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13 Implicit Differentiation

This sections highlights the difference between explicit and implicit expressions, and focuses on the differentiation of the latter, which can be a very useful tool in mathematics. By the end of this section, you should have the following skills:

- An understanding of the definition of explicit and implicit functions and differentiation.
- Find dy/dx from an implicit relation, calculated my differentiating implicitly.
- Find the tangent of a curve at a given point using implicit differentiation.

13.1 Explicit and Implicit Expressions

A function written in the form $y = f(x)$ is called an **explicit function of** x i.e. it is immediately clear what the function is. However, if y is given by an expression of the form $y^2 + 2y = x$ then we have to do some work in order to express y as a function of x. Such an expression is called an **implicit relation**. Can we find the derivative of y?

In this case we have

$$\begin{aligned} y^2 + 2y &= x \Rightarrow \\ (y+1)^2 &= x + 1 \Rightarrow \\ y &= \pm\sqrt{x+1} - 1. \end{aligned}$$

Strictly speaking y is not a function of x as it is two valued. If we choose the positive square root then we obtain a function and we can differentiate to get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}}.$$

However, there are examples where we cannot solve easily for y in terms of x e.g.

$$y^3 + y^2 + y = 3x$$

and there are examples where it is **not possible** to solve for y in terms of x e.g.

$$y + \sin(y) = x.$$

13.2 Implicit Differentiation

In the last two cases the finding of the derivative dy/dx would seem to be impossible. However this is not the case as we can make progress by using implicit differentiation. Here we differentiate both sides of the equation by not only differentiating the expressions in x but also differentiating the expressions in y. The basic idea is we are assuming that y is a function of x and so we can use the normal rules of differentiation on expressions involving y, especially the Chain Rule.

Example 1 *Differentiate the following expressions with respect to x where y is a function of x.*

(a) y^2.

(b) xy.

(c) x^2y^2.

(d) e^y.

(e) ye^y.

(f) $\sin(y)$.

(g) $\cos(xy)$.

(h) $x/(1+y)$.

Solution.

(a) *Remember that y is a function of x. Hence using the Chain Rule we have*

$$\frac{dy^2}{dx} = 2y\frac{dy}{dx}.$$

(b) *We use the Product Rule here:*

$$\frac{d(xy)}{dx} = x\frac{dy}{dx} + y.$$

(c) *We use the Product and Chain Rules here:*

$$\frac{d(x^2y^2)}{dx} = x^2 2y\frac{dy}{dx} + 2xy^2.$$

(d) *We use the Chain Rule:*

$$\frac{d(e^y)}{dx} = e^y\frac{dy}{dx}.$$

(e) *We use the Product Rule here:*

$$\begin{aligned}\frac{d(ye^y)}{dx} &= \frac{dy}{dx}e^y + y\frac{d(e^y)}{dx}\\ &= \frac{dy}{dx}e^y + ye^y\frac{dy}{dx}.\end{aligned}$$

(f) *We use the Chain Rule:*

$$\frac{d(\sin(y))}{dx} = \cos(y)\frac{dy}{dx}.$$

(g) *We use the Product and Chain Rules:*

$$\begin{aligned}\frac{d(\cos(xy))}{dx} &= -\frac{d(xy)}{dx}\sin(xy)\\ &= -(x\frac{dy}{dx} + y)\sin(xy)\end{aligned}$$

(h) *We use y' to denote dy/dx.*
Let

$$\begin{aligned} z &= x/(1+y) \Rightarrow \\ \frac{dz}{dx} &= \frac{1(1+y) - xy'}{(1+y)^2} \\ &= \frac{1+y-xy'}{(1+y)^2}. \end{aligned}$$

13.3 Finding dy/dx from an implicit relation

The following example is a simple example in finding dy/dx from a relation.

Example 2 *Consider the expression*

$$y + 2x = 3 - 2x$$

Find dy/dx.
Solution.

Differentiate both sides of the equation:

$$\begin{aligned} \frac{d}{dx}(y+2x) &= \frac{d}{dx}(3-2x) \Rightarrow \\ \frac{dy}{dx} + 2 &= -2 \Rightarrow \\ \frac{dy}{dx} &= -4. \end{aligned}$$

Of course this is an easy example. I could have rearranged the implicit expression to get

$$y = 3 - 4x$$

and then $dy/dx = -4$.

Example 3 *Consider the implicit expression for y as function of x*

$$y^2 + 2y = x$$

Find dy/dx.
Solution.

Differentiate both sides of the equation:

$$\begin{aligned} \frac{d}{dx}(y^2+2y) &= \frac{d}{dx}(x) \Rightarrow \\ 2y\frac{dy}{dx} + 2\frac{dy}{dx} &= 1 \Rightarrow \text{ on collecting terms in } dy/dx \\ (2y+2)\frac{dy}{dx} &= 1 \Rightarrow \\ \frac{dy}{dx} &= \frac{1}{2y+2}. \end{aligned}$$

Note that the derivative involves y. This is often the case in using implicit differentiation.
Note.
We differentiated y^2 using the Chain Rule as y is considered as a function of x and so

$$\frac{d(y^2)}{dx} = 2y\frac{dy}{dx}.$$

Example 4 *Consider the implicit expression for y as function of x*

$$y^3 + y^2 + xy = \sin(x).$$

Find dy/dx.
Solution.

Differentiate both sides of the equation:

$$\begin{aligned}
\frac{d}{dx}(y^3 + y^2 + xy) &= \frac{d}{dx}(\sin(x)) \Rightarrow \\
3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} + y + x\frac{dy}{dx} &= \cos(x) \Rightarrow \ \textit{on collecting terms in } dy/dx \\
(3y^2 + 2y + x)\frac{dy}{dx} &= \cos(x) - y \Rightarrow \\
\frac{dy}{dx} &= \frac{\cos(x) - y}{3y^2 + 2y + x}.
\end{aligned}$$

Notes.

(i)

$$\frac{d(y^2)}{dx} = 2y\frac{dy}{dx}.$$

(ii) We differentiated y^3 using the Chain Rule and so

$$\frac{d(y^3)}{dx} = 3y^2\frac{dy}{dx}.$$

(iii) We differentiated xy using the Product Rule and so

$$\frac{d(xy)}{dx} = y\frac{dx}{dx} + x\frac{dy}{dx} = y + x\frac{dy}{dx}.$$

Exercise 1

Consider the following implicit expressions for y as function of x. In each case find dy/dx in terms of $x,\ y$.

(a) $x^2 - y^2 = 4$.

(b) $y + x^2y^2 = x$.

(c) $y^2 + x = y^3$.

(d) $\sin(y) + \cos(x) = y$.

(e) $e^y - e^x = y^2 - x^2$.

(f) $\tan(xy) - yx^2 = 1$.

Solutions to exercise 1

Note that we use y' for dy/dx in these solutions.
In each case we differentiate both sides with respect to x

(a) $2x - 2yy' = 0 \Rightarrow 2yy' = 2x \Rightarrow y' = x/y$.

(b)

$$\begin{aligned} y' + 2xy^2 + x^2 2yy' &= 1 \Rightarrow \\ y'(1 + 2x^2y) &= 1 - 2xy^2 \Rightarrow \\ y' &= \frac{1 - 2xy^2}{1 + 2x^2y}. \end{aligned}$$

(c)

$$\begin{aligned} 2yy' + 1 &= 3y^2y' \Rightarrow \\ y'(3y^2 - 2y) &= 1 \Rightarrow \\ y' &= \frac{1}{3y^2 - 2y}. \end{aligned}$$

(d)

$$\begin{aligned} \cos(y)y' - \sin(x) &= y' \Rightarrow \\ y'(\cos(y) - 1)) &= \sin(x) \Rightarrow \\ y' &= \frac{\sin(x)}{\cos(y) - 1}. \end{aligned}$$

(e)

$$\begin{aligned} e^y y' - e^x &= 2yy' - 2x \Rightarrow \\ y'(e^y - 2y) &= e^x - 2x \Rightarrow \\ y' &= \frac{e^x - 2x}{e^y - 2y}. \end{aligned}$$

(f)

$$\begin{aligned} (y + xy')\sec^2(xy) - 2xy - x^2y' &= 0 \Rightarrow \\ y'(x\sec^2(xy) - x^2) &= 2xy - y\sec^2(xy) \Rightarrow \\ y' &= \frac{2xy - y\sec^2(xy)}{x\sec^2(xy) - x^2}. \end{aligned}$$

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13.4 Curves given by implicit expressions and their tangents

If we are given an implicit expression such as

$$y^3 + y = x$$

then we can plot a point on the curve given by this expression by giving x a value and solving for y.

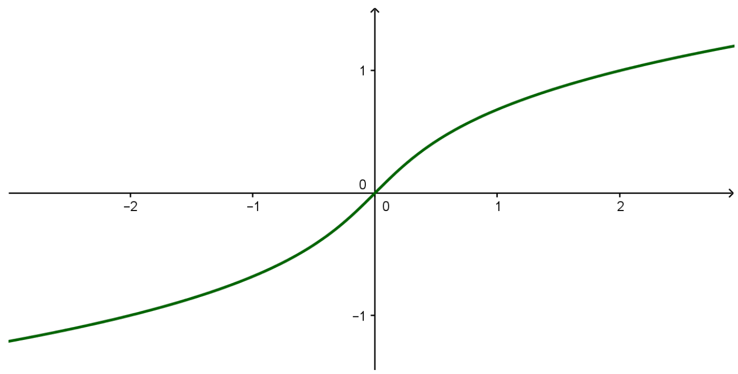
For example:

Put $x = 0$ and we get $y^3 + y = 0 \Rightarrow y(y^2 + 1) = 0 \Rightarrow y = 0$. Hence the point $(0, 0)$ lies on the curve.

Put $x = 2$ and we get $y^3 + y = 2 \Rightarrow (y - 1)(y^2 + y + 2) = 0 \Rightarrow y = 1$. Hence the point $(2, 1)$ lies on the curve.

Continuing in this way we can plot the curve given by the implicit expression.

Note it is not expected that you should be able to plot these graphs - they are displayed here for information.



Graph of $y^3 + y = x$.

We can now ask for the slope of the tangent to this curve at a particular point by finding dy/dx.

Example 5 *A curve is given by*

$$y^3 + y = x.$$

(a) Show that the point $(-2, -1)$ *lies on the curve.*

(b) Find dy/dx *by using implicit differentiation.*

(c) Find the slope of the tangent at the point $(-2, -1)$.

Solution.

(a) To show that the point $(-2, -1)$ *lies on the curve all we do is to substitute these values into the equation and see if they satisfy the equation.*
We get on putting $x = -2, \; y = -1$:

$$(-1)^3 + (-1) = -2$$

which is clearly true, hence the point lies on the curve.

(b) *Differentiate both sides of the equation to get:*

$$\begin{aligned}
\frac{d}{dx}(y^3 + y) &= \frac{d}{dx}(x) \Rightarrow \\
3y^2\frac{dy}{dx} + \frac{dy}{dx} &= 1 \Rightarrow \text{ on collecting terms in } dy/dx \\
(3y^2 + 1)\frac{dy}{dx} &= 1 \Rightarrow \\
\frac{dy}{dx} &= \frac{1}{3y^2 + 1}.
\end{aligned}$$

(c) *Putting* $x = -2, \quad y = -1$ *in the expression for* dy/dx *gives the slope of the tangent at the point* $(-2, -1)$ *i.e.*

$$\frac{dy}{dx} = \frac{1}{3 \times (-1)^2 + 1} = \frac{1}{4}.$$

Example 6 *A curve is given by*

$$y + \sin(y) = x^2 + x$$

(a) *Show that the point* $(0, 0)$ *lies on the curve.*

(b) *Find the slope of the tangent at the point* $(0, 0)$.

(c) *Find the equation of the tangent at the point* $(0, 0)$.

Solution.

(a) *To show that the point* $(0, 0)$ *lies on the curve all we do is to substitute these values into the equation and see if they satisfy the equation.*
We get on putting $x = 0, \ y = 0$:

$$0 + \sin(0) = 0^2 + 0$$

which is clearly true, hence the point lies on the curve.

(b) Differentiate both sides of the equation:

$$\begin{aligned}
\frac{d}{dx}(y+\sin(y)) &= \frac{d}{dx}(x^2+x) \Rightarrow \\
\frac{dy}{dx}+\cos(y)\frac{dy}{dx} &= 2x+1 \Rightarrow \text{ on collecting terms in } dy/dx \\
(1+\cos(y))\frac{dy}{dx} &= 2x+1 \Rightarrow \\
\frac{dy}{dx} &= \frac{2x+1}{1+\cos(y)}.
\end{aligned}$$

Hence at the point $(0,0)$ *we have the slope of the tangent is*

$$\frac{dy}{dx} = \frac{2\times 0+1}{1+\cos(0)} = \frac{1}{2}.$$

(c) The equation of the tangent at $(0,0)$ *is of the form*

$$y = \frac{1}{2}x + c$$

but at $x=0,\ \ y=0$ *hence* $c=0$.
So the equation of the tangent is

$$y = \frac{1}{2}x.$$

Note
On differentiating the implicit expression we found:

$$\frac{d(\sin(y))}{dx} = \cos(y)\frac{dy}{dx}.$$

Once again this is because we are using the Chain Rule where $\sin(y)$ *is a function of a function with* y *as the "innermost" function.*

Example 7 *A curve is given by the equation*

$$y^3 + 2y = \sin(x) + 3.$$

Find the slope of the curve at the point $(0, 1)$*.*
Also approximate the value of y *when* $x = 0.05$*.*
Solution.

In this example we use y' *as shorthand for* dy/dx*.*
Note that $(0, 1)$ *lies on the curve as we can see that* $x = 0, \quad y = 1$ *satisfies the equation.*
Differentiating both sides of the equation with respect to x *gives:*

$$\begin{aligned} 3y^2y' + 2y' &= \cos(x) \Rightarrow \\ (3y^2 + 2)y' &= \cos(x) \Rightarrow \\ y' &= \frac{\cos(x)}{3y^2 + 2}. \end{aligned}$$

At $x = 0, \quad y = 1$*:*

$$y' = \frac{1}{3+2} = \frac{1}{5}.$$

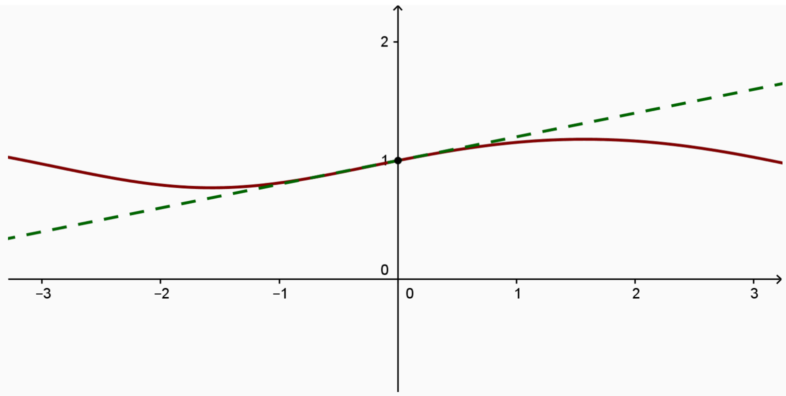
This is the slope of the curve at $(0, 1)$*.*
To approximate the value of y *at* $x = 0.05$ *we use the equation of the tangent at* $(0, 1)$ *which is of the form*

$$y = \frac{1}{5}x + c$$

and since $y = 1$ *when* $x = 0$ *we have* $c = 1$*. Hence the equation of the tangent is*

$$y = \frac{1}{5}x + 1.$$

The approximation for y *at* $x = 0.05$ *is then* $y = 0.05/5 + 1 = 1.01$*.*
The true value is $y = 1.009936398$ *and the error is* 0.000064 *to 6 decimal places.*
The graph together with the tangent at $(0, 1)$ *is Figure 2.*



Graph of $y^3 + 2y = \sin(x) + 3$ *with tangent at* $(0, 1)$.

Exercise 2

A curve is given by the equation

$$y^3 + y^2 = x^3 + 1.$$

Find the slope of the curve at the point $(-1, -1)$.
Also approximate the value of y when $x = -0.95$.

Solutions to exercise 2

Once again we use y' as shorthand for dy/dx.

Note that $(-1, -1)$ lies on the curve as we can see that $x = -1, \ \ y = -1$ satisfies the equation.
Differentiating both sides of the equation with respect to x gives:

$$\begin{aligned} 3y^2y' + 2yy' &= 3x^2 \Rightarrow \\ (3y^2 + 2y)y' &= 3x^2 \Rightarrow \\ y' &= \frac{3x^2}{3y^2 + 2y}. \end{aligned}$$

At $x=-1, \quad y=-1, \quad y'=\frac{3}{1}=3.$

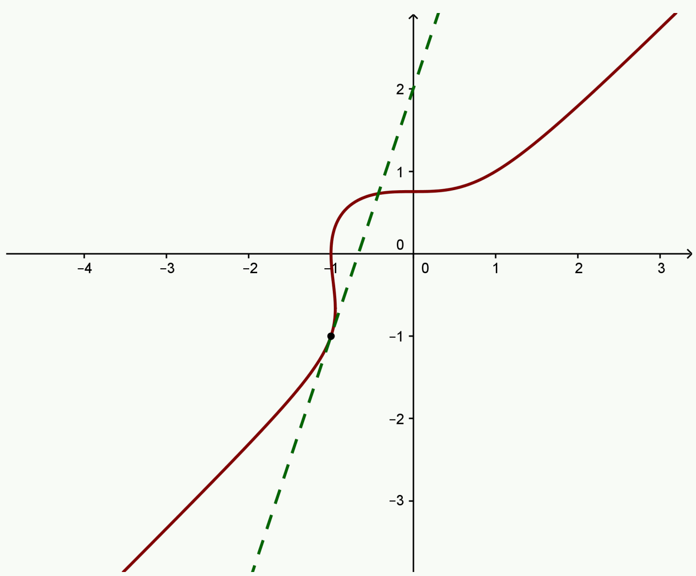
This is the slope of the curve at $(-1,-1)$.

To approximate the value of y at $x=-0.95$ we use the equation of the tangent at $(-1,-1)$ which is of the form $y=3x+c$ and since $y=-1$ when $x=-1$ we have $c=2$. Hence the equation of the tangent is $y=3x+2$.

The approximation for y at $x=-0.95$ is then

$$y = 3\times(-0.95)+2 = -0.85.$$

In fact, the true value as calculated by Maple (a powerful mathematics package) is $y=-0.7384525950$ and the error is therefore -0.1115 to 4 decimal places.
The graph together with the tangent at $(-1,-1)$ is given in below:



Graph of $y^3+y^2=x^3+1$ with tangent at $(-1,-1)$.

The next example uses the second derivative to estimate the error in using the tangent as an approximation.

Estimating the error using the second derivative.

Example 8 *Estimate the error in the tangent approximation above.*
Solution.

This question asks for an estimate of the error in the tangent approximation above. It has been stated that a reasonable estimate is:

$$\frac{(x-a)^2}{2}\frac{d^2y}{dx^2}$$

where $x = 0.05, \quad a = 0$ *and we evaluate the second derivative of* y *at* $x = 0, \quad y = 1$.
We denote the second derivative by y''.
We can find the value of the second derivative by differentiating

$$3y^2y' + 2y' = \cos(x)$$

again to get

$$\begin{aligned} 6y(y')^2 + 3y^2y'' + 2y'' &= -\sin(x) \Rightarrow \\ (3y^2+2)y'' &= -\sin(x) - 6y(y')^2 \Rightarrow \\ y'' &= -\frac{\sin(x) + 6y(y')^2}{3y^2+2}. \end{aligned}$$

Hence evaluating at $x = 0, \quad y = 1$ *and since* $y' = 1/5$ *at these values gives*

$$y'' = \frac{0 + 6 \times 1 \times 1/25}{5} = \frac{6}{125}.$$

The estimate of the error is

$$\frac{(0.05)^2}{2} \times \frac{6}{125} = 0.00006.$$

This is a good estimate as the true error is 0.000064 *(see above).*

Exercise 3

A curve is given by

$$xy + \sin(y) = x^2 + x.$$

(a) Show that the point $(0, 0)$ lies on the curve.

(b) Find the slope of the tangent at the point $(0, 0)$.

(c) Find the equation of the tangent at the point $(0, 0)$.

(d) Approximate the value of y when $x = -0.01$.

(e) Estimate the error in using the tangent approximation at $x = -0.01$.

Solutions to exercise 3

(a) We can see that $(0, 0)$ lies on the curve as $x = 0, \quad y = 0$ satisfies the equation.

(b) Differentiating both sides of the equation with respect to x gives:

$$\begin{aligned} y + xy' + \cos(y)y' &= 2x + 1 \Rightarrow \\ (x + \cos(y))y' &= 2x + 1 - y \Rightarrow \\ y' &= \frac{2x + 1 - y}{x + \cos(y)}. \end{aligned}$$

At $x = 0, \quad y = 0, \quad y' = \frac{1}{1} = 1$.
This is the slope of the curve at $(0, 0)$.

(c) The equation of the tangent at $(0, 0)$ is of the form $y = x + c$ and since $y = 0$ when $x = 0$ we have $c = 0$. Hence the equation of the tangent is $y = x$.

(d) Using the tangent, the approximate value for y is -0.01.

(e) Recall that this estimate is given by

$$\frac{(x-a)^2}{2}\frac{d^2y}{dx^2}$$

where $x = -0.01, \quad a = 0$ and we evaluate the second derivative of y at $x = 0, \quad y = 0$.
We denote the second derivative by y''.
We can find the value of the second derivative by differentiating $y + xy' + \cos(y)y' = 2x + 1$ again to get:

$$\begin{aligned} y' + y' + xy'' - \sin(y)(y')^2 + \cos(y)y'' &= 2 \Rightarrow \\ (x + \cos(y))y'' &= 2 + \sin(y)(y')^2 - 2y' \Rightarrow \\ y'' &= -\frac{2 + \sin(y)(y')^2 - 2y'}{x + \cos(y)}. \end{aligned}$$

Hence evaluating at $x = 0, \quad y = 0$ and since $y' = 1$ at these values gives $y'' = 0$ at $x = 0, \quad y = 0$.
The estimate of the error is then

$$\frac{(-0.01)^2}{2} \times 0 = 0.$$

This is in fact a good estimate as the true value of y at $x = -0.01$, as calculated by Maple, is -0.01000016836 and the error is 0.00000016836.

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13.5 Videos

Implicit Differentiation 1

Suppose you are not given y as an explicit function of x, but instead a relation between x and y. This video explains how to find the differential of y with respect to x.

Implicit Differentiation 2

This video shows how to find dy/dx for y = $\arctan(x)$.

Implicit Differentiation 3

Find the tangent at $(0, 0)$ and the equation of the normal at $(0, 0)$ for the curve given by the implicit relation $3y + 2x + x^3 = 2\sin(y)$.