AP Calculus Notes Unit 5.4 Key

Particle Motion Problems

Particle motion problems deal with particles that are moving along the x – or y – axis. Thus, we are speaking of horizontal or vertical movement. The position, velocity, or acceleration of a particle's motion are DEFINED by functions, but the particle DOES NOT move along the graph of the function. It moves along an axis. Most of the time, we speak of movement along the x – axis. In units 6 and 7, particle motion is revisited. At that time, we will deal more with vertical motion. For the time, we will focus on horizontal motion of particles.

In this lesson, we develop the ideas of velocity and acceleration in terms of position. We will speak of two types of velocities and accelerations. Let's define average and instantaneous velocity in the box below.

Average and Instantaneous Velocity → Rate of change in position

s(t) or p(t) is position

avg. velocity: $\frac{s(b)-s(a)}{b-a}$, where s(t) is position on $[a,b]$

Instantaneous velocity: $s'(t)$

A particle's position is given by the function $p(t) = e^t \sin t$, where $p(t)$ is measured in centimeters and t is measured in seconds. Answer the following questions.

What is the average velocity on the interval $t = 1$ to $t = 3$ seconds? Indicate appropriate units of measure.

avg. vel: $= \frac{s(3)-s(1)}{3-1} = \frac{e^3\sin 3 - e^1 \sin 1}{2} \approx \frac{-.547}{-2} = \boxed{0.274 \text{ cm/s}}$

What is the instantaneous velocity of the particle at time $t = 1.5$. Indicate appropriate units of measure.

$p'(t) = e^t \sin t + e^t \cos t = e^t(\sin t + \cos t)$

$p'(1.5) = e^{1.5}(\sin 1.5 + \cos 1.5)$

$\boxed{p'(1.5) = 4.787 \text{ cm/s}}$

Before we proceed, a connection needs to be made. When given a function, $f(x)$, how did we find the slope of the secant line on the interval from $x = a$ to $x = b$? In terms of position of a particle, to what does the slope of the secant line correspond? To what does the instantaneous velocity correspond?

avg. velocity on $[a, b] \Rightarrow$ slope of secant line

Inst. velocity at $t=c \Rightarrow$ slope of tangent line $\Rightarrow s'(c)$

Average and Instantaneous Acceleration

avg. acceleration $= \dfrac{v(b)-v(a)}{b-a}$, where $v(t) = s'(t)$

Instantaneous Accel. $= v'(c)$ or $s''(c)$

A particle's position is given by the function $p(t) = e^t \sin t$, where $p(t)$ is measured in centimeters and t is measured in seconds. Answer the following questions.

What is the average acceleration on the interval $t = 1$ to $t = 3$ seconds? Indicate appropriate units of measure.

$p'(t)$ or $v(t) = e^t(\sin t + \cos t)$

avg. accel : $\dfrac{v(3)-v(1)}{3-1} = \boxed{-10.403 \text{ cm/s}^2}$

What is the instantaneous acceleration of the particle at time $t = 1.5$.

$p'(t) = e^t(\sin t + \cos t)$

* use calc!

$p''(1.5) = a(1.5) = \boxed{0.634 \text{ cm/s}^2}$

In summary, let's correlate the concepts of position, velocity, and acceleration to what we already know about a function and its first and second derivative.

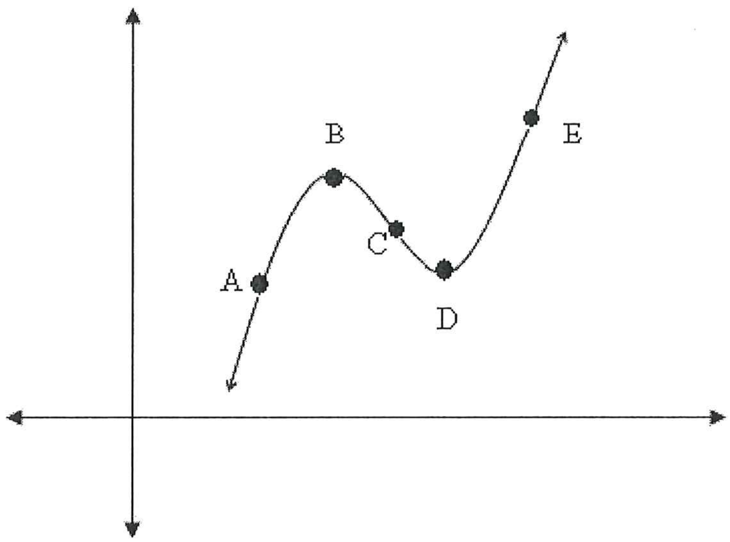
$f(x)$	corresponds with	position	
$f'(x)$	corresponds with	velocity	speed $\downarrow$ $\lvert v \rvert$
$f''(x)$	corresponds with	acceleration	

Let's summarize our relationships between position, velocity and acceleration below.

Velocity $f'(x)$	***Position (Motion of the Particle)*** $f(x)$
Is = 0 or is undefined	motion stops (potential turn around)
Is > 0	is increasing
Is < 0	is decreasing
Changes from positive to negative	Increasing → decreasing
Changes from negative to positive	Decreasing → Increasing

Acceleration	***Velocity***
Is = 0 or is undefined	potential change in velocity
Is > 0	is increasing
Is < 0	is decreasing
Changes from positive to negative	changes from inc → dec
Changes from negative to positive	changes from dec → inc.

The graph below represents the position, $s(t)$, of a particle which is moving along the x axis.



- At which point(s) is the velocity equal to zero? Justify your answer.

B, D $s(t)$ has a rel. max or min which means $v(t)=0$ there.

- At which point(s) does the acceleration equal zero? Justify your answer.

C b/c $s(t)$ has a pt. of inflection at c $\therefore$ $a(t) = 0$.

- On what interval(s) is the particle's velocity positive? Justify your answer.

$v(t) > 0$ on $(-\infty, B) \cup (D, \infty)$ b/c $s(t)$ is increasing

- On what interval(s) is the particle's velocity negative? Justify your answer.

$v(t) < 0$ on (B, D) b/c $s(t)$ is decreasing.

- On what interval(s) is the particle's acceleration positive? Justify your answer.

$s''(t)$

$a(t) > 0$ on (C, ∞) b/c $s(t)$ is concave up.

- On what interval(s) is the particle's acceleration negative? Justify your answer.

$a(t) < 0$ on $(-\infty, C)$ b/c $s(t)$ is concave down.

Five Commandments of Particle Motion → For particles moving along x-axis

1. If v(t) is +, particle is moving Right

2. If v(t) is −, particle is moving Left.

3. Speed increases when v(t) + a(t) have same signs.

4. speed decreases when v(t) + a(t) have opposite signs.

5. If v(t) = 0, but a(t) ≠ 0, the particle is stopped + changing direction.

*

Suppose the velocity of a particle is given by the function $v(t) = (t+2)(t+4)^2$ for $t \geq 0$, where t is measured in minutes and $v(t)$ is measured in inches per minute. Answer the questions that follow.

a. Find the values of $v(3)$ and $v'(3)$. Based on these values, describe the speed of the particle at $t = 3$.

$v(3) = (5)(7)^2 = 245$ in/min

$v'(t) = (t+4)^2 + (t+2)2(t+4)$

$v'(3) = (7)^2 + (5)(2)(9) = 119 \text{ in/min}^2 = a(3)$

Speed is increasing at t = 3 since both v(t) + a(t) are positive.

b. On what interval(s) is the particle moving to the left? Right? Show your analysis and justify your answer.

$v(t) = 0 = (t+2)(t+4)^2$

$t = -2 \quad t = -4$

$\boxed{* \, t \geq 0}$

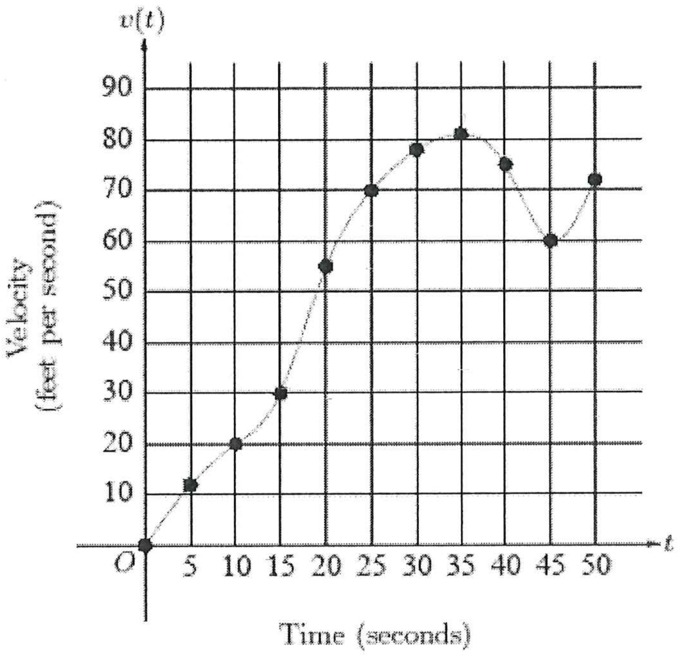
v(t) +

0

Since v(t) > 0 for all values of t ≥ 0, the particle is always moving Right.

1998 AP Calculus AB #3 (Modified)

The graph of the velocity $v(t)$, in feet per second, of a car traveling on a straight road, for $0 \le t \le 50$ is shown below. A table of values for $v(t)$, at 5 second intervals of time, is also shown to the right of the graph.



t (seconds)	$v(t)$ (feet per second)
0	0
5	12
10	20
15	30
20	55
25	70
30	78
35	81
40	75
45	60
50	72

a. During what interval(s) of time is the acceleration of the car positive? Give a reason for your answer.

$a(t) > 0$ on $(0,35) \cup (45,50)$ b/c $v(t)$ is increasing.

b. Find the average acceleration of the car over the interval $0 \le t \le 50$. Indicate units of measure.

$$\text{avg. accel} = \frac{V(50)-V(0)}{50-0} = \frac{72-0}{50} = \frac{72}{50} = \frac{36}{25}\ \text{ft/s}^2$$

c. Find one approximation for the acceleration of the car at $t = 40$. Show the computations you used to arrive at your answer. Indicate units of measure.

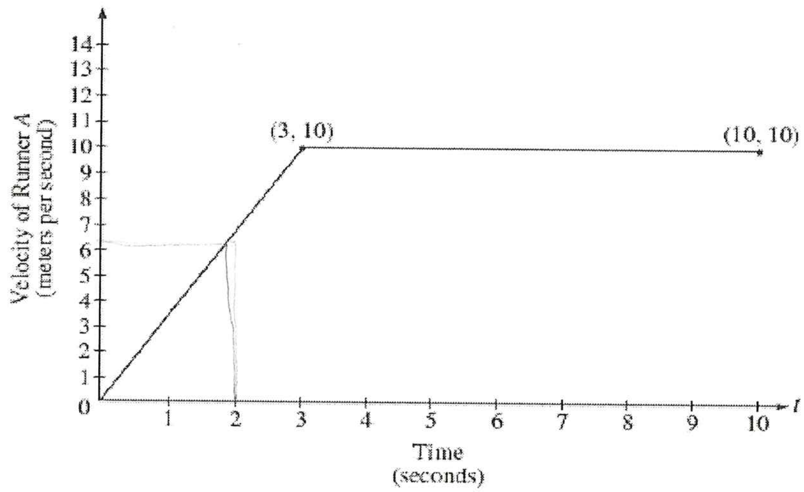
$$a(40) \approx \frac{V(45)-V(35)}{45-35} \approx \frac{60-81}{10} \approx \frac{-21}{10}\ \text{ft/s}^2$$

d. Is the speed of the car increasing or decreasing at $t = 40$? Give a reason for your answer.

Since $V(40) = 75$ ft/s > 0 and $a(40) \approx \frac{-21}{10}$ ft/s^2 < 0, speed is decreasing since $V(40)$ + $a(40)$ have different signs.

2000 AP Calculus AB #2 (Partial)

Two runners, A and B, run on a straight racetrack for $0 \le t \le 10$ seconds. The graph below, which consists of two line segments, shows the velocity, in meters per second, of Runner A. The velocity, in meters per second, of Runner B is given by the function v defined by $v(t) = \frac{24t}{2t+3}$.



a. Find the velocity of Runner A and the velocity of Runner B at $t = 2$ seconds. Indicate units of measure.

Runner A: equation of line

$v(t) = \frac{10}{3}t + 0$

$v(2) = \frac{10}{3}(2) = \boxed{20/3 \text{ m/s}}$

Runner B:

$v(t) = \frac{24t}{2t+3}$

$v(2) = \frac{24(2)}{2(2)+3} = \boxed{\frac{48}{7} \text{ m/s}}$

b. Find the acceleration of Runner A and the acceleration of Runner B at time $t = 2$ seconds. Indicate units of measure.

Runner A:

$a(t) = v'(t) = \boxed{\frac{10}{3} \text{ m/s}^2}$

Runner B:

$a(t) = v'(t) = \frac{(2t+3)(24) - 24t(2)}{(2t+3)^2}$

$a(2) = \frac{(7)(24) - 24(2)(2)}{(7)^2}$

$a(2) = \boxed{\frac{72}{49} \text{ m/s}^2}$

2002 AP Calculus AB #3 (Partial)

An object moves along the x – axis with initial position $x(0) = 2$. The velocity of the object at time $t \geq 0$ is given by the function $v(t) = \sin\left(\frac{\pi}{3}t\right)$.

a. What is the acceleration of the object at time $t = 4$?

$a(t) = v'(t) = \cos(\pi/3 t) \cdot \pi/3$

$a(4) = \cos\left(\frac{4\pi}{3}\right) \cdot \frac{\pi}{3} = \frac{-1}{2}(\pi/3) = \boxed{\frac{-\pi}{6}}$

b. Consider the following two statements.

Statement I: For $3 < t < 4.5$, the velocity of the object is decreasing. $a(t) < 0$

Statement II: For $3 < t < 4.5$, the speed of the object is decreasing.

Are either or both of these statements correct? For each statement, provide a reason why it is correct or not correct.

Statement I: since $a(4) = -\frac{\pi}{6} < 0$, the $v(4)$ must be decreasing. $\boxed{\text{TRUE}}$

Statement II: $v(4) = \sin\left(\frac{\pi}{3} \cdot 4\right) = -\frac{\sqrt{3}}{2} < 0$

since $v(4) < 0$ and $a(4) < 0$, the speed would be increasing since $v(4)$ + $a(4)$ have the same signs.

$\boxed{\text{FALSE}}$