DEFINITION of One-to-One Functions

A function $f(x)$ is ONE-TO-ONE on a domain D if $f(x_1) = f(x_2)$ whenever $x_1 \neq x_2$ in D

⇒ Some functions are one-to-one on their entire natural domain.

⇒ other functions are not one-to-one on their entire domain.

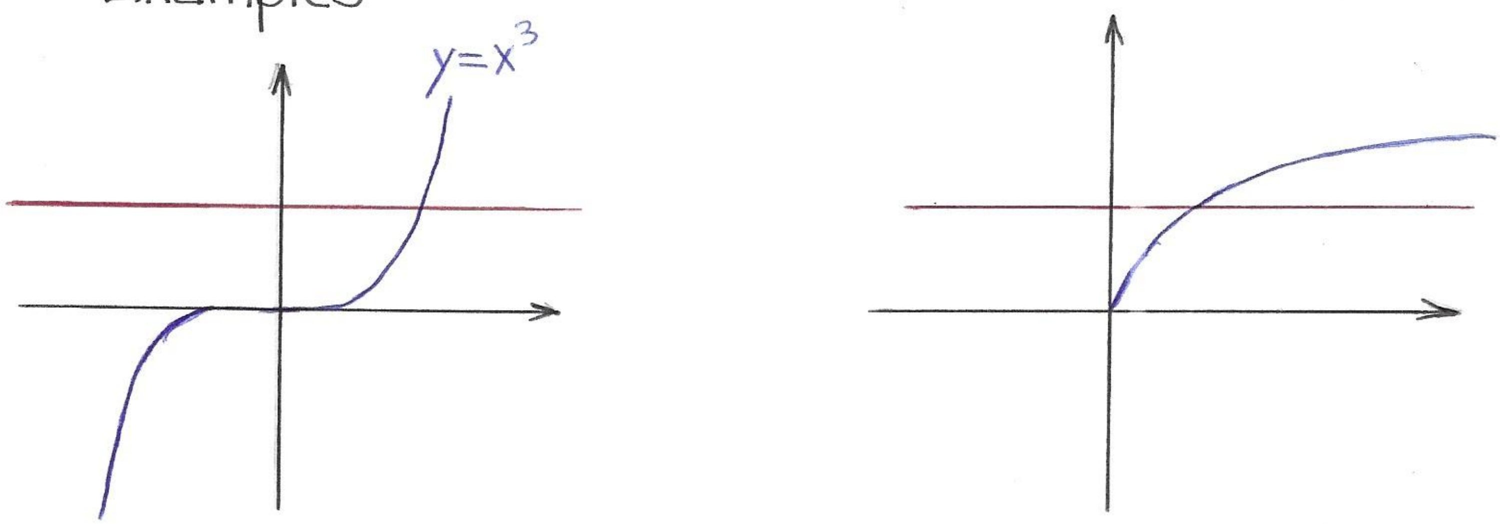
we can restrict the function to a smaller domain. By restricting the domain, we can create a function that is one-to-one.

⇒ The original and restricted functions ARE NOT the SAME functions because they have DIFFERENT DOMAINS. However, the two functions have the same values on the smaller domain.

The graph of a one-to-one function $y = f(x)$ can intersect a given horizontal line at MOST once.

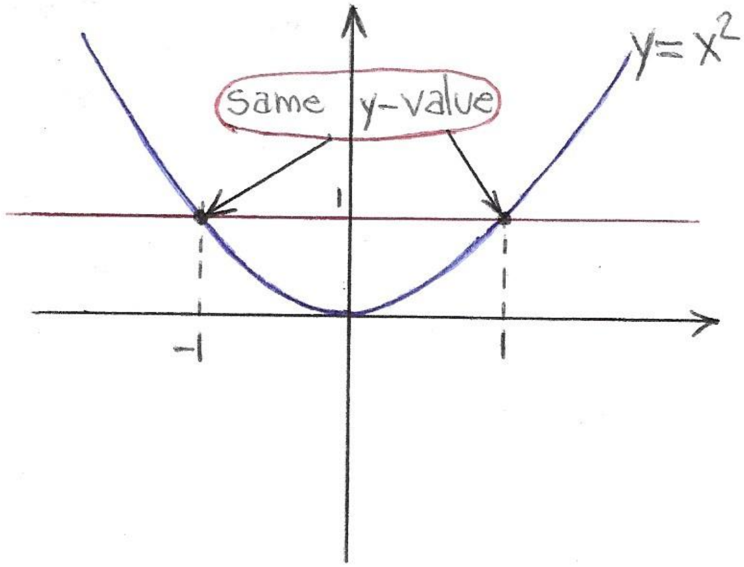
If the function intersects the line more than once, then it assumes the same y-value for at LEAST two different x-values and its therefore NOT one-to-one.

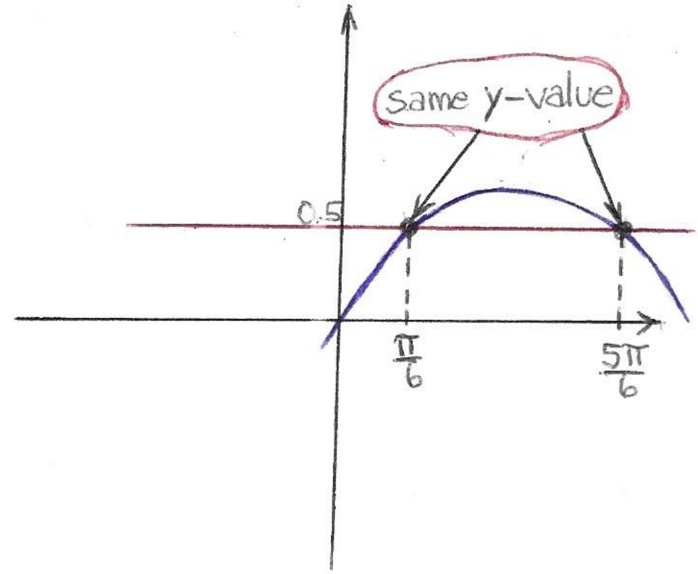
Examples



The functions above are examples of one-to-one functions.

$$f(x_1) \neq f(x_2) \text{ where } x_1 \neq x_2$$





$x_1 = -1\ ,\quad x_2 = 1$

$x_1 \neq x_2$

But

$f(x_1) = f(x_2) = 1$

$x_1 = \frac{\pi}{6}\ ,\quad x_2 = \frac{5\pi}{6}$

$x_1 \neq x_2$

$f(\pi/6) = 0.5 = f\left(\frac{5\pi}{6}\right)$

so

$f(x_1) = f(x_2)$

The functions above are NOT one-to-one.

The graph meets one or more horizontal lines more than once.

Inverse Functions

* Since each output of a one-to-one function comes from just one input, the effect of the function can be inverted to send each output back to the input from which it came.

DEFINITION ⇒ Suppose that f is one-to-one function on a domain D with range R. The INVERSE FUNCTION f^{-1} is defined by

$$f^{-1}(b) = a \quad \text{if} \quad f(a) = b$$

The domain of f^{-1} is R and the range of f^{-1} is D.

$$(f^{-1} \circ f)(x) = x \qquad \text{where } f(x) = y$$
$$(f \circ f^{-1})(y) = y \qquad \text{where } f^{-1}(y) = x$$

THE BIG IDEA

Only a one-to-one function can have an inverse.

The reason is that if $f(x_1)=y$ and $f(x_2)=y$ for two distinct inputs x_1 and x_2, then there is no way to assign a value to $f^{-1}(y)$ that satisfies both

$$f^{-1}\big(f(x_1)\big)=x_1 \quad \text{and} \quad f^{-1}\big(f(x_2)\big)=x_2$$