

2.5 Implicit Differentiation

An explicit function is a function that explicitly tells you how to find one of the variable values such as $y = f(x)$. An implicit function is less direct in that no variable has been isolated and in many cases it cannot be isolated. An example might be $xy = 6$ or $x^2 - xy + y^2 - 4 = 0$. In the first example, we could isolate either variable easily. In the second example it is not easy to isolate either variable (possible but not easy).

Guidelines for Implicit Differentiation –

1. Differentiate both sides of the equation *with respect to x*.
2. Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx .

Examples: Find dy/dx by implicit differentiation.

1. $x^2 - y^2 = 25$

Always remember $y = y(x)$. So anytime you take the derivative of y you get $y' = \frac{dy}{dx}$.

Derivative of x^2 is $2x$

Derivative of y^2 is $2y \frac{dy}{dx}$ (chain rule)

Derivative of 25 is 0

Derivative of $x^2 - y^2 = 25$ is $2x - 2y \frac{dy}{dx} = 0$. Now solve for $\frac{dy}{dx}$:

$$2x = 2y \frac{dy}{dx} \rightarrow \frac{2x}{2y} = \frac{dy}{dx} \rightarrow \boxed{\frac{dy}{dx} = \frac{x}{y}}$$

$$2. x^3 + y^3 = 64$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 0$$

$$3y^2 \frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = \frac{-3x^2}{3y^2} = \frac{-x^2}{y^2}$$

$$3. \frac{x^2 y + y^2 x}{\text{Product}} = -2$$

$$x^2 \left(\frac{dy}{dx} \right) + y(2x) + y^2(1) + x(2y \frac{dy}{dx}) = 0$$

$$x^2 \frac{dy}{dx} + 2xy + y^2 + 2xy \frac{dy}{dx} = 0$$

$$x^2 \frac{dy}{dx} + 2xy \frac{dy}{dx} = -y^2 - 2xy$$

$$\frac{dy}{dx} (x^2 + 2xy) = -y^2 - 2xy$$

$$\frac{dy}{dx} = \frac{-y^2 - 2xy}{x^2 + 2xy}$$

$$4. \cot y = x - y$$

$$-\csc^2 y \frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx} - \csc^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} (1 - \csc^2 y) = 1$$

$$\frac{dy}{dx} = \frac{1}{1 - \csc^2 y}$$

Examples: Find two explicit functions by solving the equation for y in terms of x .

1. $x^2 + y^2 = 64$

$$y^2 = 64 - x^2$$

$$\sqrt{y^2} = \pm \sqrt{64 - x^2}$$

$$y = +\sqrt{64 - x^2}$$

and

$$y = -\sqrt{64 - x^2}$$

To find $\frac{dy}{dx}$ at this point, we would have to find two derivatives. Implicit make it easier

2. $16y^2 - x^2 = 16$

$$16y^2 = 16 + x^2$$

$$y^2 = \frac{16 + x^2}{16}$$

$$y = \pm \sqrt{\frac{16 + x^2}{16}}$$

$$y = \frac{\sqrt{16 + x^2}}{4} \text{ and } y = -\frac{\sqrt{16 + x^2}}{4}$$

Examples: Find dy/dx by implicit differentiation and evaluate the derivative at the given point.

1. $xy = 6, (-6, -1)$

Product rule

$$x \frac{dy}{dx} + y(1) = 0$$

$$x \frac{dy}{dx} + y = 0$$

$$x \frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

$$\left. \frac{dy}{dx} \right|_{(-6, -1)} = \frac{-(-1)}{-6} = -\frac{1}{6}$$

2. $(x+y)^3 = x^3 + y^3 \quad (-1, 1)$

$$3(x+y)^2 \left(1 + \frac{dy}{dx}\right) = 3x^2 + 3y^2 \frac{dy}{dx}$$

$$3(x+y)^2 + 3(x+y)^2 \frac{dy}{dx} = 3x^2 + 3y^2 \frac{dy}{dx}$$

$$3(x+y)^2 \frac{dy}{dx} - 3y^2 \frac{dy}{dx} = 3x^2 - 3(x+y)^2$$

$$\frac{dy}{dx} \left[3(x+y)^2 - 3y^2 \right] = 3x^2 - 3(x+y)^2$$

$$\frac{dy}{dx} = \frac{3x^2 - 3(x+y)^2}{3(x+y)^2 - 3y^2}$$

at $(-1, 1)$

$$\frac{dy}{dx} = \frac{3(-1)^2 - 3(-1+1)^2}{3(-1+1)^2 - 3(1)^2}$$

$$\frac{dy}{dx} = \frac{3-0}{0-3}$$

$$\frac{dy}{dx} = \frac{3}{-3} = -1$$

Examples: Find d^2y/dx^2 in terms of x and y . — first find $\frac{dy}{dx}$ then take the derivative again.

1. $x^2y^2 - 2x = 3$

$$x^2(2y \frac{dy}{dx}) + y^2(2x) - 2 = 0$$

$$2x^2y \frac{dy}{dx} + 2xy^2 - 2 = 0$$

$$2x^2y \frac{dy}{dx} = 2 - 2xy^2$$

$$\frac{dy}{dx} = \frac{2 - 2xy^2}{2x^2y} = \frac{1 - xy^2}{x^2y}$$

To find $\frac{d^2y}{dx^2}$ we use the quotient rule with some product rule thrown in.

$$\frac{d^2y}{dx^2} = \frac{x^2y(0 - (x^2y \frac{dy}{dx} + y^2(1))) - (1 - xy^2)[x^2 \frac{dy}{dx} + y(2x)]}{(x^2y)^2}$$

We clean it up by substituting $\frac{dy}{dx}$ as indicated.

$$\frac{d^2y}{dx^2} = \frac{-x^2y(2xy \frac{1 - xy^2}{x^2y} + y^2) - (1 - xy^2)(x^2 \frac{1 - xy^2}{x^2y} + 2xy)}{x^4y^2}$$

2. $1 - xy = x - y$

$$0 - (x \frac{dy}{dx} + y(1)) = 1 - \frac{dy}{dx}$$

$$-x \frac{dy}{dx} - y = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx} - x \frac{dy}{dx} = 1 + y$$

$$\frac{dy}{dx}(1 - x) = 1 + y \quad \text{so} \quad \frac{dy}{dx} = \frac{1 + y}{1 - x}$$

$$\frac{d^2y}{dx^2} = \frac{(1-x) \frac{dy}{dx} - (1+y)(-1)}{(1-x)^2} = \frac{(1-x) \frac{1+y}{1-x} + (1+y)}{(1-x)^2} = \frac{1+y + 1+y}{(1-x)^2} = \frac{2+2y}{(1-x)^2}$$

$$3. y^2 = 10x$$

$$2y \frac{dy}{dx} = 10, \quad \frac{dy}{dx} = \frac{10}{2y} = \frac{5}{y}$$

$$\frac{d^2y}{dx^2} = \frac{y(0) - 5 \frac{dy}{dx}}{y^2} = \frac{-5\left(\frac{5}{y}\right)}{y^2} = \frac{-25}{y^2} = \frac{-25}{y^3}$$