Lecture 13: Differentiation – Derivatives of Trigonometric Functions

Derivative of sin

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$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \end{aligned}$$

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Recall that using the Squeeze Theorem we proved that

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Recall that using the Squeeze Theorem we proved that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Derivative of sin – continued

Further, using the same approach as used in Example 13 we can show that

$$\lim_{h \to 0} \frac{\cos h - 1}{h}$$

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Thus we have

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You use the sum formula for cos to prove the corresponding differentiation formula for $\cos x$, which is

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as we predicted.

You use the sum formula for cos to prove the corresponding differentiation formula for $\cos x$, which is

$$\frac{d}{dx}\cos x = -\sin x$$

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We can prove the formula for the derivative of cos in a different way. Two basic trigonometric identities are

$$\begin{aligned} \sin\left(\frac{\pi}{2} - x\right) &= \cos x \\ \cos\left(\frac{\pi}{2} - x\right) &= \sin x \end{aligned}$$

since x and $\frac{\pi}{2} - x$ are complementary angles in a right triangle.

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An alternative way to simplify the previous expression is

$$\frac{d}{dx} \tan x$$

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$$\begin{aligned}\frac{d}{dx}\tan x &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} \\ &= 1 + \left(\frac{\sin x}{\cos x}\right)^2\end{aligned}$$

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The equality above can also be proved using the Pythagorean identity

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Most text books use the $\sec^2 x$ formula for the derivative of $\tan x$, but Maple and other symbolic differentiating programs use the $1 + \tan^2 x$ formula.

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Using basic differentiation rules as in the derivation of the derivative formula for tan we can find derivative formulas for all of the other trigonometric functions. Also, recall that when we derived the General Rule for the Exponential Function we stated that we would give all derivative formulas in a general form using the Chain Rule. In this form we introduce an intermediate variable u assumed to represent some function of x. With this assumption the derivative rules for all six basic trigonometric functions are:

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$$\frac{d}{dx}\sec u = \sec u\tan u\frac{du}{dx} \qquad \frac{d}{dx}\csc u$$

Derivatives the Six Trigonometric Functions

Using basic differentiation rules as in the derivation of the derivative formula for tan we can find derivative formulas for all of the other trigonometric functions. Also, recall that when we derived the General Rule for the Exponential Function we stated that we would give all derivative formulas in a general form using the Chain Rule. In this form we introduce an intermediate variable u assumed to represent some function of x. With this assumption the derivative rules for all six basic trigonometric functions are:

$$\frac{d}{dx}\sin u = \cos u \frac{du}{dx} \qquad \frac{d}{dx}\cos u = -\sin u \frac{du}{dx}$$

$$\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx} = \left(1 + \tan^2 u\right)\frac{du}{dx}$$

$$\frac{d}{dx}\sec u = \sec u \tan u \frac{du}{dx} \qquad \frac{d}{dx}\csc u = -\csc u \cot u \frac{du}{dx}$$

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$$\frac{d}{dx}\cot u$$

Derivatives the Six Trigonometric Functions

Using basic differentiation rules as in the derivation of the derivative formula for tan we can find derivative formulas for all of the other trigonometric functions. Also, recall that when we derived the General Rule for the Exponential Function we stated that we would give all derivative formulas in a general form using the Chain Rule. In this form we introduce an intermediate variable u assumed to represent some function of x. With this assumption the derivative rules for all six basic trigonometric functions are:

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$$\frac{d}{dx}\sec u = \sec u\tan u\frac{du}{dx} \qquad \frac{d}{dx}\csc u = -\csc u\cot u\frac{du}{dx}$$

$$\frac{d}{dx}\cot u = -\csc^2 u\frac{du}{dx} = -\left(1+\cot^2 u\right)\frac{du}{dx}$$

Example 46 – Differentiating with Trig Functions

Find and simplify the indicated derivative(s) of each function.

(a) Find $f'(x)$ and $f''(x)$ for $f(x) = x^2 \cos(3x)$.

(b) Find $\frac{ds}{dt}$ for $s = \frac{\cos t}{\sin t + \cos t}$.

(c) Find $C'(x)$ for $C(x) = \tan\left(e^{\sqrt{1+x^2}}\right)$.

Solution: Example 46(a)

Using the Product Rule followed by the Chain Rule (for $\cos(3x)$) gives

$$f'(x)$$

Solution: Example 46(a)

Using the Product Rule followed by the Chain Rule (for $\cos(3x)$) gives

$$f'(x) = 2x\cos(3x) + x^2[-3\sin(3x)]$$

Solution: Example 46(a)

Using the Product Rule followed by the Chain Rule (for $\cos(3x)$) gives

$$\begin{aligned} f'(x) &= 2x\cos(3x) + x^2\left[-3\sin(3x)\right] \\ &= 2x\cos(3x) - 3x^2\sin(3x) \end{aligned}$$

Solution: Example 46(a)

Using the Product Rule followed by the Chain Rule (for $\cos(3x)$) gives

$$\begin{aligned} f'(x) &= 2x\cos(3x) + x^2\left[-3\sin(3x)\right] \\ &= 2x\cos(3x) - 3x^2\sin(3x) \\ &= x\left[2\cos(3x) - 3x\sin(3x)\right] \end{aligned}$$

Solution: Example 46(a)

Using the Product Rule followed by the Chain Rule (for $\cos(3x)$) gives

$$\begin{aligned} f'(x) &= 2x\cos(3x) + x^2\left[-3\sin(3x)\right] \\ &= 2x\cos(3x) - 3x^2\sin(3x) \\ &= x\left[2\cos(3x) - 3x\sin(3x)\right] \end{aligned}$$

For $f''(x)$ use the expression in the second line. Again using the Product and Chain Rules gives

$$f''(x)$$

Solution: Example 46(a)

Using the Product Rule followed by the Chain Rule (for $\cos(3x)$) gives

$$\begin{aligned} f'(x) &= 2x\cos(3x) + x^2[-3\sin(3x)] \\ &= 2x\cos(3x) - 3x^2\sin(3x) \\ &= x[2\cos(3x) - 3x\sin(3x)] \end{aligned}$$

For $f''(x)$ use the expression in the second line. Again using the Product and Chain Rules gives

$$f''(x) = 2\cos(3x) - 6x\sin(3x) - 6x\sin(3x) - 9x^2\cos(3x)$$

Solution: Example 46(a)

Using the Product Rule followed by the Chain Rule (for $\cos(3x)$) gives

$$\begin{aligned} f'(x) &= 2x\cos(3x) + x^2\left[-3\sin(3x)\right] \\ &= 2x\cos(3x) - 3x^2\sin(3x) \\ &= x\left[2\cos(3x) - 3x\sin(3x)\right] \end{aligned}$$

For $f''(x)$ use the expression in the second line. Again using the Product and Chain Rules gives

$$\begin{aligned} f''(x) &= 2\cos(3x) - 6x\sin(3x) - 6x\sin(3x) - 9x^2\cos(3x) \\ &= \left(2 - 9x^2\right)\cos(3x) - 12x\sin(3x) \end{aligned}$$

Solution: Example 46(b)

Using the Quotient Rule gives

$$\frac{ds}{dt}$$

Solution: Example 46(b)

Using the Quotient Rule gives

$$\frac{ds}{dt} = \frac{-\sin t\,(\sin t + \cos t) - \cos t\,(\cos t - \sin t)}{(\sin t + \cos t)^2}$$

Solution: Example 46(b)

Using the Quotient Rule gives

$$\begin{aligned}\frac{ds}{dt} &= \frac{-\sin t\,(\sin t + \cos t) - \cos t\,(\cos t - \sin t)}{(\sin t + \cos t)^2} \\ &= \frac{-\sin^2 t - \sin t \cos t - \cos^2 t + \cos t \sin t}{(\sin t + \cos t)^2}\end{aligned}$$

Solution: Example 46(b)

Using the Quotient Rule gives

$$\begin{aligned}\frac{ds}{dt} &= \frac{-\sin t\,(\sin t + \cos t) - \cos t\,(\cos t - \sin t)}{(\sin t + \cos t)^2} \\ &= \frac{-\sin^2 t - \sin t \cos t - \cos^2 t + \cos t \sin t}{(\sin t + \cos t)^2} \\ &= \frac{-\left(\sin^2 t + \cos^2 t\right)}{(\sin t + \cos t)^2}\end{aligned}$$

Solution: Example 46(b)

Using the Quotient Rule gives

$$\begin{aligned}\frac{ds}{dt} &= \frac{-\sin t\,(\sin t+\cos t)-\cos t\,(\cos t-\sin t)}{(\sin t+\cos t)^2}\\ &= \frac{-\sin^2 t-\sin t\cos t-\cos^2 t+\cos t\sin t}{(\sin t+\cos t)^2}\\ &= \frac{-\left(\sin^2 t+\cos^2 t\right)}{(\sin t+\cos t)^2}\\ &= -\frac{1}{(\sin t+\cos t)^2}\end{aligned}$$

Solution: Example 46(b)

Using the Quotient Rule gives

$$\begin{aligned}\frac{ds}{dt} &= \frac{-\sin t\,(\sin t+\cos t)-\cos t\,(\cos t-\sin t)}{(\sin t+\cos t)^2}\\ &= \frac{-\sin^2 t-\sin t\cos t-\cos^2 t+\cos t\sin t}{(\sin t+\cos t)^2}\\ &= \frac{-\left(\sin^2 t+\cos^2 t\right)}{(\sin t+\cos t)^2}\\ &= -\frac{1}{(\sin t+\cos t)^2}\end{aligned}$$

This example illustrates the fact that when simplifying derivatives involving trig functions, you sometimes need to use standard trigonometric identities.

Solution: Example 46(c)

This is a composite function with

Solution: Example 46(c)

This is a composite function with

$$C(x)$$

Solution: Example 46(c)

This is a composite function with

$$C(x) = f\left(g\left(h\left(k(x)\right)\right)\right)$$

Solution: Example 46(c)

This is a composite function with

$$C(x) = f\left(g\left(h\left(k(x)\right)\right)\right)$$

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$$f(x)$$

Solution: Example 46(c)

This is a composite function with

$$C(x) = f\left(g\left(h\left(k(x)\right)\right)\right)$$

where

$$f(x) = \tan x,$$

Solution: Example 46(c)

This is a composite function with

$$C(x) = f\left(g\left(h\left(k(x)\right)\right)\right)$$

where

$$f(x) = \tan x,\ g(x)$$

Solution: Example 46(c)

This is a composite function with

$$C(x) = f\left(g\left(h\left(k(x)\right)\right)\right)$$

where

$$f(x) = \tan x,\ g(x) = e^{x},$$

Solution: Example 46(c)

This is a composite function with

$$C(x) = f\left(g\left(h\left(k(x)\right)\right)\right)$$

where

$$f(x) = \tan x,\ g(x) = e^x,\ h(x)$$

Solution: Example 46(c)

This is a composite function with

$$C(x) = f\left(g\left(h\left(k(x)\right)\right)\right)$$

where

$$f(x) = \tan x,\ g(x) = e^x,\ h(x) = \sqrt{x} = x^{1/2},$$

Solution: Example 46(c)

This is a composite function with

$$C(x) = f\left(g\left(h\left(k(x)\right)\right)\right)$$

where

$$f(x) = \tan x,\ g(x) = e^x,\ h(x) = \sqrt{x} = x^{1/2},\ k(x)$$

Solution: Example 46(c)

This is a composite function with

$$C(x) = f\left(g\left(h\left(k(x)\right)\right)\right)$$

where

$$f(x) = \tan x,\ g(x) = e^x,\ h(x) = \sqrt{x} = x^{1/2},\ k(x) = 1 + x^2$$

Solution: Example 46(c)

This is a composite function with

$$C(x) = f\left(g\left(h\left(k(x)\right)\right)\right)$$

where

$$f(x) = \tan x,\ g(x) = e^x,\ h(x) = \sqrt{x} = x^{1/2},\ k(x) = 1 + x^2$$

Using the Chain Rule three times gives

$$C'(x)$$

Solution: Example 46(c)

This is a composite function with

$$C(x) = f\left(g\left(h\left(k(x)\right)\right)\right)$$

where

$$f(x) = \tan x,\ g(x) = e^x,\ h(x) = \sqrt{x} = x^{1/2},\ k(x) = 1 + x^2$$

Using the Chain Rule three times gives

$$C'(x) = f'\left(g\left(h\left(k(x)\right)\right)\right) g'\left(h\left(k(x)\right)\right) h'\left(k(x)\right) k'(x)$$

Solution: Example 46(c)

This is a composite function with

$$C(x) = f\left(g\left(h\left(k(x)\right)\right)\right)$$

where

$$f(x) = \tan x,\ g(x) = e^x,\ h(x) = \sqrt{x} = x^{1/2},\ k(x) = 1 + x^2$$

Using the Chain Rule three times gives

$$\begin{aligned} C'(x) &= f'\left(g\left(h\left(k(x)\right)\right)\right) g'\left(h\left(k(x)\right)\right) h'\left(k(x)\right) k'(x) \\ &= \sec^2\left(e^{\sqrt{1+x^2}}\right)\left(e^{\sqrt{1+x^2}}\right)\left(\frac{1}{2}\right)\left(1+x^2\right)^{-1/2}(2x) \end{aligned}$$

Solution: Example 46(c)

This is a composite function with

$$C(x) = f\left(g\left(h\left(k(x)\right)\right)\right)$$

where

$$f(x) = \tan x,\ g(x) = e^x,\ h(x) = \sqrt{x} = x^{1/2},\ k(x) = 1 + x^2$$

Using the Chain Rule three times gives

$$\begin{aligned} C'(x) &= f'\left(g\left(h\left(k(x)\right)\right)\right) g'\left(h\left(k(x)\right)\right) h'\left(k(x)\right) k'(x) \\ &= \sec^2\left(e^{\sqrt{1+x^2}}\right)\left(e^{\sqrt{1+x^2}}\right)\left(\frac{1}{2}\right)\left(1+x^2\right)^{-1/2}(2x) \\ &= \frac{xe^{\sqrt{1+x^2}}\sec^2\left(e^{\sqrt{1+x^2}}\right)}{\sqrt{1+x^2}} \end{aligned}$$

Example 47 – Damped Oscillations

Consider the function

$$q(t) = e^{-7t} \sin(24t)$$

This function describes **damped simple harmonic motion**. It gives the position of a mass attached to a spring relative to the equilibrium (resting) position of the spring. A frictional force acts to gradually slow the mass.

(a) Find $q'(t)$ and $q''(t)$ and explain their meaning in terms of the damped oscillatory motion.

(b) Note that $q(0) = 0$. This means that the **initial position** of the mass is at the equilibrium position of the spring. Find the **initial velocity** of the mass. Also find the velocity when the mass first returns to the equilibrium position.

(c) Draw a graph of the function $q(t)$.

(d) Find the first two times when the oscillating mass turns around. Show the corresponding points on the graph of $q(t)$.

(e) Show that the function $q(t)$ satisfies the differential equation

$$\frac{d^2q}{dt^2} + 14\frac{dq}{dt} + 625q = 0$$

Solution: Example 47(a)

Using the Product and Chain Rules gives

Solution: Example 47(a)

Using the Product and Chain Rules gives

$$q'(t)$$

Solution: Example 47(a)

Using the Product and Chain Rules gives

$$q'(t) = -7e^{-7t}\sin(24t) + e^{-7t}\left[24\cos(24t)\right]$$

Solution: Example 47(a)

Using the Product and Chain Rules gives

$$\begin{aligned} q'(t) &= -7e^{-7t}\sin(24t) + e^{-7t}[24\cos(24t)] \\ &= e^{-7t}[24\cos(24t) - 7\sin(24t)] \end{aligned}$$

Solution: Example 47(a)

Using the Product and Chain Rules gives

$$\begin{aligned} q'(t) &= -7e^{-7t} \sin(24t) + e^{-7t} [24 \cos(24t)] \\ &= e^{-7t} [24 \cos(24t) - 7 \sin(24t)] \end{aligned}$$

From our interpretation of the derivative as a rate of change, we know that this is the velocity of the mass at time t.

Solution: Example 47(a)

Using the Product and Chain Rules gives

$$\begin{aligned} q'(t) &= -7e^{-7t}\sin(24t) + e^{-7t}[24\cos(24t)] \\ &= e^{-7t}[24\cos(24t) - 7\sin(24t)] \end{aligned}$$

From our interpretation of the derivative as a rate of change, we know that this is the velocity of the mass at time t.
Taking the derivative of the expression above for $q'(t)$ gives

Solution: Example 47(a)

Using the Product and Chain Rules gives

$$\begin{aligned} q'(t) &= -7e^{-7t}\sin(24t) + e^{-7t}\left[24\cos(24t)\right] \\ &= e^{-7t}\left[24\cos(24t) - 7\sin(24t)\right] \end{aligned}$$

From our interpretation of the derivative as a rate of change, we know that this is the velocity of the mass at time t.
Taking the derivative of the expression above for $q'(t)$ gives

$q''(t)$

Solution: Example 47(a)

Using the Product and Chain Rules gives

$$\begin{aligned} q'(t) &= -7e^{-7t}\sin(24t) + e^{-7t}\left[24\cos(24t)\right] \\ &= e^{-7t}\left[24\cos(24t) - 7\sin(24t)\right] \end{aligned}$$

From our interpretation of the derivative as a rate of change, we know that this is the velocity of the mass at time t.
Taking the derivative of the expression above for $q'(t)$ gives

$$q''(t) = -7e^{-7t}\left[24\cos(24t) - 7\sin(24t)\right] + e^{-7t}\left[-24^2\sin(24t) - 7(24)\cos(24t)\right]$$

Solution: Example 47(a)

Using the Product and Chain Rules gives

$$\begin{aligned} q'(t) &= -7e^{-7t}\sin(24t) + e^{-7t}\left[24\cos(24t)\right] \\ &= e^{-7t}\left[24\cos(24t) - 7\sin(24t)\right] \end{aligned}$$

From our interpretation of the derivative as a rate of change, we know that this is the velocity of the mass at time t.
Taking the derivative of the expression above for $q'(t)$ gives

$$\begin{aligned} q''(t) &= -7e^{-7t}\left[24\cos(24t) - 7\sin(24t)\right] + e^{-7t}\left[-24^2\sin(24t) - 7(24)\cos(24t)\right] \\ &= -e^{-7t}\left[14(24)\cos(24t) + \left(24^2 - 7^2\right)\sin(24t)\right] \end{aligned}$$

Solution: Example 47(a)

Using the Product and Chain Rules gives

$$\begin{aligned} q'(t) &= -7e^{-7t}\sin(24t) + e^{-7t}[24\cos(24t)] \\ &= e^{-7t}[24\cos(24t) - 7\sin(24t)] \end{aligned}$$

From our interpretation of the derivative as a rate of change, we know that this is the velocity of the mass at time t.
Taking the derivative of the expression above for $q'(t)$ gives

$$\begin{aligned} q''(t) &= -7e^{-7t}[24\cos(24t) - 7\sin(24t)] + e^{-7t}\left[-24^2\sin(24t) - 7(24)\cos(24t)\right] \\ &= -e^{-7t}\left[14(24)\cos(24t) + \left(24^2 - 7^2\right)\sin(24t)\right] \\ &= -e^{-7t}[336\cos(24t) + 527\sin(24t)] \end{aligned}$$

Solution: Example 47(a)

Using the Product and Chain Rules gives

$$\begin{aligned} q'(t) &= -7e^{-7t}\sin(24t) + e^{-7t}\left[24\cos(24t)\right] \\ &= e^{-7t}\left[24\cos(24t) - 7\sin(24t)\right] \end{aligned}$$

From our interpretation of the derivative as a rate of change, we know that this is the velocity of the mass at time t.
Taking the derivative of the expression above for $q'(t)$ gives

$$\begin{aligned} q''(t) &= -7e^{-7t}\left[24\cos(24t) - 7\sin(24t)\right] + e^{-7t}\left[-24^2\sin(24t) - 7(24)\cos(24t)\right] \\ &= -e^{-7t}\left[14(24)\cos(24t) + \left(24^2 - 7^2\right)\sin(24t)\right] \\ &= -e^{-7t}\left[336\cos(24t) + 527\sin(24t)\right] \end{aligned}$$

The rate of change of velocity is acceleration, so this gives the acceleration of the mass at time t.

Solution: Example 47(b)

Substituting $t = 0$ into the expression for the velocity $v(t) = q'(t)$ gives

$$v(0)$$

Solution: Example 47(b)

Substituting $t = 0$ into the expression for the velocity $v(t) = q'(t)$ gives

$$v(0) = e^0 \left[24 \cos(0) - 7 \sin(0)\right]$$

Solution: Example 47(b)

Substituting $t = 0$ into the expression for the velocity $v(t) = q'(t)$ gives

$$v(0) = e^0 \left[24 \cos(0) - 7 \sin(0)\right] = 24$$

Solution: Example 47(b)

Substituting $t = 0$ into the expression for the velocity $v(t) = q'(t)$ gives

$$v(0) = e^0 \left[24\cos(0) - 7\sin(0)\right] = 24$$

The mass returns to the equilibrium position when .

Solution: Example 47(b)

Substituting $t = 0$ into the expression for the velocity $v(t) = q'(t)$ gives

$$v(0) = e^0 \left[24\cos(0) - 7\sin(0)\right] = 24$$

The mass returns to the equilibrium position when $q(t) = 0$.

Solution: Example 47(b)

Substituting $t = 0$ into the expression for the velocity $v(t) = q'(t)$ gives

$$v(0) = e^0 \left[24\cos(0) - 7\sin(0)\right] = 24$$

The mass returns to the equilibrium position when $q(t) = 0$. The first time after $t = 0$ when this happens is when

Solution: Example 47(b)

Substituting $t = 0$ into the expression for the velocity $v(t) = q'(t)$ gives

$$v(0) = e^0 \left[24 \cos(0) - 7 \sin(0)\right] = 24$$

The mass returns to the equilibrium position when $q(t) = 0$. The first time after $t = 0$ when this happens is when

$$24t = \pi \Rightarrow t$$

Solution: Example 47(b)

Substituting $t = 0$ into the expression for the velocity $v(t) = q'(t)$ gives

$$v(0) = e^0 \left[24\cos(0) - 7\sin(0)\right] = 24$$

The mass returns to the equilibrium position when $q(t) = 0$. The first time after $t = 0$ when this happens is when

$$24t = \pi \Rightarrow t = \frac{\pi}{24}$$

Solution: Example 47(b)

Substituting $t = 0$ into the expression for the velocity $v(t) = q'(t)$ gives

$$v(0) = e^0 \left[24 \cos(0) - 7 \sin(0)\right] = 24$$

The mass returns to the equilibrium position when $q(t) = 0$. The first time after $t = 0$ when this happens is when

$$24t = \pi \Rightarrow t = \frac{\pi}{24}$$

The velocity at this time is

$$v\left(\frac{\pi}{24}\right)$$

Solution: Example 47(b)

Substituting $t = 0$ into the expression for the velocity $v(t) = q'(t)$ gives

$$v(0) = e^0 \left[24\cos(0) - 7\sin(0)\right] = 24$$

The mass returns to the equilibrium position when $q(t) = 0$. The first time after $t = 0$ when this happens is when

$$24t = \pi \Rightarrow t = \frac{\pi}{24}$$

The velocity at this time is

$$v\left(\frac{\pi}{24}\right) = e^{-7\pi/24}\left[24\cos(\pi) - 7\sin(\pi)\right]$$

Solution: Example 47(b)

Substituting $t = 0$ into the expression for the velocity $v(t) = q'(t)$ gives

$$v(0) = e^0 \left[24\cos(0) - 7\sin(0)\right] = 24$$

The mass returns to the equilibrium position when $q(t) = 0$. The first time after $t = 0$ when this happens is when

$$24t = \pi \Rightarrow t = \frac{\pi}{24}$$

The velocity at this time is

$$v\left(\frac{\pi}{24}\right) = e^{-7\pi/24}\left[24\cos(\pi) - 7\sin(\pi)\right] = -24e^{-7\pi/24}$$

Solution: Example 47(b)

Substituting $t = 0$ into the expression for the velocity $v(t) = q'(t)$ gives

$$v(0) = e^0 \left[24\cos(0) - 7\sin(0)\right] = 24$$

The mass returns to the equilibrium position when $q(t) = 0$. The first time after $t = 0$ when this happens is when

$$24t = \pi \Rightarrow t = \frac{\pi}{24}$$

The velocity at this time is

$$v\left(\frac{\pi}{24}\right) = e^{-7\pi/24}\left[24\cos(\pi) - 7\sin(\pi)\right] = -24e^{-7\pi/24} = -9.6$$

Solution: Example 47(b)

Substituting $t = 0$ into the expression for the velocity $v(t) = q'(t)$ gives

$$v(0) = e^0 \left[24\cos(0) - 7\sin(0)\right] = 24$$

The mass returns to the equilibrium position when $q(t) = 0$. The first time after $t = 0$ when this happens is when

$$24t = \pi \Rightarrow t = \frac{\pi}{24}$$

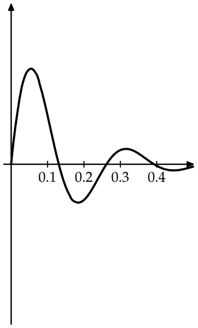
The velocity at this time is

$$v\left(\frac{\pi}{24}\right) = e^{-7\pi/24}\left[24\cos(\pi) - 7\sin(\pi)\right] = -24e^{-7\pi/24} = -9.6$$

This velocity is less than the initial velocity and in the opposite direction.

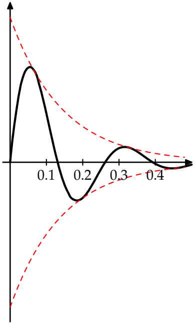
Solution: Example 47(c)

The graph of the function $q(t)$ looks like this. The amplitude of the motion decreases following an **envelope** given by the decaying exponential function e^{-7t}, as shown.



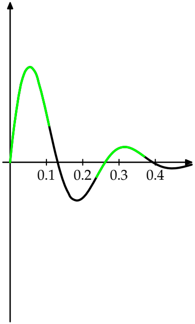
Solution: Example 47(c)

The graph of the function $q(t)$ looks like this. The amplitude of the motion decreases following an **envelope** given by the decaying exponential function e^{-7t}, as shown. As we saw in the last part, not only does the amplitude decrease, but so does the velocity.



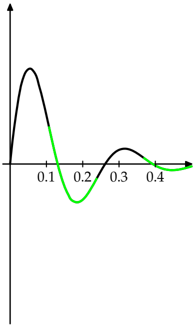
Solution: Example 47(c)

The graph of the function $q(t)$ looks like this. The amplitude of the motion decreases following an **envelope** given by the decaying exponential function e^{-7t}, as shown. As we saw in the last part, not only does the amplitude decrease, but so does the velocity. Further, note that where the graph is concave down, $q''(t) < 0$, the mass is decelerating. The velocity is getting less positive or more negative.



Solution: Example 47(c)

The graph of the function $q(t)$ looks like this. The amplitude of the motion decreases following an **envelope** given by the decaying exponential function e^{-7t}, as shown. As we saw in the last part, not only does the amplitude decrease, but so does the velocity. Further, note that where the graph is concave down, $q''(t) < 0$, the mass is decelerating. The velocity is getting less positive or more negative. And, where the graph is concave up, $q''(t) > 0$, the mass is accelerating. The velocity is getting more positive or less negative.



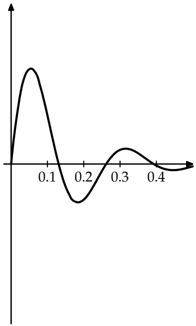
Solution: Example 47(d)

The mass turns around when $v(t) = q'(t) = 0$.
This happens when

Solution: Example 47(d)

The mass turns around when $v(t) = q'(t) = 0$. This happens when

$$24\cos(24t) - 7\sin(24t) = 0$$



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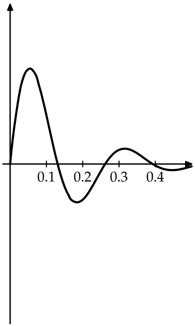
$$\Rightarrow \tan(24t) = \frac{24}{7}$$



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$$24\cos(24t) - 7\sin(24t) = 0$$
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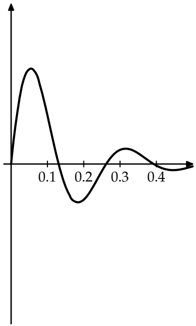
$$\Rightarrow 24t = \arctan\left(\frac{24}{7}\right) + k\pi$$



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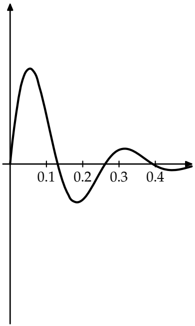
where k is any integer.



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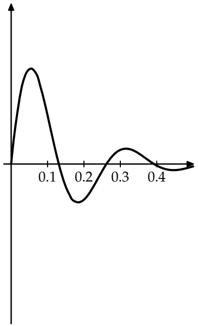

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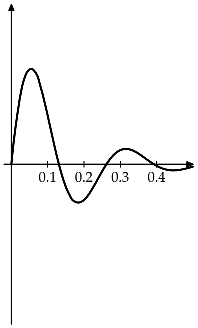

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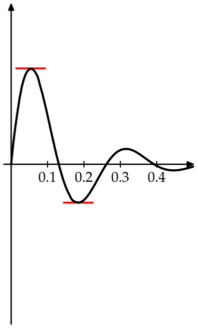

Solution: Example 47(e)

From part (a) we have

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&\quad = 0
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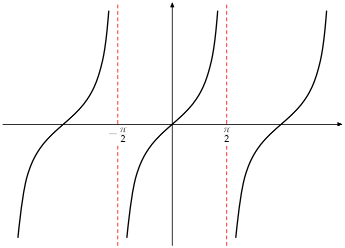

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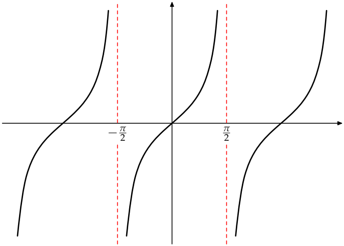
Recall that the graph of the tangent function looks like this. From this graph we realize that the tangent function is not one-to-one, and so does not have an inverse function.



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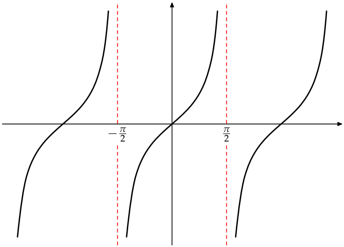

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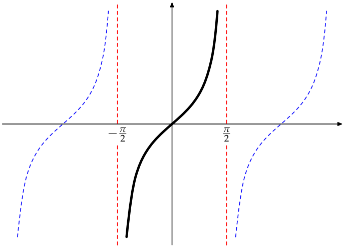

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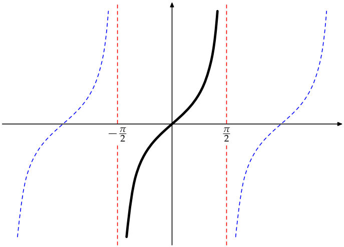

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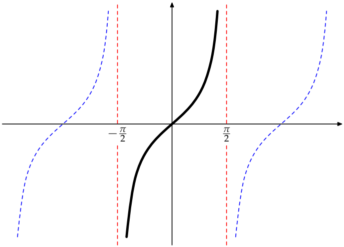

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So the inverse of the tangent function is defined as

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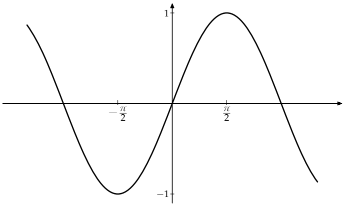

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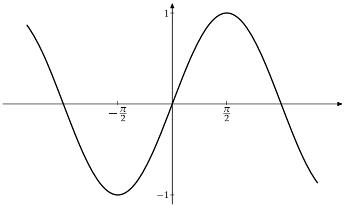

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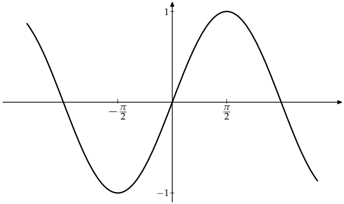

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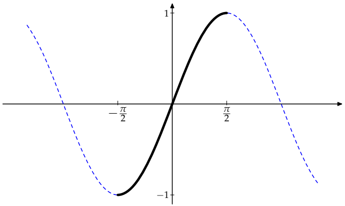

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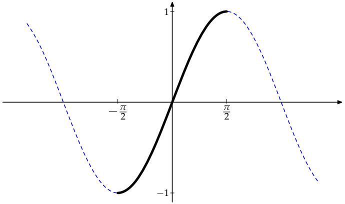

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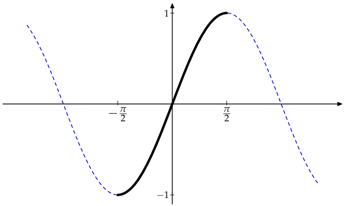

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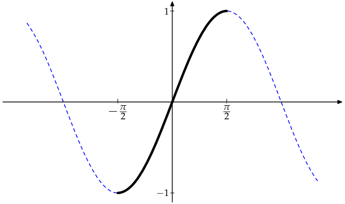

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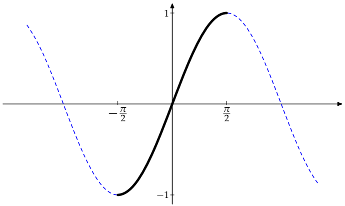

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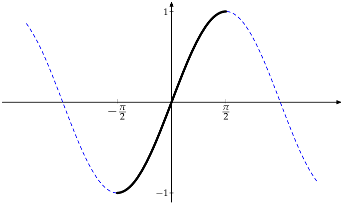

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Now for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ we have $\cos x \ge 0$, and using the basic Pythagorean identity $\cos^2 x + \sin^2 x = 1$, gives $\cos x = \sqrt{1 - \sin^2 x}$.

The Derivative of the arcsin Function

As with the arctan function, we have

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin x) = x \quad \text{for } -\tfrac{\pi}{2} \leq x \leq \tfrac{\pi}{2}$$

Taking the derivative of the first of the formulas above, as we just did for the arctan, gives

$$\frac{d}{dx}\sin(\arcsin x) = \cos(\arcsin x)\frac{d}{dx}\arcsin x = \frac{d}{dx}x = 1$$

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$$\frac{d}{dx}\sin(\arcsin x)$$

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Solving for $\frac{d}{dx}\arcsin x$ gives

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$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1 - x^2}}$$

Example 48 – Differentiating with Inverse Trig Functions

Find and simplify the indicated derivative(s) of each function.

(a) Find $f'(x)$ and $f''(x)$ for $f(x) = \left(1 + x^2\right) \arctan x$.

(b) Find $\frac{dy}{dx}$ for $y = \arcsin\left(\sqrt{1 - x^2}\right)$.

Solution: Example 48(a)

Using the Product Rule gives

$$f'(x)$$

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Using the Product Rule gives

$$f'(x) = 2x \arctan x + \left(1 + x^2\right)\left(\frac{1}{1 + x^2}\right)$$

Solution: Example 48(a)

Using the Product Rule gives

$$f'(x) = 2x \arctan x + \left(1 + x^2\right)\left(\frac{1}{1 + x^2}\right) = 2x \arctan x + 1$$

Solution: Example 48(a)

Using the Product Rule gives

$$f'(x) = 2x \arctan x + \left(1 + x^2\right)\left(\frac{1}{1 + x^2}\right) = 2x \arctan x + 1$$

Taking the derivative of the expression above, using the Product Rule again, gives

$$f''(x)$$

Solution: Example 48(a)

Using the Product Rule gives

$$f'(x) = 2x \arctan x + \left(1 + x^2\right)\left(\frac{1}{1 + x^2}\right) = 2x \arctan x + 1$$

Taking the derivative of the expression above, using the Product Rule again, gives

$$f''(x) = 2 \arctan x + 2x \left(\frac{1}{1 + x^2}\right)$$

Solution: Example 48(a)

Using the Product Rule gives

$$f'(x) = 2x \arctan x + \left(1 + x^2\right)\left(\frac{1}{1+x^2}\right) = 2x \arctan x + 1$$

Taking the derivative of the expression above, using the Product Rule again, gives

$$f''(x) = 2 \arctan x + 2x\left(\frac{1}{1+x^2}\right) = 2 \arctan x + \frac{2x}{1+x^2}$$

Solution: Example 48(b)

Using the Chain Rule gives

$$\frac{dy}{dx}$$

Solution: Example 48(b)

Using the Chain Rule gives

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1 - \left(\sqrt{1 - x^2}\right)^2}} \right) \left(\frac{1}{2}\right) \left(1 - x^2\right)^{-1/2} (-2x)$$

Solution: Example 48(b)

Using the Chain Rule gives

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1 - \left(\sqrt{1 - x^2}\right)^2}} \right) \left(\frac{1}{2} \right) \left(1 - x^2 \right)^{-1/2} (-2x)$$

$$= \left(\frac{1}{\sqrt{1 - (1 - x^2)}} \right) \left(\frac{-x}{\sqrt{1 - x^2}} \right)$$

Solution: Example 48(b)

Using the Chain Rule gives

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1 - \left(\sqrt{1-x^2}\right)^2}} \right) \left(\frac{1}{2} \right) \left(1 - x^2\right)^{-1/2} (-2x)$$

$$= \left(\frac{1}{\sqrt{1 - (1 - x^2)}} \right) \left(\frac{-x}{\sqrt{1-x^2}} \right) = \left(\frac{1}{\sqrt{x^2}} \right) \left(\frac{-x}{\sqrt{1-x^2}} \right)$$

Solution: Example 48(b)

Using the Chain Rule gives

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\right)\left(\frac{1}{2}\right)\left(1-x^2\right)^{-1/2}(-2x)$$

$$= \left(\frac{1}{\sqrt{1-(1-x^2)}}\right)\left(\frac{-x}{\sqrt{1-x^2}}\right) = \left(\frac{1}{\sqrt{x^2}}\right)\left(\frac{-x}{\sqrt{1-x^2}}\right) = -\frac{x}{|x|\sqrt{1-x^2}}$$

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Thus, if $0 \leq x < 1$, then

$$\frac{d}{dx}\arcsin\left(\sqrt{1-x^2}\right) = -\frac{1}{\sqrt{1-x^2}}$$

Solution: Example 48(b)

Using the Chain Rule gives

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\right)\left(\frac{1}{2}\right)\left(1-x^2\right)^{-1/2}(-2x)$$

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But you can show that

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$

Solution: Example 48(b)

Using the Chain Rule gives

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\right)\left(\frac{1}{2}\right)\left(1-x^2\right)^{-1/2}(-2x)$$

$$= \left(\frac{1}{\sqrt{1-(1-x^2)}}\right)\left(\frac{-x}{\sqrt{1-x^2}}\right) = \left(\frac{1}{\sqrt{x^2}}\right)\left(\frac{-x}{\sqrt{1-x^2}}\right) = -\frac{x}{|x|\sqrt{1-x^2}}$$

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But you can show that

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So is it true that $\arccos x = \arcsin\left(\sqrt{1-x^2}\right)$?