AP Calc Notes: MD – 6A Derivative of Inverse Functions from Equations, Graphs and Tables

Review of inverses:

1. A function f will have an inverse function f^{-1} if and only if f is one-to-one

   - To be one-to-one, a continuous function must be either increasing or decreasing

2. The graph of f^{-1} is the reflection of the graph of f over the line $y = x$

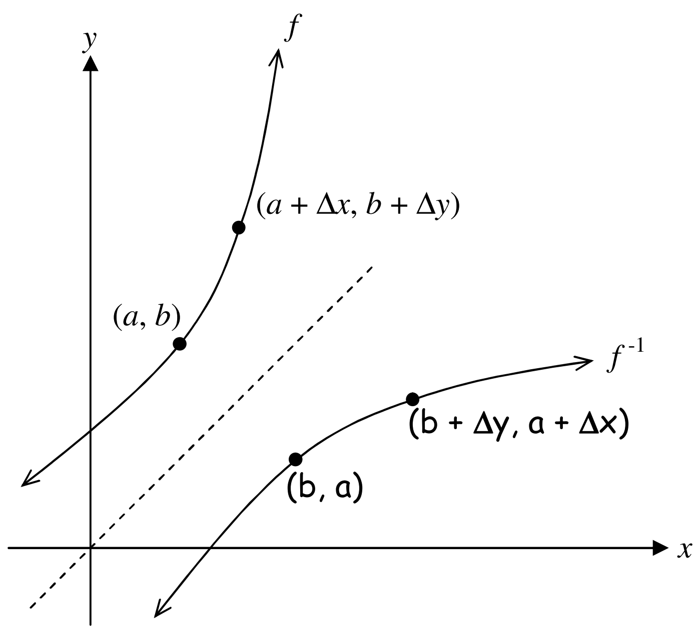
3. If f is continuous, then f^{-1} is also continuous.

4. If f is increasing(decreasing), then f^{-1} is also increasing(decreasing).

5. If f is differentiable at $x = c$ and $f'(c) \neq 0$, then f^{-1} is differentiable at $x = f(c)$.

We want a formula for $(f^{-1})'$.

Geometric Approach



$$f: \ m_{sec} = \frac{\Delta y}{\Delta x}$$

$$f^{-1}: \ m_{sec} = \frac{\Delta x}{\Delta y}$$

Reciprocals

$$\therefore (f^{-1}(b))' = \frac{1}{f'(a)}$$

Inverse functions have reciprocal slopes at reflected points.

Note: $f(a) = b \rightarrow a = f^{-1}(b)$

$$\therefore (f^{-1}(b))' = \frac{1}{f'(f^{-1}(b))} = \frac{1}{f'(a)}$$

We may substitute x for b, so

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}$$

Analytic Approach

$y = f'(x) \rightarrow x = f^{-1}(y) \rightarrow 1 = f'(y)y' \rightarrow y' = \dfrac{1}{f'(y)}$

$\therefore \dfrac{d}{dx}(f^{-1}(x)) = \dfrac{1}{f'(f^{-1}(x))}$

(2) Reflect the point, find x for $y = a$ $f(b) = a$	(1) To find $(f^{-1})'(a)$
(3) Find the derivative, evaluate at $x = b$ $f'(b)$	(4) Take the reciprocal of $f'(b)$ $(f^{-1})'(a) = \dfrac{1}{f'(b)}$

Ex: If $f(4) = 6$ and $f'(4) = \dfrac{2}{3}$ and $f'(6) = \dfrac{3}{5}$, find $(f^{-1})'(6)$

(2) Reflect the point, find x for $y = 6$ $f(4) = 6$ so $x = 4$	(1) To find $(f^{-1})'(6)$
(3) Find the derivative, evaluate at $x = 4$ $f'(4) = \dfrac{2}{3}$	(4) Take the reciprocal of $f'(4)$ $(f^{-1})'(6) = \dfrac{1}{\frac{2}{3}} = \dfrac{3}{2}$

Ex: If $f(x) = \dfrac{x^3}{32} + x - 5$, find $(f^{-1})'(1)$.

(2) Reflect the point, find x for $y = 1$ (use calc) $\dfrac{x^3}{32} + x - 5 = 1$ so $x = 4$	(1) To find $(f^{-1})'(1)$
(3) Find the derivative, evaluate at $x = 4$ $f'(x) = \dfrac{3x^2}{32} + 1$ $f'(4) = \dfrac{5}{2}$	(4) Take the reciprocal of $f'(4)$ $(f^{-1})'(1) = \dfrac{2}{5}$

Ex: If $g(x) = x^5 + 2x - 1$, find $\frac{d}{dx} g^{-1}(2)$.

(2) Reflect the point, find x for $y = 2$ (use calc) $x^5 + 2x - 1 = 2$ so $x = 1$	(1) To find $\left(g^{-1}\right)'(2)$
(3) Find the derivative, evaluate at $x = 1$ $g'(x) = 5x^4 + 2$ $g'(1) = 7$	(4) Take the reciprocal of $g'(1)$ $\left(g^{-1}\right)'(2) = \frac{1}{7}$

Ex. Selected values for the functions $f(x)$ and $f'(x)$ are shown in the table below. Let $g(x) = f^{-1}(x)$.

x	1	2	3	4
$f(x)$	2	4	1	3
$f'(x)$	1	3	4	2

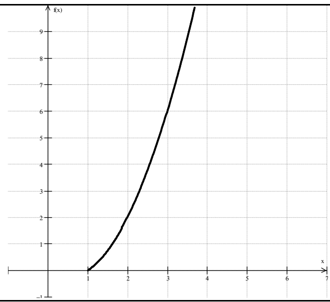
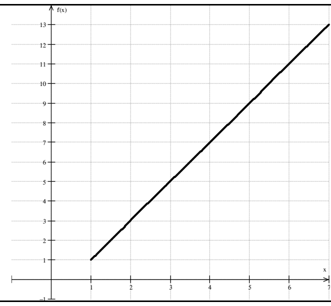
a) Find $g'(3)$

(2) Reflect the point, find x for $y = 3$ $f(x) = 3$ so $x = 4$	(1) To find $(g)'(3)$
(3) Find the derivative, evaluate at $x = 4$ $f'(4) = 2$	(4) Take the reciprocal of $f'(4)$ $(g)'(3) = \frac{1}{2}$

b) Find $g'(4)$

(2) Reflect the point, find x for $y = 4$ $f(x) = 4$ so $x = 2$	(1) To find $(g)'(4)$
(3) Find the derivative, evaluate at $x = 2$ $f'(2) = 3$	(4) Take the reciprocal of $f'(2)$ $(g)'(4) = \frac{1}{3}$

Example with a graph

$f(x)$	$f'(x)$
	

If $h(x) = f^{-1}(x)$, find $h'(6)$

(2) Reflect the point, find x for $y = 6$ $f(x) = 6$ so $x = 3$	(1) To find $(h)'(6)$
(3) Find the derivative at $x = 3$ $f'(3) = 5$	(4) Take the reciprocal of $f'(3)$ $(h)'(6) = \frac{1}{5}$

1) y = 6 → x = 3
2) f′(3) = 5
3) h′(6) = 1/5