p332 Section 5.3: Inverse Functions

By switching the x & y-coordinates of an ordered pair, the inverse function can be formed. (The domain and range switch places)

Denoted by f^{-1}

Definition of Inverse Function

A function g is the inverse function of the function f if $f(g(x)) = x$ for each x in the domain of g and $g(f(x)) = x$ for each x in the domain of f.
The function g is denoted by f^{-1} (read "f inverse")

If g is the inverse function of f, then f is the inverse function of g

The domain of f^{-1} is equal to the range of f, and the range of f^{-1} is equal to the domain of f

A function need not have an inverse function, but if it does, the inverse function is unique

Example 1: Inverse Functions

Show that the functions are inverse functions of each other
(if they are inverses, $f(g(x)) = x$)

$$f(x) = 2x^3 - 1 \qquad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

$f(g(x)) = x$

$$f(g(x)) = f\left(\sqrt[3]{\frac{x+1}{2}}\right)$$

$$= 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1$$

$$= 2\left(\frac{x+1}{2}\right) - 1 = x + 1 - 1 = x$$

$g(f(x)) = x$

$$g(f(x)) = g(2x^3 - 1)$$

$$= \sqrt[3]{\frac{(2x^3-1)+1}{2}} = \sqrt[3]{\frac{2x^3}{2}}$$

$$= \sqrt[3]{x^3} = x$$

Theorem 5.6: Reflective Property of Inverse Functions

The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a)

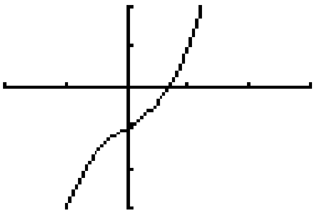
Theorem 5.7: The Existence of an Inverse Function

1. A function has an inverse function if and only if it is one-to-one
    a. no repeating x or y coordinates
    b. the function passes the Horizontal Line Test - a horizontal line only passes through one point on the graph
2. If f is strictly monotonic (only increasing or decreasing) on its entire domain, then it is one-to-one and therefore has an inverse function.

Example 2: The Existence of an Inverse Function

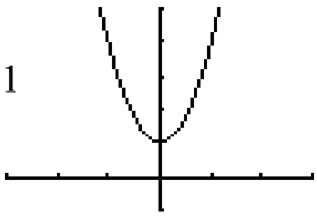
Which of the functions has an inverse function?

(a). $f(x) = x^3 + x - 1$

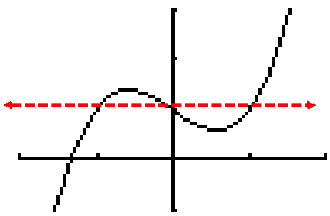


From the graph of f it appears that f is increasing over its entire domain. This can be verified by the first derivative test. The derivative is positive for all real values of x. Therefore, f is strictly monotonic and it must have an inverse function

$f'(x) = 3x^2 + 1$



(b). $f(x) = x^3 - x + 1$



This graph doesn't pass the horizontal line test, so f is not one-to-one and does not have an inverse function

Guidelines for Finding an Inverse Function

1. Use Theorem 5.7 to determine whether the function given by y = f(x) has an inverse function.
2. Switch the x & y variables (not exponents or coefficients - just the variables)
3. Solve for y - the resulting equation is y = $f^{-1}(x)$
4. Define the domain of f^{-1} to be the range of f.
5. Verify that $f(f^{-1}(x)) = x$

   $f^{-1}(f(x)) = x$

Example 3: Finding an Inverse Function

$f(x) = \sqrt{2x-3}$ Let f (x) = y

$y = \sqrt{2x-3}$ Switch x & y

$x = \sqrt{2y-3}$ Solve for y - square both sides to get rid of the radical

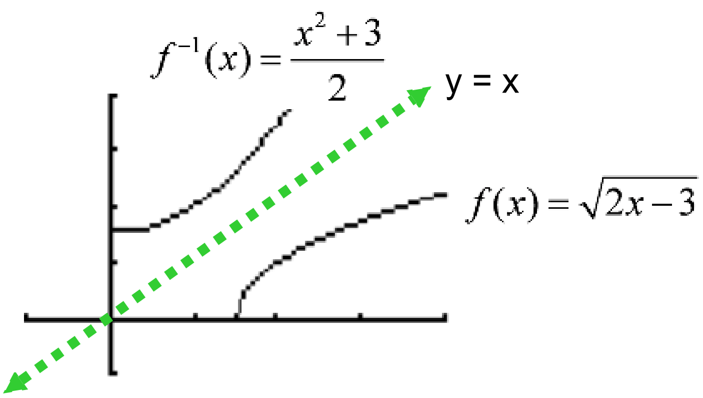
$\left(x = \sqrt{2y-3}\right)^2$

$x^2 = 2y-3$

$x^2 + 3 = 2y$

$\frac{x^2+3}{2} = y = f^{-1}(x)$ Replace y with $f^{-1}(x)$

The domain of f -1 is the range of f, which is [0, ∞)



Inverses reflect across the line y = x

#41. Find the inverse function of f. Use a graphing utility to graph f and f $^{-1}$ in the same viewing window. Describe the relationship between the graph.

$$f(x) = \frac{x}{\sqrt{x^2+7}}$$

$$y = \frac{x}{\left(x^2+7\right)^{1/2}}$$

$$x = \frac{y}{\left(y^2+7\right)^{1/2}}$$

$$x\left(y^2+7\right)^{1/2} = y$$

$$\left(x\left(y^2+7\right)^{1/2} = y\right)^2$$

$$x^2(y^2+7) = y^2$$

$$x^2y^2 + 7x^2 = y^2$$

$$x^2y^2 - y^2 = -7x^2$$

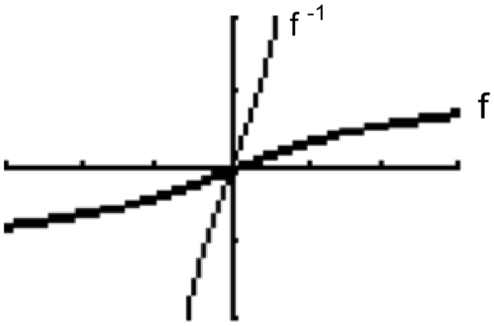
$$-x^2y^2 + y^2 = 7x^2$$

$$y^2(1-x^2) = 7x^2$$

$$y^2 = \frac{7x^2}{1-x^2}$$

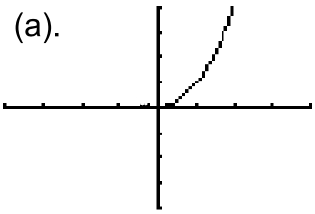
$$y = \sqrt{\frac{7x^2}{1-x^2}} = \frac{x\sqrt{7}}{\sqrt{1-x^2}}$$

$$f^{-1}(x) = \frac{x\sqrt{7}}{\sqrt{1-x^2}}, \qquad < x < 1$$

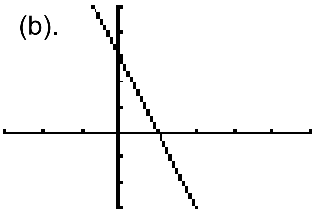


#9 - 12 Match the graph of the function with the graph of its inverse function.
The graph of the inverse functions are labels (a), (b), (c) and (d)

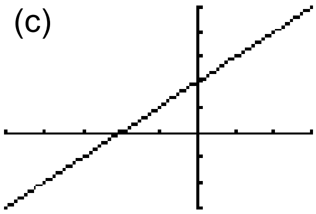
(a).



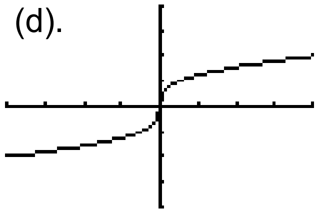
(b).



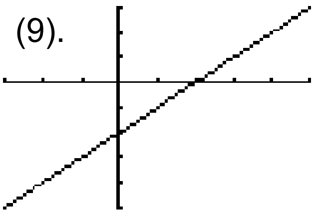
(c)



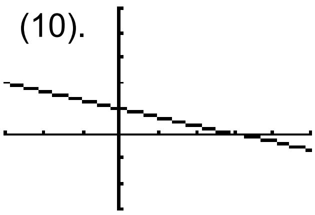
(d).



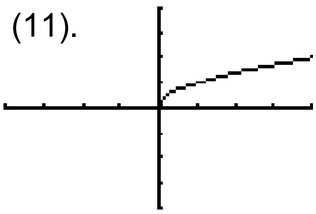
(9).



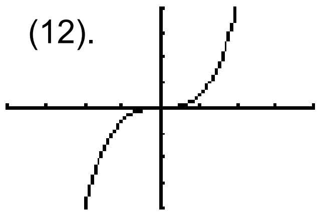
(10).



(11).



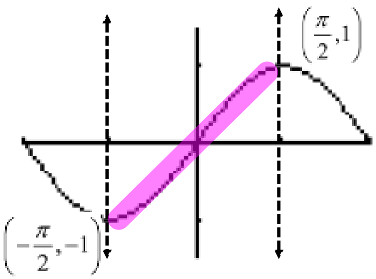
(12).



Sometimes you have to restrict the domain to an interval on which the function is strictly monotonic. Then you can conclude that the new function is one-to-one on the restricted domain.

Example 4: Testing Whether a Function is One-to-One

Show that the function is not one-to-one on the entire real line. Then show that $[-\pi/2, \pi/2]$ is the largest interval, centered at the origin, for which f is strictly monotonic. $f(x) = \sin x$

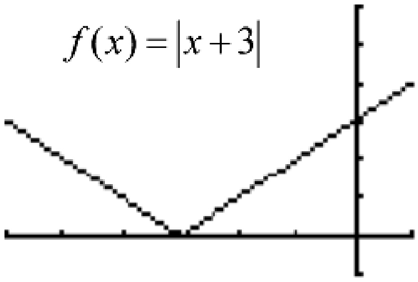


It is obvious that f is not one-to-one, because it does not pass the horizontal line test.
f is increasing on the open interval, $(-\pi/2, \pi/2)$ because its derivative is positive there.
Since the left and right endpoints correspond to relative extrema of the since function, you can conclude that f is increasing on the closed interval $[-\pi/2, \pi/2]$

- in a larger interval the function would not be strictly monotonic.

#65. Delete part of the domain so that the function that remains is one-to-one. Find the inverse function of the remaining function and give the domain of the inverse function

$f(x) = |x+3|$



f (x) is one-to-one for x ≥ -3

$$x+3=y$$
$$y+3=x$$
$$y=x-3$$
$$f^{-1}(x)=x-3, \quad x \geq 0$$

Could this problem have done another way?

Derivative of an Inverse Function

Theorem 5.8: Continuity and Differentiability of Inverse Functions

Let f be a function whose domain is an interval, I. If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is increasing on its domain, then f^{-1} is increasing on its domain.
3. If f is decreasing on its domain, then f^{-1} is decreasing on its domain.
4. If f is differentiable at c and $f'(c) \neq 0$, then f^{-1} is differentiable at f (c).

Theorem 5.9: The Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

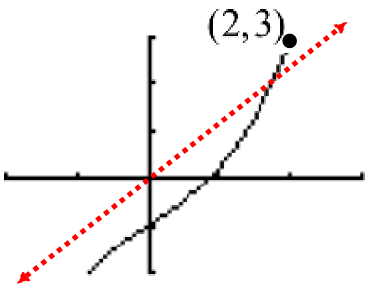
$$g'(x) = \frac{1}{f'(g(x))}, \qquad f'(g(x)) \neq 0$$

<u>Exploration:</u> Graph the inverse functions and calculate the slope of f at (1, 1), (2, 8), and (3, 27) and the slope of g at (1, 1), (8, 2), and (27, 3). What do you observe? What happens at (0, 0)?

$$f(x) = x^3$$
$$f'(x) = 3x^2$$
$$(1,1): 3(1)^2 = 3$$
$$(2,8): 3(2)^2 = 12$$
$$(3,27): 3(3)^2 = 27$$
$$(0,0): 3(0)^2 = 0$$

$$g(x) = x^{1/3}$$
$$g'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$
$$(1,1): \frac{1}{3(1)^{2/3}} = \frac{1}{3}$$
$$(8,2): \frac{1}{3(8)^{2/3}} = \frac{1}{12}$$
$$(27,3): \frac{1}{3(27)^{2/3}} = \frac{1}{27}$$
$$(0,0): \frac{1}{3(0)^{2/3}} = undefined$$

Example 5: Evaluating the Derivative of an Inverse Function



$$f(x) = \frac{1}{4}x^3 + x - 1$$

a. What is the value of $f^{-1}(x)$ when x = 3?
b. What is the value of $(f^{-1})'(x)$ when x = 3?

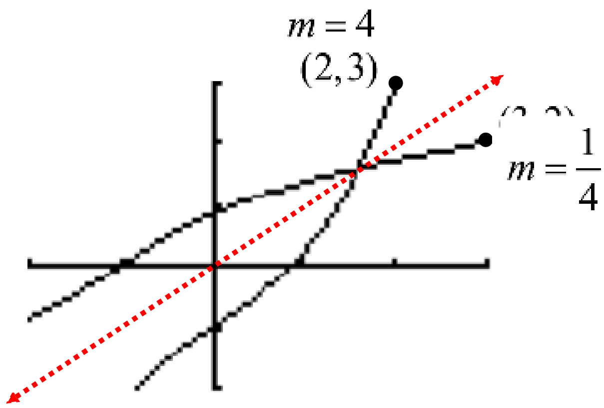
Notice that f is one-to-one, therefore it has an inverse function

a. Because f(x) = 3 when x = 2, your know that $f^{-1}(3) = 2$
b. Because the function f is differentiable and has an inverse function, you can apply Theorem 5.9 (with $g = f^{-1}$) to write

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(2)}$$

Using the derivative $f'(x) = \frac{3}{4}x^2 + 1$ you can conclude that

$$(f^{-1})'(3) = \frac{1}{f'(2)} = \frac{1}{\frac{3}{4}(2)^2 + 1} = \frac{1}{4}$$



Note that at the point (2, 3) the slope of the graph of f is 4 and at the point (3, 2) the slope of the graph of f -1 is 1/4. This reciprocal relationship is sometimes written as

$$\frac{dy}{dx} = \frac{1}{dx/dy}$$

Example 6: Graphs of Inverse Functions Have Reciprocal Slopes

Let $f(x) = x^2$ (for $x \geq 0$) and let $f^{-1}(x) = \sqrt{x} = x^{1/2}$. Show that the slopes of the graphs of f and f^{-1} are reciprocals at each of the following points.

a. (2, 4) and (4, 2)   b. (3, 9) and (9, 3)

$$f'(x) = 2x \qquad (f^{-1})'(x) = \frac{1}{2\sqrt{x}}$$

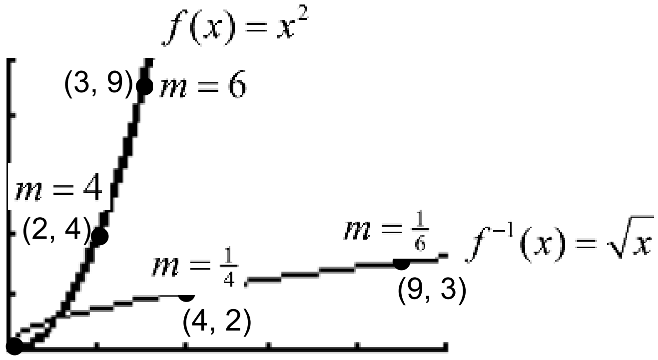
(a). $(2,4): f'(2) = 2(2) = 4$

$(4,2): (f^{-1})'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

(b). $(3,9): f'(3) = 2(3) = 6$

$(9,3): (f^{-1})'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$

In both cases the slopes are reciprocals



At (0, 0), the derivative of f is 0, and the derivative of f^{-1} does not exist

HW p338 #26, 28, 32, 36, 38, 42, 44, 64, 66, 72, 74, 80, 81