Implicit Differentiation
Selected Problems

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Implicit Differentiation : Selected Problems

1. Find dy/dx.

(a)
$$\begin{aligned} y &= \sqrt[3]{2x-5} \\ &= (2x-5)^{1/3} \\ \frac{dy}{dx} &= \frac{1}{3}(2x-5)^{1/3-1}(2) \\ &= \boxed{\frac{2}{3}(2x-5)^{-2/3}} \end{aligned}$$

(b)
$$\begin{aligned} y &= \sqrt[3]{2+\tan(x^2)} \\ &= (2+\tan(x^2))^{1/3} \\ \frac{dy}{dx} &= \frac{1}{3}(2+\tan(x^2))^{(1/3-1)}(\sec^2(x^2))(2x) \\ &= \boxed{\frac{2}{3}x\sec^2(x^2)(2+\tan(x^2))^{-2/3}} \end{aligned}$$

(c)
$$\begin{aligned} y &= x^3(5x^2+1)^{-2/3} \\ \frac{dy}{dx} &= x^3\left(-\frac{2}{3}(5x^2+1)^{-5/3}(10x)\right) + 3x^2(5x^2+1)^{-2/3} \\ &= \frac{-20x^4}{3(5x^2+1)^{5/3}} + \frac{3x^2}{(5x^2+1)^{2/3}} \\ &= \frac{-20x^4}{3(5x^2+1)^{5/3}} + \frac{3x^2}{(5x^2+1)^{2/3}} \cdot \frac{3(5x^2+1)}{3(5x^2+1)} \\ &= \frac{-20x^4}{3(5x^2+1)^{5/3}} + \frac{9x^2(5x^2+1)}{3(5x^2+1)^{5/3}} \\ &= \frac{-20x^4+45x^4+9x^2}{3(5x^2+1)^{5/3}} \\ &= \frac{25x^4+9x^2}{3(5x^2+1)^{5/3}} \\ &= \boxed{\frac{1}{3}x^2(5x^2+1)^{-5/3}(25x^2+9)} \end{aligned}$$

2. Given $x + xy - 2x^3 = 2$.

(a) Find dy/dx by differentiating implicitly.

$$\begin{aligned} \frac{d}{dx}(x + xy - 2x^3) &= \frac{d}{dx}(2) \\ \frac{d}{dx}(x) + \frac{d}{dx}(xy) - \frac{d}{dx}(2x^3) &= 0 \\ 1 + x\frac{d}{dx}(y) + y - 6x^2 &= 0 \\ 1 + x\frac{dy}{dx} + y - 6x^2 &= 0 \end{aligned}$$

(b) Solve the equation for y as a function of x, and find dy/dx from that equation.

$$\begin{aligned} x + xy - 2x^3 &= 2 \\ xy &= 2 + 2x^3 - x \\ y &= 2x^{-1} + 2x^2 - 1 \quad (*) \\ \frac{dy}{dx} &= -2x^{-2} + 4x \end{aligned}$$

(c) Confirm that the two results are consistent by expressing the derivative in part (a) as a function of x alone.

$$\begin{aligned} 1 + x\frac{dy}{dx} + y - 6x^2 &= 0 \\ x\frac{dy}{dx} &= 6x^2 - y - 1 \\ \frac{dy}{dx} &= 6x - x^{-1}(y) - x^{-1} \quad \text{Sub } (*) \text{ in part (b) into } y \text{ here.} \\ &= 6x - x^{-1}(2x^{-1} + 2x^2 - 1) - x^{-1} \\ &= 6x - 2x^{-2} - 2x + x^{-1} - x^{-1} \\ &= -2x^{-2} + 4x \end{aligned}$$

We see that (b) and (c) each give $\boxed{\dfrac{dy}{dx} = 4x - \dfrac{2}{x^2}}$.

3. Find dy/dx by implicit differentiation.

(a)
$$x^3 + y^3 = 3xy^2$$

$$\begin{aligned}
\frac{d}{dx}(x^3 + y^3) &= \frac{d}{dx}(3xy^2) \\
3x^2 + 3y^2\frac{dy}{dx} &= 3y^2 + 3x(2y)\frac{dy}{dx} \\
\frac{dy}{dx}(3y^2 - 6xy) &= 3y^2 - 3x^2 \\
\frac{dy}{dx} &= \frac{3y^2 - 3x^2}{3y^2 - 6xy} = \boxed{\frac{y^2 - x^2}{y^2 - 2xy}}
\end{aligned}$$

(b)
$$x^2y + 3xy^3 - x = 3$$

$$\begin{aligned}
\frac{d}{dx}(x^2y + 3xy^3 - x) &= \frac{d}{dx}(3) \\
2xy + x^2\frac{dy}{dx} + 3y^3 + 3x(3y^2)\frac{dy}{dx} - 1 &= 0 \\
\frac{dy}{dx}(x^2 + 9xy^2) &= 1 - 2xy - 3y^3 \\
\frac{dy}{dx} &= \boxed{\frac{1 - 2xy + 3y^3}{x^2 + 9xy^2}}
\end{aligned}$$

(c)
$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = 1$$

$$\begin{aligned}
\frac{d}{dx}\left(x^{-1/2} + y^{-1/2}\right) &= \frac{d}{dx}(1) \\
-\frac{1}{2}x^{-3/2} - \frac{1}{2}y^{-3/2}\frac{dy}{dx} &= 0 \\
-\frac{1}{2y^{3/2}}\frac{dy}{dx} &= \frac{1}{2x^{3/2}} \\
\frac{dy}{dx} &= \boxed{-\frac{y^{3/2}}{x^{3/2}}}
\end{aligned}$$

(d)

$$\sin(x^2y^2) = x$$

$$\frac{d}{dx}(\sin(x^2y^2)) = \frac{d}{dx}(x)$$

$$\cos(x^2y^2)\frac{d}{dx}(x^2y^2) = 1$$

$$\cos(x^2y^2)\left(2xy^2 + x^2(2y)\frac{dy}{dx}\right) = 1$$

$$2xy^2 + 2x^2y\frac{dy}{dx} = \frac{1}{\cos(x^2y^2)}$$

$$2x^2y\frac{dy}{dx} = \frac{1}{\cos(x^2y^2)} - 2xy^2$$

$$\frac{dy}{dx} = \frac{1}{2x^2y\cos(x^2y^2)} - \frac{2xy^2}{2x^2y}$$

$$= \frac{1}{2x^2y\cos(x^2y^2)} - \frac{y}{x}$$

$$= \frac{1}{2x^2y\cos(x^2y^2)} - \frac{y}{x}\cdot\frac{2xy\cos(x^2y^2)}{2xy\cos(x^2y^2)}$$

$$= \boxed{\frac{1 - 2xy^2\cos(x^2y^2)}{2x^2y\cos(x^2y^2)}}$$

4. Find d^2y/dx^2 by implicit differentiation.

(a) $$2x^2 - 3y^2 = 4$$

$$\begin{aligned}
\frac{d}{dx}(2x^2 - 3y^2) &= \frac{d}{dx}(4) \\
4x - 6y\frac{dy}{dx} &= 0 \\
-6y\frac{dy}{dx} &= -4x \\
\frac{dy}{dx} &= \frac{2x}{3y}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{d}{dx}\left(\frac{2x}{3y}\right) \\
\frac{d^2y}{dx^2} &= \frac{3y(2) - 2x(3)\frac{dy}{dx}}{(3y)^2} \\
&= \frac{6y - 6x\left(\frac{2x}{3y}\right)}{9y^2} \\
&= \frac{\left(\frac{2}{y}\right)(3y^2 - x^2)}{9y^2} \\
&= \boxed{\frac{2(3y^2 - x^2)}{9y^3}}
\end{aligned}$$

(b) $$xy + y^2 = 2$$

$$\begin{aligned}
\frac{d}{dx}(xy + y^2) &= \frac{d}{dx}(2) \\
y + x\frac{dy}{dx} + 2y\frac{dy}{dx} &= 0 \quad (*) \\
\frac{dy}{dx}(x + 2y) &= -y \\
\frac{dy}{dx} &= -\frac{y}{x + 2y}
\end{aligned}$$

Now it is easier to differentiate $(*)$ again instead of the above expression for $dy/dx\ldots$

$$\frac{d}{dx}\left(y + x\frac{dy}{dx} + 2y\frac{dy}{dx}\right) = \frac{d}{dx}(0)$$

$$\frac{dy}{dx} + x\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\cdot\frac{dy}{dx} = 0$$

$$2\frac{dy}{dx} + \frac{d^2y}{dx^2}(x+2y) + 2\left(\frac{dy}{dx}\right)^2 = 0$$

$$\begin{aligned}
\frac{d^2y}{dx^2}(x+2y) &= -2\left(\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2\right) \\
&= -2\left(-\frac{y}{x+2y} + \left(-\frac{y}{x+2y}\right)^2\right) \\
&= -2\left(-\frac{y}{x+2y} + \frac{y^2}{(x+2y)^2}\right) \\
&= -2\left(-\frac{y(x+2y)+y^2}{(x+2y)^2}\right) \\
&= \frac{2xy+4y^2-2y^2}{(x+2y)^2}
\end{aligned}$$

$$\frac{d^2y}{dx^2}(x+2y) = \frac{2xy+2y^2}{(x+2y)^2}$$

$$\rightarrow \quad \boxed{\frac{d^2y}{dx^2} = \frac{2y(x+y)}{(x+2y)^3}}$$

5. Use implicit differentiation to find the slope of the tangent line to the curve at the specified point.

(a) $$x^4 + y^4 = 16; \qquad \left(1, \sqrt[4]{15}\right)$$

$$\begin{aligned}
\frac{d}{dx}(x^4 + y^4) &= \frac{d}{dx}(16) \\
4x^3 + 4y^3\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= -\frac{x^3}{y^3} \\
&= -\frac{(1)^3}{(\sqrt[4]{15})^3} \approx \boxed{-0.1312}
\end{aligned}$$

(b) $$2(x^2 + y^2)^2 = 25(x^2 - y^2); \qquad (3, 1)$$

$$\begin{aligned}
\frac{d}{dx}(2(x^2 + y^2)^2) &= \frac{d}{dx}(25(x^2 - y^2)) \\
4(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) &= 50x - 50y\frac{dy}{dx}
\end{aligned}$$

Rather than try and solve for dy/dx now, we can just go ahead and plug in $x = 3,\ y = 1$ and solve for dy/dx with numbers which is much easier...

$$\begin{aligned}
4(3^2 + 1^2)\left(2(3) + 2(1)\frac{dy}{dx}\right) &= 50(3) - 50(1)\frac{dy}{dx} \\
40\left(6 + 2\frac{dy}{dx}\right) &= 150 - 50\frac{dy}{dx} \\
240 + 80\frac{dy}{dx} &= 150 - 50\frac{dy}{dx} \\
130\frac{dy}{dx} &= -90 \\
\frac{dy}{dx} &= \boxed{-\frac{9}{13}}
\end{aligned}$$