use a graphing utility to graph the function. Determine whether the function is one-to-one on its entire domain.**

17. $h(s) = \dfrac{1}{s - 2} - 3$
18. $g(t) = \dfrac{1}{\sqrt{t^2 + 1}}$
19. $f(x) = \ln x$
20. $f(x) = 5x\sqrt{x - 1}$
21. $g(x) = (x + 5)^3$
22. $h(x) = |x + 4| - |x - 4|$

**In Exercises 23–28, use the derivative to determine whether the function is strictly monotonic on its entire domain and therefore has an inverse function.**

23. $f(x) = \ln(x - 3)$
24. $f(x) = \cos \dfrac{3x}{2}$
25. $f(x) = \dfrac{x^4}{4} - 2x^2$
26. $f(x) = x^3 - 6x^2 + 12x$
27. $f(x) = 2 - x - x^3$
28. $f(x) = (x + a)^3 + b$

**In Exercises 29–36, find the inverse function of $f$. Graph (by hand) $f$ and $f^{-1}$. Describe the relationship between the graphs.**

29. $f(x) = 2x - 3$
30. $f(x) = 3x$
31. $f(x) = x^5$
32. $f(x) = x^3 - 1$
33. $f(x) = \sqrt{x}$
34. $f(x) = x^2$, $x \geq 0$
35. $f(x) = \sqrt{4 - x^2}$, $x \geq 0$
36. $f(x) = \sqrt{x^2 - 4}$, $x \geq 2$

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**In Exercises 37–42, find the inverse function of $f$. Use a graphing utility to graph $f$ and $f^{-1}$ in the same viewing window. Describe the relationship between the graphs.**

**37.** $f(x) = \sqrt[3]{x-1}$

**38.** $f(x) = 3\sqrt[5]{2x-1}$

**39.** $f(x) = x^{2/3}, \quad x \geq 0$

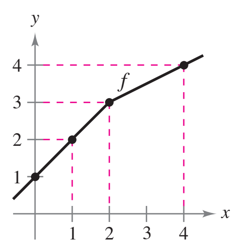
**40.** $f(x) = x^{3/5}$

**41.** $f(x) = \dfrac{x}{\sqrt{x^2+7}}$

**42.** $f(x) = \dfrac{x+2}{x}$

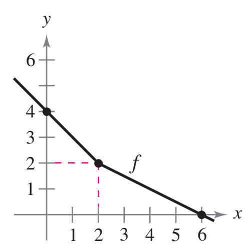
**In Exercises 43 and 44, use the graph of the function $f$ to complete the table and sketch the graph of $f^{-1}$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

**43.**




| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f^{-1}(x)$ | | | | |

**44.**




| $x$ | 0 | 2 | 4 |
|---|---|---|---|
| $f^{-1}(x)$ | | | |

**45.** ***Cost*** You need 50 pounds of two commodities costing $1.25 and $1.60 per pound.

(a) Verify that the total cost is $y = 1.25x + 1.60(50 - x)$, where $x$ is the number of pounds of the less expensive commodity.

(b) Find the inverse function of the cost function. What does each variable represent in the inverse function?

(c) Use the context of the problem to determine the domain of the inverse function.

(d) Determine the number of pounds of the less expensive commodity purchased if the total cost is $73.

**46.** ***Temperature*** The formula $C = \frac{5}{9}(F - 32)$, where $F \geq -459.6$, represents Celsius temperature $C$ as a function of Fahrenheit temperature $F$.

(a) Find the inverse function of $C$.

(b) What does the inverse function represent?

(c) Determine the domain of the inverse function.

(d) The temperature is 22°C. What is the corresponding temperature in degrees Fahrenheit?

**In Exercises 47–52, show that $f$ is strictly monotonic on the given interval and therefore has an inverse function on that interval.**

**47.** $f(x) = (x-4)^2, \quad [4, \infty)$

**48.** $f(x) = |x+2|, \quad [-2, \infty)$

**49.** $f(x) = \dfrac{4}{x^2}, \quad (0, \infty)$

**50.** $f(x) = \cot x, \quad (0, \pi)$

**51.** $f(x) = \cos x, \quad [0, \pi]$

**52.** $f(x) = \sec x, \quad \left[0, \dfrac{\pi}{2}\right)$

**In Exercises 53 and 54, find the inverse function of $f$ over the given interval. Use a graphing utility to graph $f$ and $f^{-1}$ in the same viewing window. Describe the relationship between the graphs.**

**53.** $f(x) = \dfrac{x}{x^2-4}, \quad (-2, 2)$

**54.** $f(x) = 2 - \dfrac{3}{x^2}, \quad (0, 10)$

***Graphical Reasoning*** **In Exercises 55–58, (a) use a graphing utility to graph the function, (b) use the *drawing* feature of a graphing utility to draw the inverse function of the function, and (c) determine whether the graph of the inverse relation is an inverse function. Explain your reasoning.**

**55.** $f(x) = x^3 + x + 4$

**56.** $h(x) = x\sqrt{4-x^2}$

**57.** $g(x) = \dfrac{3x^2}{x^2+1}$

**58.** $f(x) = \dfrac{4x}{\sqrt{x^2+15}}$

**In Exercises 59–62, determine whether the function is one-to-one. If it is, find its inverse function.**

**59.** $f(x) = \sqrt{x-2}$

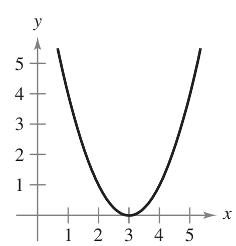
**60.** $f(x) = -3$

**61.** $f(x) = |x-2|, \quad x \leq 2$

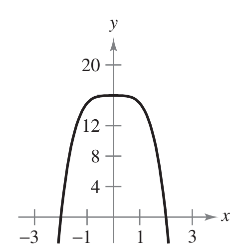
**62.** $f(x) = ax + b, \quad a \neq 0$

**In Exercises 63–66, delete part of the domain so that the function that remains is one-to-one. Find the inverse function of the remaining function and give the domain of the inverse function. (*Note:* There is more than one correct answer.)**

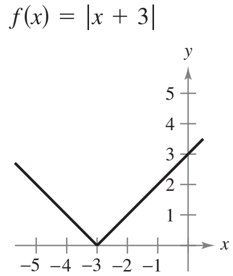
**63.** $f(x) = (x-3)^2$



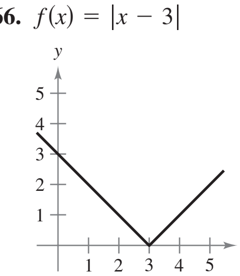

**64.** $f(x) = 16 - x^4$




**65.** $f(x) = |x+3|$




**66.** $f(x) = |x-3|$




***Think About It*** **In Exercises 67–70, decide whether the function has an inverse function. If so, what is the inverse function?**

**67.** $g(t)$ is the volume of water that has passed through a water line $t$ minutes after a control valve is opened.

**68.** $h(t)$ is the height of the tide $t$ hours after midnight, where $0 \leq t < 24$.

**69.** $C(t)$ is the cost of a long distance call lasting $t$ minutes.

**70.** $A(r)$ is the area of a circle of radius $r$.

**In Exercises 71–76, find $(f^{-1})'(a)$ for the function $f$ and the given real number $a$.**

**71.** $f(x) = x^3 + 2x - 1, \quad a = 2$

**72.** $f(x) = \frac{1}{27}(x^5 + 2x^3), \quad a = -11$

**73.** $f(x) = \sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}, \quad a = \frac{1}{2}$

**74.** $f(x) = \cos 2x, \quad 0 \le x \le \frac{\pi}{2}, \quad a = 1$

**75.** $f(x) = x^3 - \frac{4}{x}, \quad a = 6$

**76.** $f(x) = \sqrt{x - 4}, \quad a = 2$

**In Exercises 77–80, (a) find the domains of $f$ and $f^{-1}$, (b) find the ranges of $f$ and $f^{-1}$, (c) graph $f$ and $f^{-1}$, and (d) show that the slopes of the graphs of $f$ and $f^{-1}$ are reciprocals at the given points.**

| | *Functions* | *Point* |
|---|---|---|
| **77.** | $f(x) = x^3$ | $\left(\frac{1}{2}, \frac{1}{8}\right)$ |
| | $f^{-1}(x) = \sqrt[3]{x}$ | $\left(\frac{1}{8}, \frac{1}{2}\right)$ |
| **78.** | $f(x) = 3 - 4x$ | $(1, -1)$ |
| | $f^{-1}(x) = \frac{3 - x}{4}$ | $(-1, 1)$ |
| **79.** | $f(x) = \sqrt{x - 4}$ | $(5, 1)$ |
| | $f^{-1}(x) = x^2 + 4, \quad x \ge 0$ | $(1, 5)$ |
| **80.** | $f(x) = \frac{4}{1 + x^2}, \quad x \ge 0$ | $(1, 2)$ |
| | $f^{-1}(x) = \sqrt{\frac{4 - x}{x}}$ | $(2, 1)$ |

**In Exercises 81 and 82, find $dy/dx$ at the given point for the equation.**

**81.** $x = y^3 - 7y^2 + 2, \quad (-4, 1)$

**82.** $x = 2\ln(y^2 - 3), \quad (0, 4)$

**In Exercises 83–86, use the functions $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3$ to find the given value.**

**83.** $(f^{-1} \circ g^{-1})(1)$

**84.** $(g^{-1} \circ f^{-1})(-3)$

**85.** $(f^{-1} \circ f^{-1})(6)$

**86.** $(g^{-1} \circ g^{-1})(-4)$

**In Exercises 87–90, use the functions $f(x) = x + 4$ and $g(x) = 2x - 5$ to find the given function.**

**87.** $g^{-1} \circ f^{-1}$

**88.** $f^{-1} \circ g^{-1}$

**89.** $(f \circ g)^{-1}$

**90.** $(g \circ f)^{-1}$

## Writing About Concepts

**91.** Describe how to find the inverse function of a one-to-one function given by an equation in $x$ and $y$. Give an example.

**92.** Describe the relationship between the graph of a function and the graph of its inverse function.

## Writing About Concepts *(continued)*

**In Exercises 93 and 94, the derivative of the function has the same sign for all $x$ in its domain, but the function is not one-to-one. Explain.**

**93.** $f(x) = \tan x$

**94.** $f(x) = \frac{x}{x^2 - 4}$

**95.** ***Think About It*** The function $f(x) = k(2 - x - x^3)$ is one-to-one and $f^{-1}(3) = -2$. Find $k$.

**96.** (a) Show that $f(x) = 2x^3 + 3x^2 - 36x$ is not one-to-one on $(-\infty, \infty)$.

(b) Determine the greatest value $c$ such that $f$ is one-to-one on $(-c, c)$.

**97.** Let $f$ and $g$ be one-to-one functions. Prove that (a) $f \circ g$ is one-to-one and (b) $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

**98.** Prove that if $f$ has an inverse function, then $(f^{-1})^{-1} = f$.

**99.** Prove that if a function has an inverse function, then the inverse function is unique.

**100.** Prove that a function has an inverse function if and only if it is one-to-one.

***True or False?*** **In Exercises 101–104, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

**101.** If $f$ is an even function, then $f^{-1}$ exists.

**102.** If the inverse function of $f$ exists, then the $y$-intercept of $f$ is an $x$-intercept of $f^{-1}$.

**103.** If $f(x) = x^n$ where $n$ is odd, then $f^{-1}$ exists.

**104.** There exists no function $f$ such that $f = f^{-1}$.

**105.** Is the converse of the second part of Theorem 5.7 true? That is, if a function is one-to-one (and therefore has an inverse function), then must the function be strictly monotonic? If so, prove it. If not, give a counterexample.

**106.** Let $f$ be twice-differentiable and one-to-one on an open interval $I$. Show that its inverse function $g$ satisfies

$$g''(x) = -\frac{f''(g(x))}{[f'(g(x))]^3}.$$

If $f$ is increasing and concave downward, what is the concavity of $f^{-1} = g$?

**107.** If $f(x) = \int_2^x \frac{dt}{\sqrt{1 + t^4}}$, find $(f^{-1})'(0)$.

**108.** Show that $f(x) = \int_2^x \sqrt{1 + t^2}\, dt$ is one-to-one and find $(f^{-1})'(0)$.

**109.** Let $y = \frac{x - 2}{x - 1}$. Show that $y$ is its own inverse function. What can you conclude about the graph of $f$? Explain.