DEFINITION of One-to-One Functions

A function $f(x)$ is ONE-TO-ONE on a domain D if $f(x_1) = f(x_2)$ whenever $x_1 \neq x_2$ in D

⇒ Some functions are one-to-one on their entire natural domain.

⇒ other functions are not one-to-one on their entire domain.

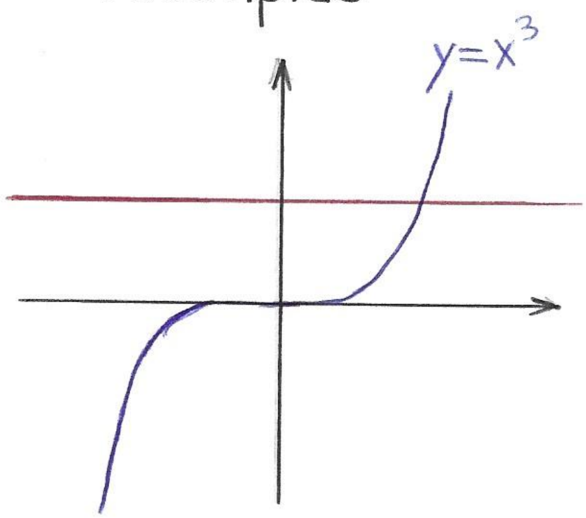
we can restrict the function to a smaller domain. By restricting the domain, we can create a function that is one-to-one.

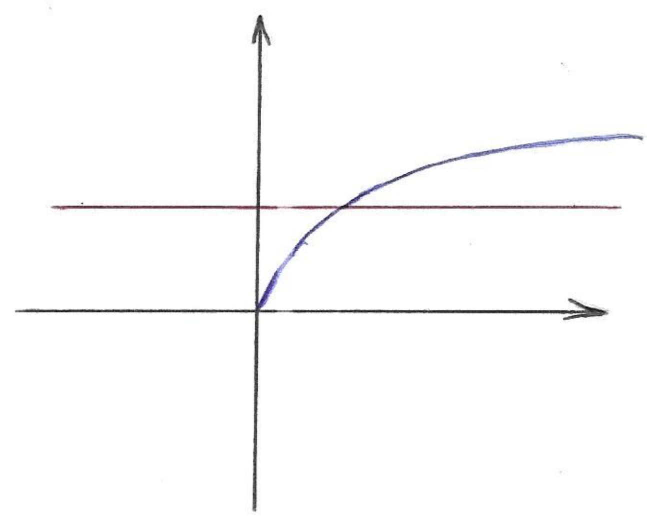
⇒ The original and restricted functions ARE NOT the SAME functions because they have DIFFERENT DOMAINS. However, the two functions have the same values on the smaller domain.

The graph of a one-to-one function $y = f(x)$ can intersect a given horizontal line at MOST once.

If the function intersects the line more than once, then it assumes the same y-value for at LEAST two different x-values and its therefore NOT one-to-one.

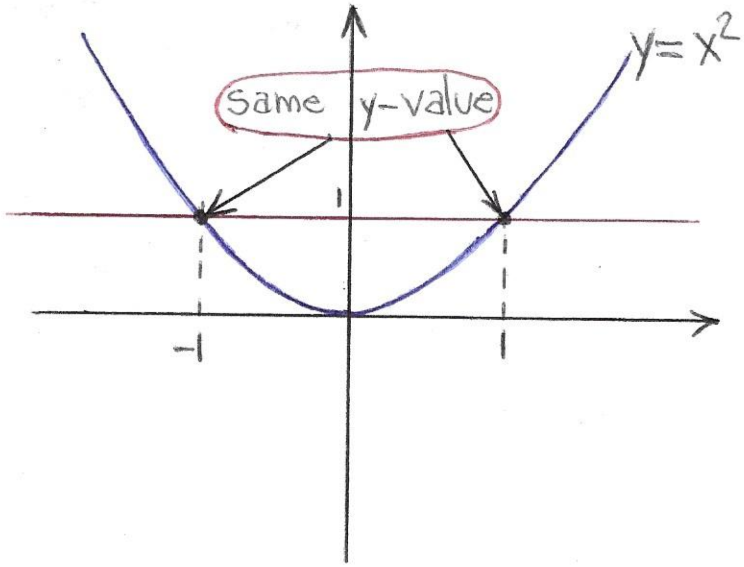
Examples





The functions above are examples of one-to-one functions.

$$f(x_1) \neq f(x_2) \quad \text{where} \quad x_1 \neq x_2$$

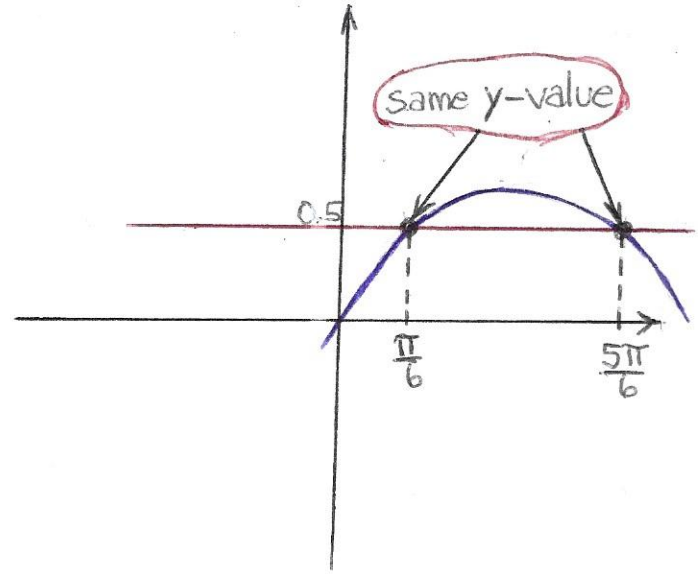


$x_1 = -1$, $x_2 = 1$

$x_1 \neq x_2$

But

$f(x_1) = f(x_2) = 1$



$x_1 = \frac{\pi}{6}$, $x_2 = \frac{5\pi}{6}$

$x_1 \neq x_2$

$f(\pi/6) = 0.5 = f\left(\frac{5\pi}{6}\right)$

so

$f(x_1) = f(x_2)$

The functions above are NOT one-to-one.

The graph meets one or more horizontal lines more than once.

Inverse Functions

* Since each output of a one-to-one function comes from just one input, the effect of the function can be inverted to send each output back to the input from which it came.

DEFINITION ⇛ Suppose that f is one-to-one function on a domain D with range R. The INVERSE FUNCTION f^{-1} is defined by

$$f^{-1}(b) = a \quad \text{if} \quad f(a) = b$$

The domain of f^{-1} is R and the range of f^{-1} is D.

$$(f^{-1} \circ f)(x) = x \qquad \text{where } f(x) = y$$
$$(f \circ f^{-1})(y) = y \qquad \text{where } f^{-1}(y) = x$$

THE BIG IDEA

Only a one-to-one function can have an inverse.

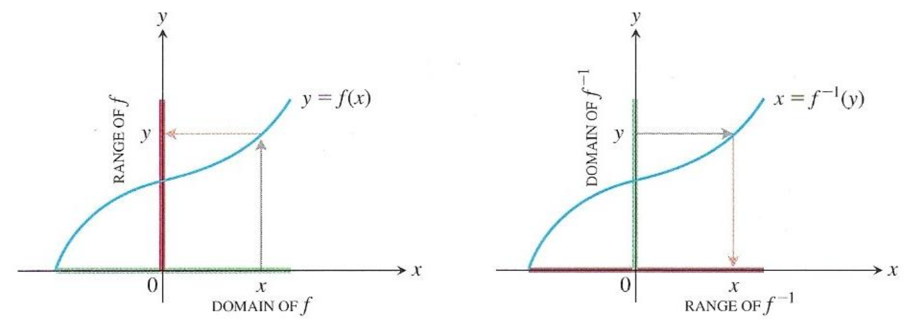
The reason is that if $f(x_1)=y$ and $f(x_2)=y$ for two distinct inputs x_1 and x_2, then there is no way to assign a value to $f^{-1}(y)$ that satisfies both

$$f^{-1}(f(x_1)) = x_1 \quad \text{and} \quad f^{-1}(f(x_2)) = x_2$$

A function that is increasing on an interval satisfies the inequality $f(x_2) > f(x_1)$ when $x_2 > x_1$, so it is one-to-one and has an inverse. A function that is decreasing on an interval also has an inverse. Functions that are neither increasing nor decreasing may still be one-to-one and have an inverse, as with the function $f(x) = 1/x$ for $x \neq 0$ and $f(0) = 0$, defined on $(-\infty, \infty)$ and passing the horizontal line test.

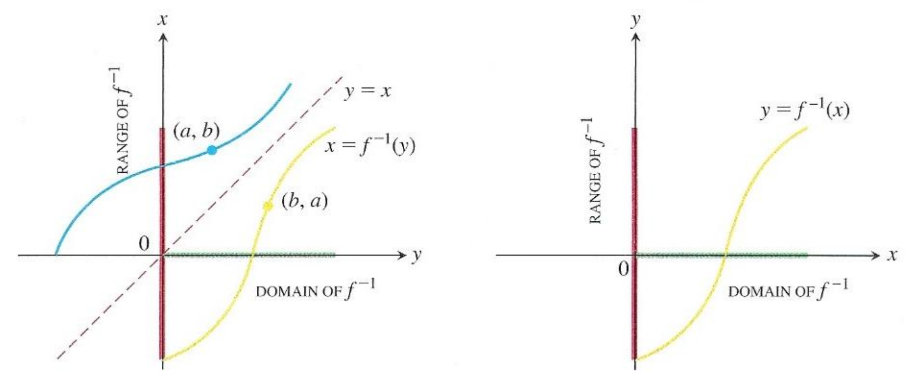
Finding Inverses

The graphs of a function and its inverse are closely related. To read the value of a function from its graph, we start at a point x on the x-axis, go vertically to the graph, and then move horizontally to the y-axis to read the value of y. The inverse function can be read from the graph by reversing this process. Start with a point y on the y-axis, go horizontally to the graph of $y = f(x)$, and then move vertically to the x-axis to read the value of $x = f^{-1}(y)$ (Figure 1.57).



(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.

(b) The graph of f^{-1} is the graph of f, but with x and y interchanged. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.



(c) To draw the graph of f^{-1} in the more usual way, we reflect the system across the line $y = x$.

(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

FIGURE 1.57 The graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ about the line $y = x$.

We want to set up the graph of f^{-1} so that its input values lie along the x-axis, as is usually done for functions, rather than on the y-axis. To achieve this we interchange the x- and y-axes by reflecting across the 45° line $y = x$. After this reflection we have a new graph that represents f^{-1}. The value of $f^{-1}(x)$ can now be read from the graph in the usual way, by starting with a point x on the x-axis, going vertically to the graph, and then horizontally to