Part I: Theory

1. True or false.

   (a) False. If $f'(c) = 0$, then f must have either a relative minimum or a relative maximum at c.

   (b) False. Since function $f(x) = -\dfrac{1}{x}$ is increasing on the intervals $(-\infty, 0)$ and $(0, \infty)$, it is impossible to find x_1 and x_2 in the domain of f such that $x_1 > x_2$ and $f(x_1) > f(x_2)$.

   (c) True. On the interval $(-1, 1]$, the function $f(x) = x^4$ has an absolute maximum and an absolute minimum.

2. Fill in the blank.

   (a) $\cos^2\theta + \underline{\sin^2\theta} = 1$

   (b) $1 + \underline{\tan^2\theta} = \sec^2\theta$

   (c) If $y = \sin x$, then $y' = \underline{\cos x}$.

   (d) If $r = \cos\theta$, then $\dfrac{dr}{d\theta} = \underline{-\sin\theta}$.

   (e) For $y = x^9 - 7x^3 + 2$, $\dfrac{d^2y}{dx^2} = \underline{72x^7 - 42x}$.

   (f) *Extreme Values:* If a function $f(x)$ has a derivative at every point in the interval $a \leq x \leq b$, calculate $f(x)$ at

   - all points in the interval $a \leq x \leq b$, where $f'(x) = \underline{0}$
   - the endpoints $x = a$ and $x = b$

   The maximum value of $f(x)$ on the interval $a \leq x \leq b$ is the largest of these values, and the minimum value of $f(x)$ on the interval is the smallest of these values.

   (g) If the position of an object, $s(t)$, is a function of time, t, then the first derivative of this function represents the velocity of the object at time t. *i.e.*, $s'(t) = v(t)$.

   (h) Acceleration is the second derivative of the position function.

   (i) $a(t) < 0$ indicates that the velocity is decreasing.

   (j) $a(t) = 0$ indicates that the velocity is constant.

3. Multiple Choice.

   (a) If $f(x) = \sec x + \csc x$, then $f'(x) =$

   (a) $\sec^2 x + \csc^2 x$ (b) $\csc x - \sec x$ (c) $\sec x \tan x - \csc x \cot x$

   Solution: The derivative of $\sec x$ is $\sec x \tan x$, and the derivative of $\csc x$ is $-\csc x \cot x$. That makes the derivative here $\sec x \tan x - \csc x \cot x$. **Answer: (c)** □

   (b) If $f(x) = \cos^2 x$, then $f''(\pi) =$

   (a) -2 (b) 0 (c) 1 (d) 2π

Solution:

$$
\begin{aligned}
f(x) &= \cos^2 x \\
f'(x) &= 2(\cos x)(-\sin x) \\
&= -2\cos x \sin x \\
f''(x) &= -2(\cos x \cos x - \sin x \sin x) \\
&= -2(\cos^2 x - \sin^2 x) \\
f''(\pi) &= -2(\cos^2(\pi) - \sin^2(\pi)) \\
&= -2(1-0) \\
&= -2
\end{aligned}
$$

Answer: (a) □

(c) A particle's position is given by $s(t) = t^3 - 6t^2 + 9t$. What is its acceleration at time $t = 4$? (a) 0 (b) 9 (c) -9 (d) 12

Solution: Answer: (e) □

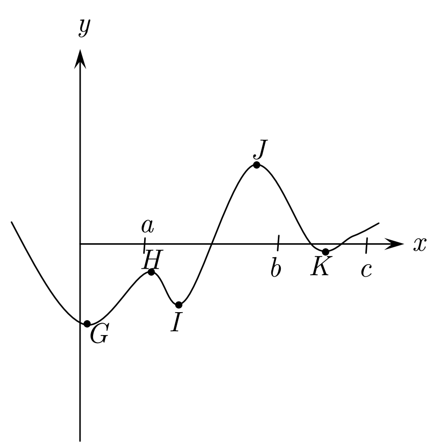
(d) The maximum velocity attained on the interval $0 \leq t \leq 5$ by the particle whose displacement is given by $s(t) = 2t^3 - 12t^2 + 16t + 2$ is
(a) 286 (b) 46 (c) 16 (d) 0

Solution: Answer: (b) □

(e) If $f(x) = \sec(4x)$, then $f'\left(\frac{\pi}{16}\right)$ is
(a) $4\sqrt{2}$ (b) $\sqrt{2}$ (c) 0 (d) $\frac{1}{\sqrt{2}}$

Solution: Answer: (a) □

(f) Using the graph below, on the interval $[a, b]$ absolute minimum value is at
(a) point G (b) point H (c) point I (d) point J (e) point K



Solution: Answer: (c) □

(g) Using the graph from (3f), on the interval $[a, b]$, the absolute maximum value is at
(a) point G (b) point H (c) point I (d) point J (e) point K

Solution: Answer: (d) □

(h) Using the graph from (3f), on the interval $[a, c]$, the absolute minimum is at
(a) point G (b) point H (c) point I (d) point J (e) point K

Solution: Answer: (c) □

4. The critical numbers for $f(x) = \sqrt[3]{x} - 1$ are found at
(a) $x = -1$ (b) $x = 0$ (c) $x = 1$ (d) $x = \{-1, 0, 1\}$

Solution:

$$\begin{aligned} f(x) &= x^{1/3} - 1 \\ f'(x) &= \frac{1}{3}x^{-2/3} \\ 0 &= \frac{1}{3}x^{-2/3} \end{aligned}$$

This equation has no solutions. $f'(x) \neq 0$. Therefore, $x = 0$ is a critical point. **Answer: (b)** □

Part II: Application

1. The position of a spring is described by the function $s(t) = 3\cos(2x)$.
   (a) The velocity function is $v(t) = \underline{-6\sin(2x)}$.
   (b) The acceleration function is $a(t) = \underline{-12\cos(2x)}$.

2. Given $f(x) = -2x^3 + 9x^2 + 4$, find the extreme values of $f(x)$ on $[-1, 5]$.

Solution:

$$\begin{aligned} f(x) &= -2x^3 + 9x^2 + 4 \\ f'(x) &= -6x^2 + 18x \end{aligned}$$

We set $f'(x) = 0$ to the find the critical points.

$$-6x^2 + 18x = 0 \to x = 0,\ 3.$$

Both values lie inside the given interval. We can then evaluate $f(x)$ for those values and at the endpoints $x = -1$ and $x = 5$ to obtain

x	$f(x)$
-1	15
0	4
3	31
5	-21

Thus, the maximum value of $f(x)$ on the interval $x \in [-1, 5]$ is $f(3) = 31$, and the minimum value is $f(5) = -21$. □

3. The position function of a particle is given by

$$s(t) = \sqrt{t}(3t^2 - 35t + 90)$$

where s is measured in metres and t is in seconds.

(a) Find the velocity and acceleration as a function of time.

Solution:

$$\begin{aligned} s(t) &= \sqrt{t}(3t^2 - 35t + 90) \\ &= 3t^{5/2} - 35t^{3/2} + 90t^{1/2} \\ v(t) &= s'(t) \\ &= \frac{15}{2}t^{3/2} - \frac{105}{2}t^{1/2} + 45t^{-1/2} \\ &= \frac{15}{2}t^{-1/2}(t^2 - 7t + 6) \\ &= \frac{15}{2\sqrt{t}}(t-1)(t-6) \end{aligned}$$

□

(b) Find the velocity of the particle at $t = 3$.

Solution:

$$\begin{aligned} v(3) &= \frac{15}{2\sqrt{3}}(2)(-3) \\ &= -15\sqrt{3} \text{ m/s} \end{aligned}$$

□

(c) When is the particle at rest?

Solution: It is at rest when $v = 0 \to t = 1$ s or 6 s. □

(d) When is the particle moving in a positive direction.

Solution: It moves in the positive direction when $v > 0 \to (t-1)(t-6) > 0 \to 0 \leq t < 1$ or $t > 6$. □

4. Mrs McCombs grows tomatoes along the side of her house. The squirrels are getting at them, and she's decided to put a fence around the tomatoes to save them. If she has 40 m of fencing available to lay out using the straight side of her house as one line of the rectangle, what dimensions will maximize the garden area?

Solution: Let x represent the width, and $x \in [0, 20]$, and y represent the length.

$$\begin{aligned} 2x + y &= 40 \\ y &= 40 - 2x \end{aligned}$$

Let's derive a formula for the area.

$$\begin{aligned} A &= lw \\ &= y(x) \\ &= (40 - 2x)x \\ &= -2x^2 + 40x \end{aligned}$$

From $A = -2x^2 + 40x$, we see that it is a parabola that is concave down. To maximize A, set $\frac{dA}{dx} = 0$.

$$\frac{dA}{dx} = -4x + 40$$

For critical numbers

$$\begin{aligned} \frac{dA}{dx} &= 0 \\ -4x + 40 &= 0 \\ 4x &= 40 \\ x &= 10 \end{aligned}$$

Check the endpoints: if $x = 0$ then $A = 0$, and if $x = 20$ then $A = 0$. Therefore, the maximum area is $A = -2(10)^2 + 40(10) = 200 \text{ m}^2$, when $x = 10$ m and $y = 40 - 2(10) = 20$ m. so the dimensions that will maximize the garden area is $\boxed{10 \text{ m} \times 20 \text{ m}}$. □

5. Rick's American Cafe sells cheeseburgers with a yearly demand function of

$$p = \frac{800000 - x}{200000}$$

and cost function

$$C(x) = 125000 + 0.42x$$

(a) State the revenue function.

Solution:

$$\begin{aligned} R(x) &= xp(x) \\ &= x\left(\frac{800000 - x}{200000}\right) \\ &= \frac{1}{200000}(800000x - x^2) \end{aligned}$$

□

(b) Find the marginal revenue function.

Solution: The marginal revenue function is

$$R'(x) = \frac{1}{200000}(800000 - 2x)$$

□

(c) What level of cheeseburger sales will maximize profits?

Solution: The marginal cost is

$$C'(x) = 0.42$$

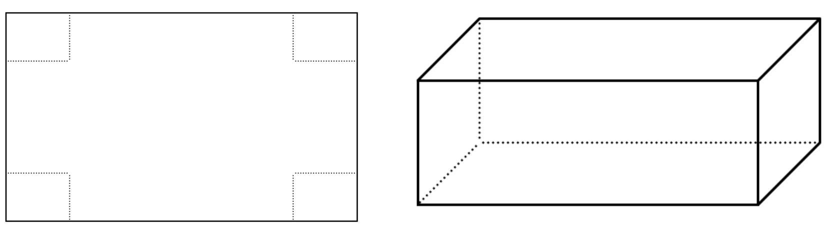
Profits are maximized when marginal revenue = marginal cost.

$$\begin{aligned} R'(x) &= C'(x) \\ \frac{1}{100000}(400000 - x) &= 0.42 \\ 400000 - x &= 42000 \\ x &= 358000 \end{aligned}$$

Sales of 358 000 cheeseburgers per year will maximize profits.

□

6. A piece of sheet metal, 60 cm by 30 cm, is to be used to make a rectangular box with an open top. Determine the dimensions that will give the box with the largest volume.



Solution: From the diagram, making the box requires the four corner squares to be cut out and discarded. Folding up the sides creates the box. Let each side of the squares by x in centimetres.

Then, the height is x, the length is $60 - 2x$, and the width is $30 - 2x$. Since all the dimensions must be positive, $0 < x < 15$. The volume of the box is the product of its dimensions and is given by the function $V(x)$, where

$$\begin{aligned} V(x) &= x(60 - 2x)(30 - 2x) \\ &= 4x^3 - 180x^2 + 1800x \end{aligned}$$

For extreme values, set $V'(x) = 0$.

$$\begin{aligned} V'(x) &= 12x^2 - 360x + 1800 \\ &= 12(x^2 - 30x + 150) \end{aligned}$$

Setting $V'(x) = 0$, we obtain $x^2 - 30x + 150 = 0$. Solving for x using the quadratic formula yields

$$\begin{aligned} x &= \frac{30 \pm \sqrt{300}}{2} \\ &= 15 \pm 5\sqrt{3} \\ &\approx \{23.7, 6.3\} \end{aligned}$$

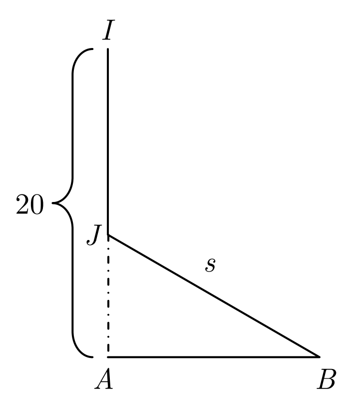
Since $0 < x < 15$, $x = 15 - 5\sqrt{3} \approx 6.3$. This is the only place within the interval where the derivative is 0. To find the largest volume, sub $x = 6.3$ into $V(x)$.

$$V(6.3) = 4(6.3)^3 - 180(6.3)^2 + 1800(6.3) = 5196$$

Note that we do not have to test the endpoints, since it is impossible to have a box that uses $x = 0$ or $x = 15$. The maximum volume is obtained by cutting out corner squares of side length 6.3 cm. The length of the box is $60 - 2 \times 6.3 = 47.4$ cm, the width is about $30 - 2 \times 6.3 = 17.4$ cm, and the height is about 6.3 cm.

□

7. Ian and Ada are both training for a marathon. Ian's house is located 20 km north of Ada's house. At 9:00 am one Saturday, Ian laves his house and jogs south at 8 km/h. At the same time, Ada leaves her house and jogs east at 6 km/h. When are Ian and Ada closest together, given that they both run for 2.5 h?



Solution: If Ian starts at point I, he reaches point J after time t hours. Then $IJ = 8t$ km, and $JA = (20 - 8t)$ km. If Ada starts at point A, she reaches point B after t hours, and $AB = 6t$ km. Now the distance they are apart is $s = JB$, and s can be expressed as a function of t by

$$\begin{aligned} s(t) &= \sqrt{JA^2 + AB^2} \\ &= \sqrt{(20 - 8t)^2 + (6t)^2} \\ &= \sqrt{100t^2 - 320t + 400} \\ &= (100t^2 - 320t + 400)^{1/2} \end{aligned}$$

The domain for $t \in [0, 2.5]$.

$$\begin{aligned} s'(t) &= \frac{1}{2}(100t^2 - 320t + 400)^{-1/2}(200t - 320) \\ &= \frac{100t - 160}{\sqrt{100t^2 - 320t + 400}} \end{aligned}$$

To obtain a minimum or maximum value, let $s'(t) = 0$.

$$\begin{aligned} \frac{100t - 160}{\sqrt{100t^2 - 320t + 400}} &= 0 \\ 100t - 160 &= 0 \\ t &= 1.6 \end{aligned}$$

Using the algorithm for finding extreme values,

$$\begin{aligned} s(0) &= \sqrt{400} = 20 \\ s(1.6) &= \sqrt{100(1.6)^2 - 320(1.6) + 400} = 12 \\ s(2.5) &= \sqrt{225} = 15 \end{aligned}$$

Therefore, the minimum value of $s(t)$ is 12 km, which occurs at time 10:36 a.m. □

8. Determine the $\frac{dy}{dx}$ for each of the following

   (a) $y = \tan x$

   Solution:

$$\begin{aligned} y &= \tan x \\ &= \frac{\sin x}{\cos x} \\ \frac{dy}{dx} &= \frac{\cos x \cos x - \sin x \sin x(-1)}{(\cos x)^2} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x \end{aligned}$$

□

(b) $y = \dfrac{\sin x}{1 + \cos x}$

Solution:

$$\begin{aligned} y &= \frac{\sin x}{1 + \cos x} \\ \frac{dy}{dx} &= \frac{\cos x(1 + \cos x) - (\sin x)(-\sin x)}{(1 + \cos x)^2} \\ &= \frac{\cos x \cos^2 x + \sin^2 x}{(1 + \cos x)^2} \\ &= \frac{1 + \cos x}{(1 + \cos x)^2} \\ &= \frac{1}{1 + \cos x} \end{aligned}$$

$\square$

(c) $r = \tan^3(4\theta + \pi)$

Solution:

$$\begin{aligned} r &= \tan^3(4\theta + \pi) \\ &= (\tan(4\theta + \pi))^3 \\ \frac{dr}{d\theta} &= 3(\tan(4\theta + \pi))^2(\sec^2(4\theta + \pi)(4)) \\ &= 12 \tan^2(4\theta + \pi) \sec^2(4\theta + \pi) \end{aligned}$$

$\square$

(d) $y = 6 \cos(\sqrt{x})$

Solution:

$$\begin{aligned} \frac{dy}{dx} &= -6 \sin(\sqrt{x}) \left(\frac{1}{2\sqrt{x}} \right) \\ &= \frac{-3 \sin(\sqrt{x})}{\sqrt{x}} \end{aligned}$$

$\square$

9. The position of a particle as it moves horizontally is described by the function

$$s(t) = \sin^2 t - 2 \cos^2 t$$

where $t \in [-\pi, \pi]$. If s is the displacement in metres and t is the time in seconds, find the absolute maximum and minimum displacements.

Solution: The maximum and the minimum values will occur at the critical points.

$$\begin{aligned} s(t) &= (\sin t)^2 - 2(\cos t)^2 \\ s'(t) &= 2\sin t\cos t + 4\cos t\sin t \\ s'(t) &= 6\sin t\cos t \end{aligned}$$

To minimize or maximize, we set the derivative equal to zero.

$$s'(t) = 0 = (6\sin t)(\cos t)$$

By the zero property, either $6\sin t = 0$ or $\cos t = 0$. From $6\sin t = 0$, $t = 0, -\pi, \pi$, and from $\cos t = 0$, $t = -\pi/2, \pi/2$.

We make a table and compare the values to find the minimum and maximum values.

t	$s(t)$
$-\pi$	-2
$-\pi/2$	1
0	-2
$\pi/2$	1
π	-2

We see the absolute minimum values is -2 when $t = -\pi, 0, \pi$, and the absolute maximum values is 1 when $t = -\pi/2, \pi/2$. □

10. Find an equation of the tangent line to the curve $y = \dfrac{\cos^2 x}{\sin^2 x}$ at $\left(\frac{\pi}{4}, 1\right)$.

Solution:

$$\begin{aligned} y &= \frac{\cos^2 x}{\sin^2 x} \\ &= \frac{(\cos x)^2}{(\sin x)^2} \\ \frac{dy}{dx} &= \frac{2(\cos x)(-\sin x)(\sin x)^2 - (\cos x)^2(2)(\sin x)(\cos x)}{(\sin x)^4} \\ &= \frac{-2\cos x\sin^3 x - 2\cos^3 x\sin x}{\sin^4 x} \\ &= \frac{-2\cos x\sin x(\sin^2 x + \cos^2 x)}{\sin^4 x} \\ &= \frac{-2\cos x\sin x}{\sin^4 x} \\ &= \frac{-2\cos x}{\sin^3 x} \\ &= -2\frac{\cos x}{\sin x}\frac{1}{\sin^2 x} \\ &= -2\cot x\sec^2 x \end{aligned}$$

Then
$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = -2\cot\left(\frac{\pi}{4}\right)\sec^2\left(\frac{\pi}{4}\right) = -4$$
Using the point-slope formula
$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= -4\left(x - \frac{\pi}{4}\right) \\ y - 1 &= -4x + \pi \\ 4x + y - 1 - \pi &= 0. \end{aligned}$$

$\square$