Practice problems for sections on September 27th and 29th.

Here are some example problems about the product, fraction and chain rules for derivatives and implicit differentiation. If you notice any errors please let me know.

1. (easy) Find the equation of the tangent line of $f(x) = 2x^{3/2}$ at $x = 1$.

Solution: The derivative of f at $x = 1$ is $f'(1) = 3$ and so the equation of the tangent line is $y = 3x + b$, where b is determined by $y(1) = 3 + b = 2$, i.e. $b = -1$.

2. (easy) Differentiate $f(x) = e^x/\sqrt{x}$ and find the equation of the tangent line at $x = 1$.

Solution: The derivative of f is

$$f'(x) = \frac{e^x\sqrt{x} - e^x/(2\sqrt{x})}{x}$$

and at $x = 1$, $f'(1) = e - (e/2) = e/2$. The value of the function at $x = 1$ is e, so the tangent line has equation $y = (e/2)x + b$, where b is determined from $y(1) = (e/2) + b = e$, i.e. $b = e/2$.

3. (easy) Find the derivative of $f(x) = (x^2 \sin x)/(x^2 + 1)$.

Solution: This is a straightforward application of the quotient and product rules for derivatives:

$$f'(x) = \frac{(2x\sin x + x^2\cos x)(x^2+1) - 2x^3\sin x}{(x^2+1)^2} = \frac{2x\sin x + (x^4+x^2)\cos x}{(x^2+1)^2}\,.$$

4. (hard) Calculate $f^{(n)}(x)$ for $f(x) = x^n\, e^x$ at $x = 0$.

Solution: The n derivatives will produce a huge number of terms but after evaluation at $x = 0$ all with any x in front will vanish. Hence the only contribution to $f'(0)$ comes from the term where we have differentiated x^n n times (giving $n!$):

$$f^{(n)}(x) = n!\, e^x + (\text{terms which vanish at } x = 0)\,.$$

Therefore $f^{(n)}(0) = n!$.

5. (easy) Differentiate $f(x) = \sin x \cos x \tan x$.

Solution: First notice that the function is $f(x) = \sin^2 x$. Then using the product rule or chain rule $f'(x) = 2\sin x\cos x = \sin(2x)$.

6. (medium) Suppose $f(x)$ is a twice differentiable function satisfying $f(x^2) = f(x) + x^2$. What are $f'(1)$ and $f''(1)$?

Solution: Using the chain rule we get $2xf'(x^2) = f'(x) + 2x$ and $2f'(x^2) + 4x^2f''(x^2) = f''(x) + 2$. Plugging in $x = 1$ gives $f'(1) = 2$ and $4 + 4f''(1) = f''(1) + 2$, i.e. $f''(1) = -2/3$.

7. (easy) Differentiate $\sqrt{x}\,\cot x$.

Solution: The derivative of $\cot x$ is $-1/\sin^2 x$ and so

$$(\sqrt{x}\,\cot x)' = \frac{\cot x}{2\sqrt{x}} - \frac{\sqrt{x}}{\sin^2 x}\,.$$

8. (easy) Differentiate $f(x) = \exp\sqrt{x+1}$.

Solution: Write $u(x) = \sqrt{x+1}$ so that $f(x) = \exp u$. The chain rule gives

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \frac{e^u}{2\sqrt{x}} = \frac{1}{2\sqrt{x+1}}\,e^{\sqrt{x+1}}\,.$$

9. (medium) Differentiate $\exp(\sin(\exp x))$.

Solution: Let $v(x) = \exp x$ and $u(v) = \sin v$. Then $f(x) = \exp u$ and the chain rule gives

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dv}\frac{dv}{dx} = e^u\,\cos(v)\,e^x = \cos(\exp x)\,\exp(\sin(\exp x) + x)\,.$$

10. (medium) Differentiate $f(x) = \sqrt[3]{\sin \sqrt[3]{x}}$.

Solution: Let $v(x) = \sqrt[3]{x}$ and $u(v) = \sin v$. Then $f(x) = \sqrt[3]{u}$ and the chain rule gives

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dv}\frac{dv}{dx} = \frac{\cos v}{3u^{2/3}}\,\frac{1}{3x^{2/3}} = \frac{\cos \sqrt[3]{x}}{9\,(x \sin \sqrt[3]{x})^{2/3}}\,.$$

11. (medium) Suppose the derivative of $\ln x$ exists. Find it using the chain rule. Then show that the derivative of x^r is $r\,x^{r-1}$ for any real number r.

Solution: If the derivative of $\ln x$ exists, then since $\exp(\ln x) = x$, differentiation using the chain rule yields

$$(\ln x)' \exp(\ln x) = 1\,,$$

that is $(\ln x)' = 1/x$. This is indeed correct (since the derivative exists). Now $x^r = \exp(r \ln x)$ and so

$$(x^r)' = \frac{r}{x}\,\exp(r \ln x) = r\,x^{r-1}\,.$$

12. (hard) Using $(\ln x)' = 1/x$, differentiate the function $f(x) = x^x$.

Solution: Write $f(x) = \exp(x \ln x)$. Then the chain rule and the product rule give

$$f'(x) = (\ln x + 1)\exp(x \ln x) = (\ln x + 1)\,x^x\,.$$

13. (medium) What is the equation of the tangent line of the curve $xy^5 + 2x^2y - x^3 + y + 1 = 0$ at $x = 0$?

Solution: The curve passes through the point $(x, y) = (0, -1)$. Using implicit differentiation, we get

$$y^5 + 5xy^4y' + 4xy + 2x^2y' - 3x^2 + y' = 0$$

and at $(x, y) = (0, -1)$ this becomes

$$-1 + y'(0) = 0\,,$$

i.e. the slope of the tangent line at $x = 0$ is $y'(0) = 1$. Since the tangent line passes through $(0, -1)$, its equation is $y = x - 1$.

14. (medium) Find the derivative y' for the curve $x^2 + y^2 = \sin(xy)$.

Solution: Using implicit differentiation we get

$$2x + 2yy' = (y + xy')\cos(xy)$$

from which we can solve

$$y'(2y - x\cos(xy)) = y\cos(xy) - 2x$$

and further

$$y' = -\frac{x\cos(xy) - 2y}{y\cos(xy) - 2x}\,.$$

15. (medium) Find the points on the graph of $x^{5/2} + y^{5/2} = 1$ where the slope of the tangent line is -1.

Solution: Implicit differentiation gives

$$\frac{5}{2}\,x^{3/2} + \frac{5}{2}\,y^{3/2}y' = 0$$

and further

$$y' = -\frac{x^{3/2}}{y^{3/2}}\,.$$

This is -1 when $y = x$. Finally inserting $y = x$ into the equation of the graph we get $2x^{5/2} = 1$ from which we can easily solve $x = 2^{-2/5}$.

16. (medium) Find the derivative y' for the curve $\exp(xy) - \exp(xy^2) = \sin y$.

Solution: Implicit differentiation gives

$$(y + xy')e^{xy} - (y^2 + 2xyy')e^{xy^2} = y' \cos y \,,$$

from which we can solve

$$y'(xe^{xy} - 2xye^{xy^2} - \cos y) = y^2 e^{xy^2} - ye^{xy}$$

and finally

$$y' = \frac{y^2 e^{xy^2} - ye^{xy}}{xe^{xy} - 2xye^{xy^2} - \cos y} \,.$$