DEFINITION The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

Thus, instantaneous rates are limits of average rates.

It is conventional to use the word *instantaneous* even when x does not represent time. The word is, however, frequently omitted. When we say *rate of change*, we mean *instantaneous rate of change*.

Motion Along a Line: Displacement, Velocity, Speed, Acceleration, and Jerk

Suppose that an object (or body, considered as a whole mass) is moving along a coordinate line (an s-axis), usually horizontal or vertical, so that we know its position s on that line as a function of time t:

$$s = f(t).$$

The **displacement** of the object over the time interval from t to $t + \Delta t$ (Figure 3.15) is

$$\Delta s = f(t + \Delta t) - f(t),$$

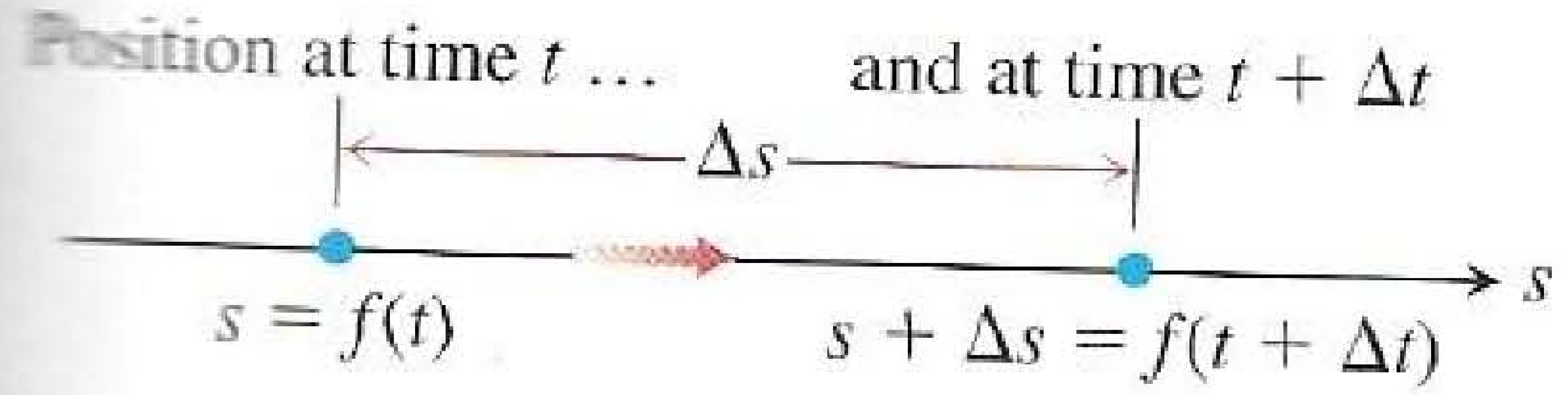


FIGURE 3.15 The positions of a body moving along a coordinate line at time t and shortly later at time $t + \Delta t$. Here the coordinate line is horizontal.

and the **average velocity** of the object over that time interval is

$$v_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

To find the body's velocity at the exact instant t, we take the limit of the average velocity over the interval from t to $t + \Delta t$ as Δt shrinks to zero. This limit is the derivative of f with respect to t.

DEFINITION **Velocity (instantaneous velocity)** is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

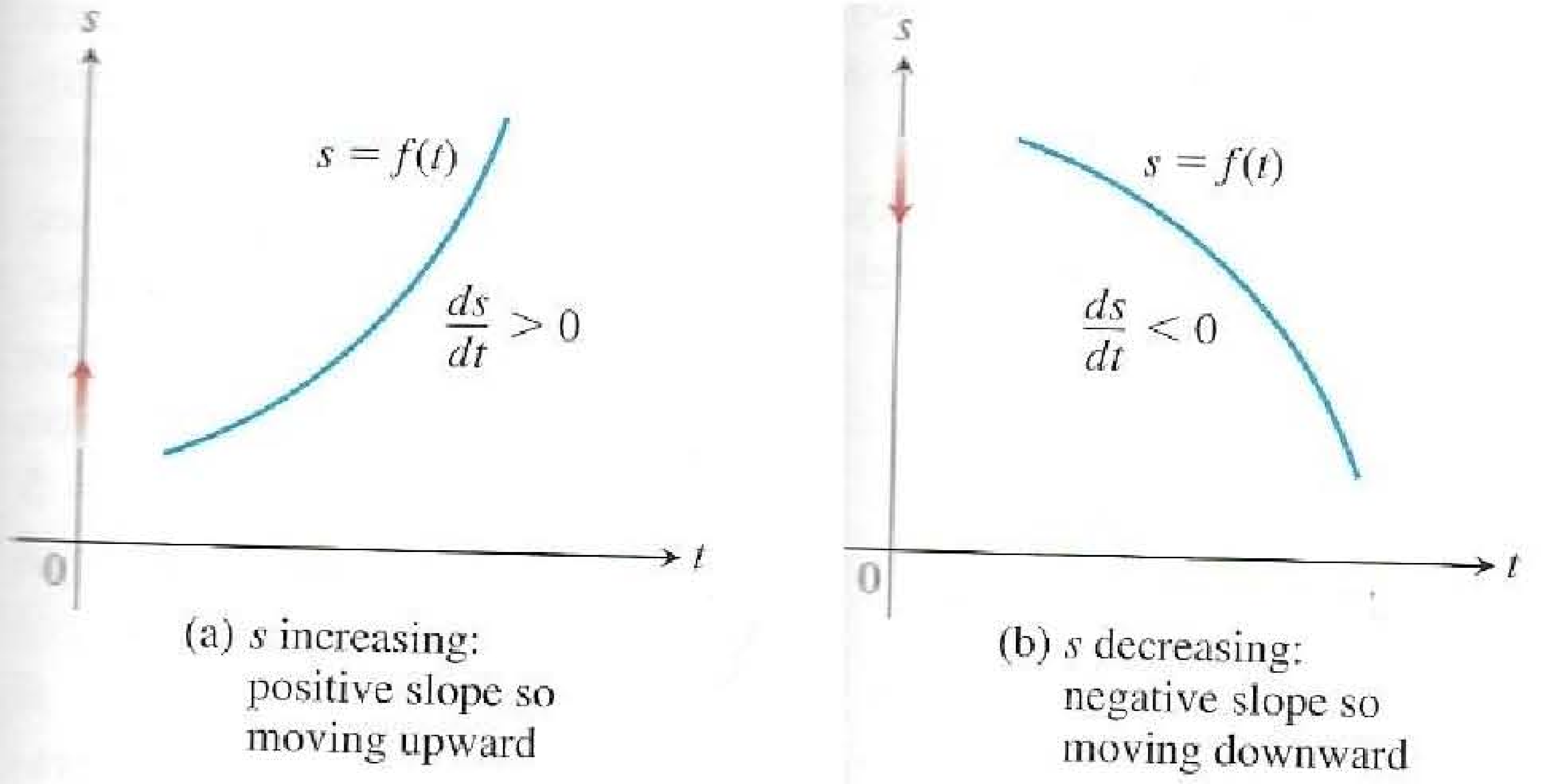


FIGURE 3.16 For motion $s = f(t)$ along a straight line (the vertical axis), $v = ds/dt$ is (a) positive when s increases and (b) negative when s decreases.

DEFINITION **Speed** is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

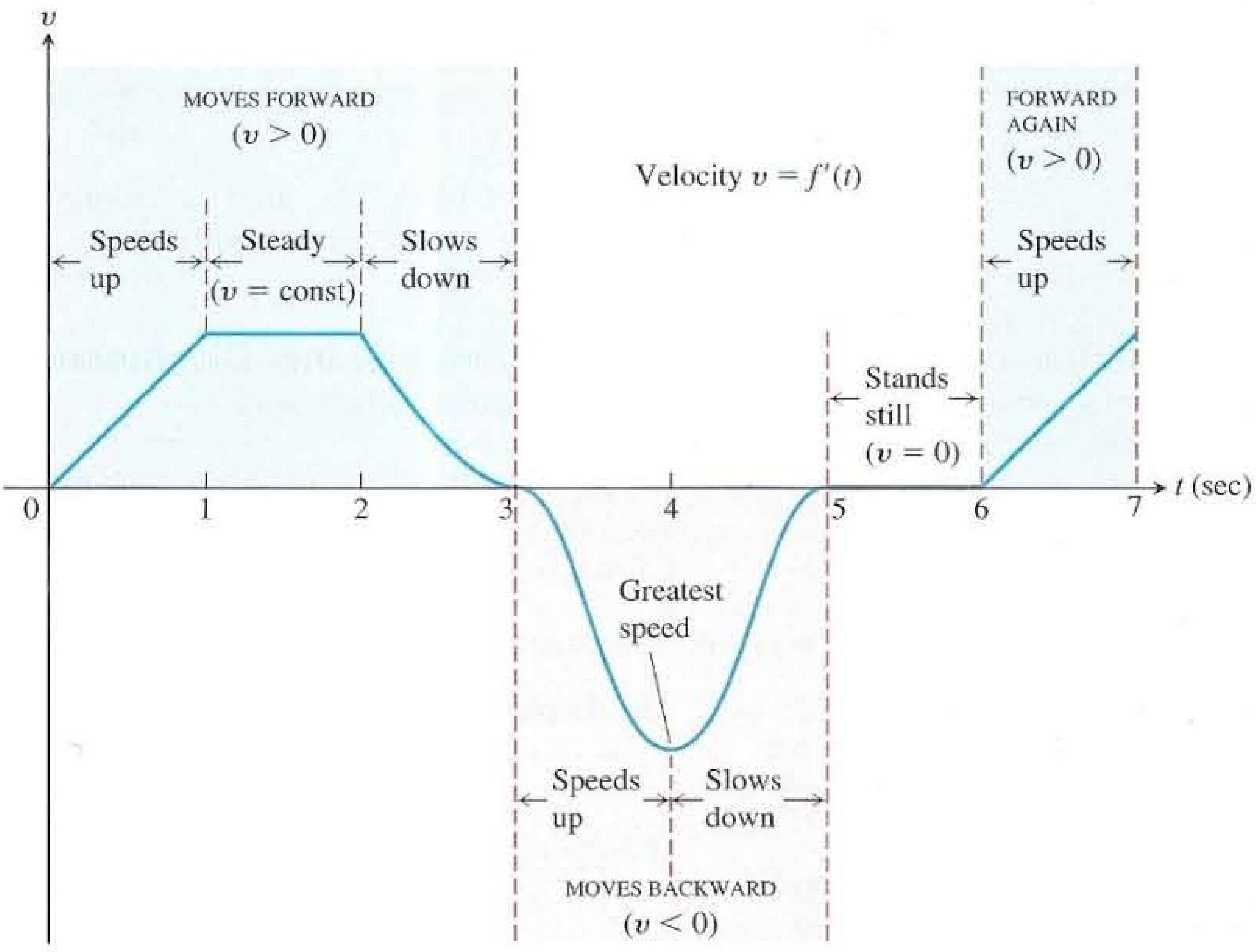


FIGURE 3.17 The velocity graph of a particle moving along a horizontal line, discussed in Example 2.

DEFINITIONS **Acceleration** is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

3.4 Applications of the Derivative

Physics

position, velocity, and acceleration

$s(t)$ $\quad s'(t)$ $\quad s''(t)$

$$s(t)=t^3-12t^2+45t$$

$a)$ Find the velocity at time t.

$s'(t)=3t^2-24t+45$

$b)$ Find the initial velocity and the velocity after 2sec.

$s'(0)$ $\qquad s'(2)$

$s'(0)=45$

$s'(2)=3(2)^2-24(2)+45=12-48+45=9$

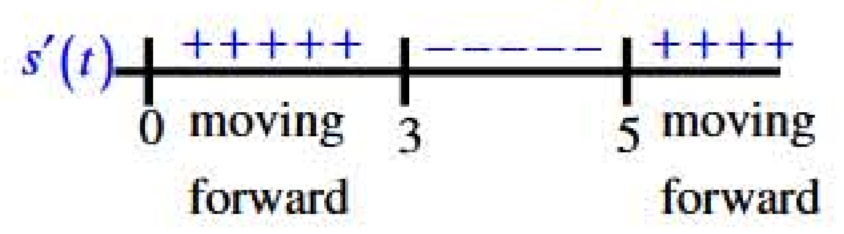
$c)$ When is the particle at rest?

$s'(t)=0$

$s'(t)=3t^2-24t+45=0 \Rightarrow 3(t^2-8t+15)=0 \Rightarrow 3(t-3)(t-5)=0 \Rightarrow t=3, t=5$

$d)$ When is the particle moving forward?

$s'(t)>0$



Moving forward $(0,3)\cup(5,\infty)$

Moving backwards $(3,5)$

$s(t)=t^3-12t^2+45t$

$s'(t)=3t^2-24t+45$

$e)$ Find the acceleration at time t.

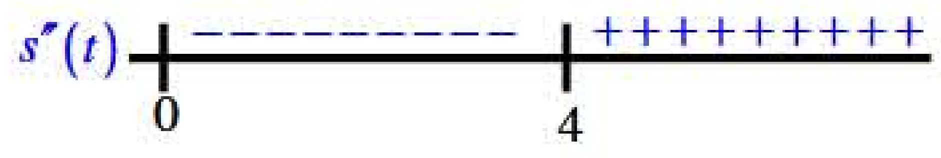
$s''(t)=6t-24$

$f)$ Find the acceleration after 2 sec.

$s''(2)=12-24=-12$

$g)$ When is the acceleration 0?

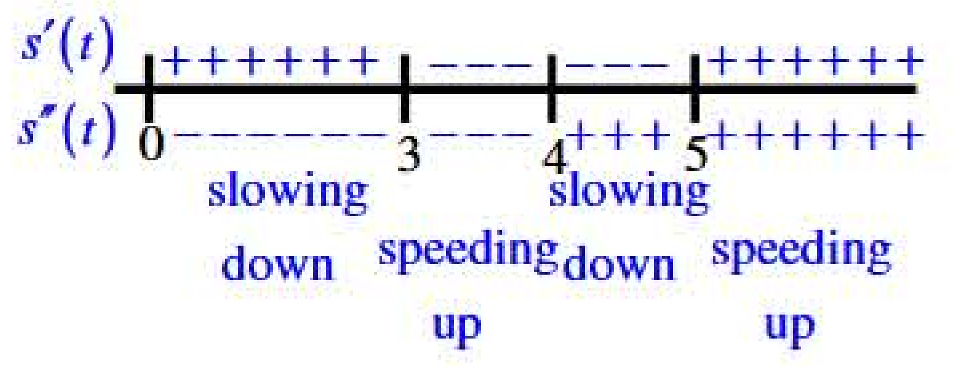
$s''(t)=6t-24=0 \Rightarrow t=4$



$h)$ When is the particle speeding up, when is it slowing down?

velocity and acceleration have same sign

velocity and acceleration have different signs



slowing down $(0,3)$ $(4,5)$

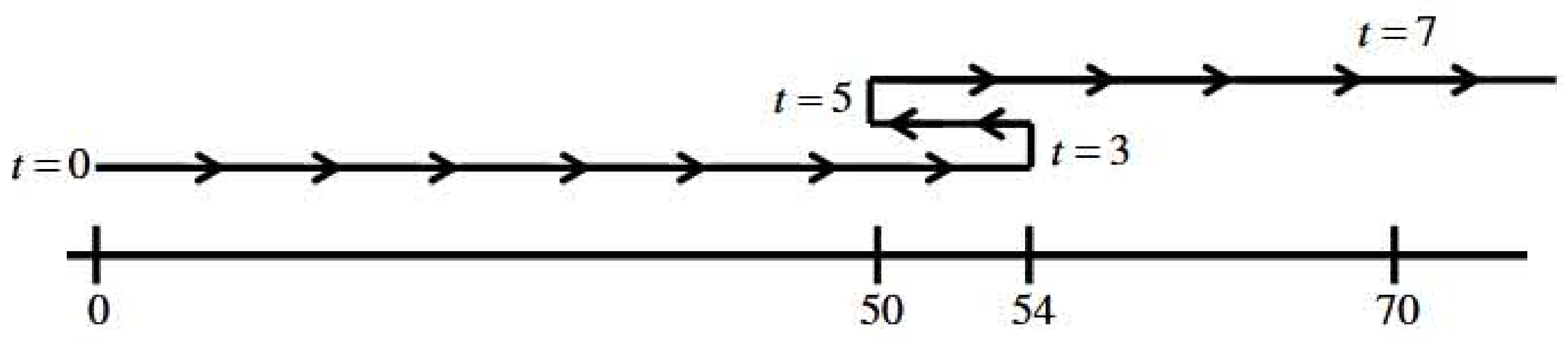
speeding up $(3,4)$ $(5,\infty)$

$s(t)=t^3-12t^2+45t$

$i)$ Draw a picture that describes the motion

$s(3)=3^3-12(3)^2+45(3)=27-12\cdot 9+45\cdot 3=27-108+135=-108+162=54$

$s(5)=5^3-12(5)^2+45(5)=125-12\cdot 25+45\cdot 5=125-300+225=50$



$j)$ Find the total distance traveled by the particle during the first 7 seconds.

$[0,3]$: distance traveled $=54$

$[3,5]$: distance traveled $=4$

$[5,7]$: distance traveled $=20$

$\overline{78}$

$s(7)=7^3-12(7)^2+45(7)$

$=343-588+315=658-588=70$

Economics

$C(x)=$ the cost of producing x units of a product

$C'(x)=$ measures the rate of change of the cost function, it is called the **marginal cost**.

$C(x)=0.000003x^3-0.04x^2+200x+70000$

$C'(x)=0.000009x^2-0.08x+200$

Find $C'(3000)$ and interpret it.

$$C'(3000)=\frac{9}{1,000,000}(3000)^2-\frac{8}{100}(3000)+200=\frac{9\cdot 9,\cancel{000,000}}{1,\cancel{000,000}}-\frac{8\cdot 30\cancel{00}}{1\cancel{00}}+200$$

$C'(3000)=81-240+200=41$ $\qquad C'(3000)=\dfrac{\Delta C}{\Delta x}=\dfrac{41}{1}$

When producing 3000 units, if you decide to produce 1 more unit, cost will increase by \$ 41.

The cost of the 3001st unit is approximately \$ 41.

Actual cost of the 3001st unit $=C(3001)-C(3000)=\$40.99$

$R(x)=$ the revenue gained by selling x units of a product

$R'(x)=$ measures the rate of change of the revenue function, it is called the **marginal revenue**.

$R(x)=-0.02x^2+300x$

$R'(x)=-0.04x+300$

Find $R'(3000)$ and interpret it.

$$R'(3000)=-\frac{4\cdot 30\cancel{00}}{1\cancel{00}}+300$$

$R'(3000)=-120+300=180$

When producing 3000 units, if you decide to produce (and sell) 1 more unit, revenue will increase by \$ 180.

The revenue from selling the 3001st unit is approximately \$ 180.

Actual revenue from selling the 3001st unit $=R(3001)-R(3000)=\$179.98$

Motion along a Line

Another use for the derivative is to analyze motion along a line. We have described velocity as the rate of change of position. If we take the derivative of the velocity, we can find the acceleration, or the rate of change of velocity. It is also important to introduce the idea of **speed**, which is the magnitude of velocity. Thus, we can state the following mathematical definitions.

Definition

Let $s(t)$ be a function giving the position of an object at time t.

The velocity of the object at time t is given by $v(t) = s'(t)$.

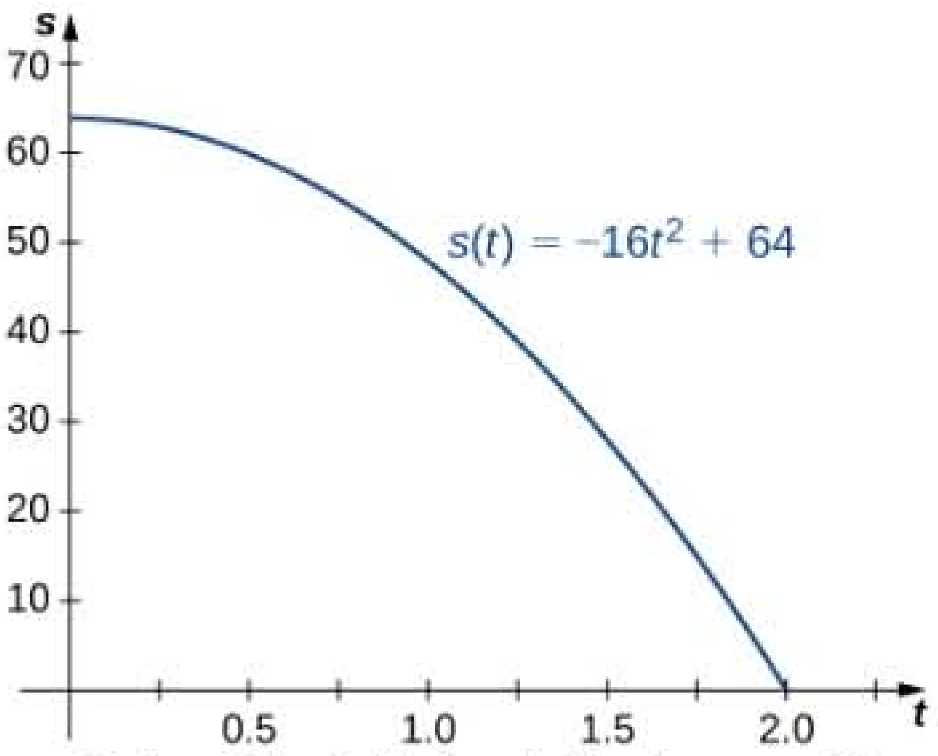
The speed of the object at time t is given by $|v(t)|$.

The acceleration of the object at t is given by $a(t) = v'(t) = s''(t)$.

Example 3.34

Comparing Instantaneous Velocity and Average Velocity

A ball is dropped from a height of 64 feet. Its height above ground (in feet) t seconds later is given by $s(t) = -16t^2 + 64$.



a. What is the instantaneous velocity of the ball when it hits the ground?

b. What is the average velocity during its fall?

Solution

The first thing to do is determine how long it takes the ball to reach the ground. To do this, set $s(t) = 0$. Solving $-16t^2 + 64 = 0$, we get $t = 2$, so it take 2 seconds for the ball to reach the ground.

a. The instantaneous velocity of the ball as it strikes the ground is $v(2)$. Since $v(t) = s'(t) = -32t$, we obtain $v(t) = -64$ ft/s.

b. The average velocity of the ball during its fall is

$$v_{ave} = \frac{s(2) - s(0)}{2 - 0} = \frac{0 - 64}{2} = -32 \text{ ft/s}.$$

Example 3.35

Interpreting the Relationship between $v(t)$ and $a(t)$

A particle moves along a coordinate axis in the positive direction to the right. Its position at time t is given by $s(t) = t^3 - 4t + 2$. Find $v(1)$ and $a(1)$ and use these values to answer the following questions.

a. Is the particle moving from left to right or from right to left at time $t = 1$?

b. Is the particle speeding up or slowing down at time $t = 1$?

Solution

Begin by finding $v(t)$ and $a(t)$.

and $a(t) = v'(t) = s''(t) = 6t$.

Evaluating these functions at $t = 1$, we obtain $v(1) = -1$ and $a(1) = 6$.

a. Because $v(1) < 0$, the particle is moving from right to left.

b. Because $v(1) < 0$ and $a(1) > 0$, velocity and acceleration are acting in opposite directions. In other words, the particle is being accelerated in the direction opposite the direction in which it is traveling, causing $|v(t)|$ to decrease. The particle is slowing down.

Position and Velocity

The position of a particle moving along a coordinate axis is given by $s(t) = t^3 - 9t^2 + 24t + 4,\ t \geq 0.$

a. Find $v(t)$.

b. At what time(s) is the particle at rest?

c. On what time intervals is the particle moving from left to right? From right to left?

d. Use the information obtained to sketch the path of the particle along a coordinate axis.

Solution

a. The velocity is the derivative of the position function:

$$v(t) = s'(t) = 3t^2 - 18t + 24.$$

b. The particle is at rest when $v(t) = 0,$ so set $3t^2 - 18t + 24 = 0.$ Factoring the left-hand side of the equation produces $3(t-2)(t-4) = 0.$ Solving, we find that the particle is at rest at $t = 2$ and $t = 4.$

c. The particle is moving from left to right when $v(t) > 0$ and from right to left when $v(t) < 0.$ Figure 3.23 gives the analysis of the sign of $v(t)$ for $t \geq 0,$ but it does not represent the axis along which the particle is moving.

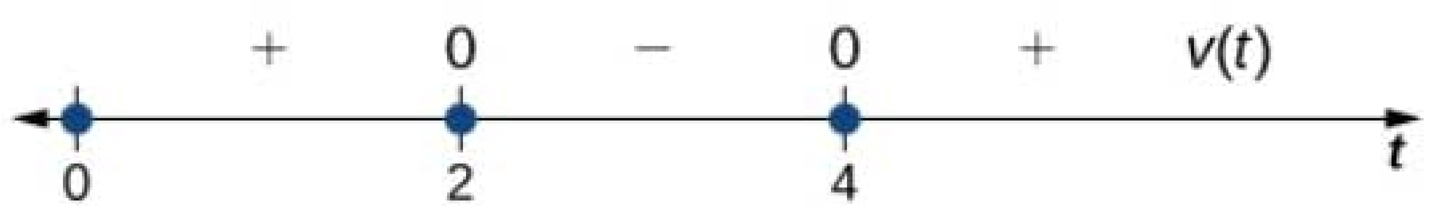


Figure 3.23 The sign of v(t) determines the direction of the particle.

Since $3t^2 - 18t + 24 > 0$ on $[0, 2) \cup (2, +\infty)$, the particle is moving from left to right on these intervals.

Since $3t^2 - 18t + 24 < 0$ on $(2, 4)$, the particle is moving from right to left on this interval.

d. Before we can sketch the graph of the particle, we need to know its position at the time it starts moving $(t = 0)$ and at the times that it changes direction $(t = 2, 4)$. We have $s(0) = 4$, $s(2) = 24$, and $s(4) = 20$. This means that the particle begins on the coordinate axis at 4 and changes direction at 0 and

20 on the coordinate axis. The path of the particle is shown on a coordinate axis in Figure 3.24.

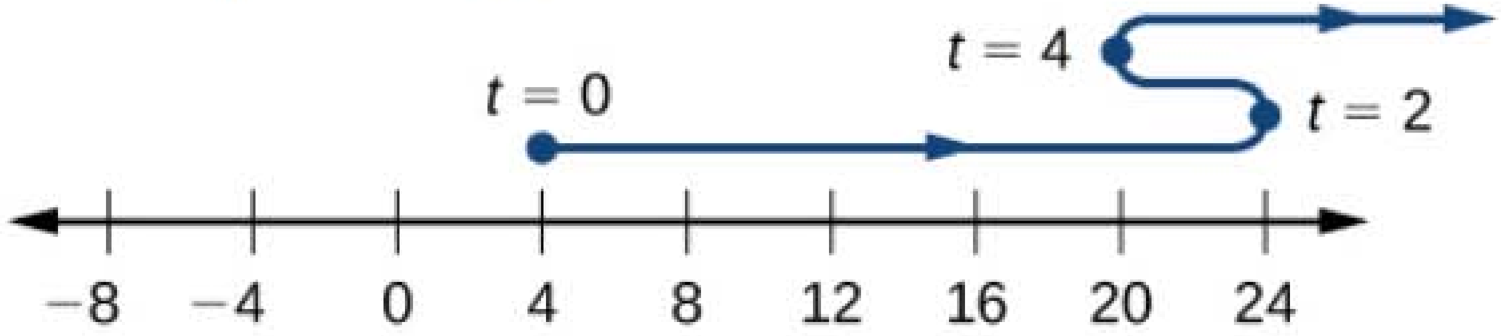


Figure 3.24 The path of the particle can be determined by analyzing v(t).

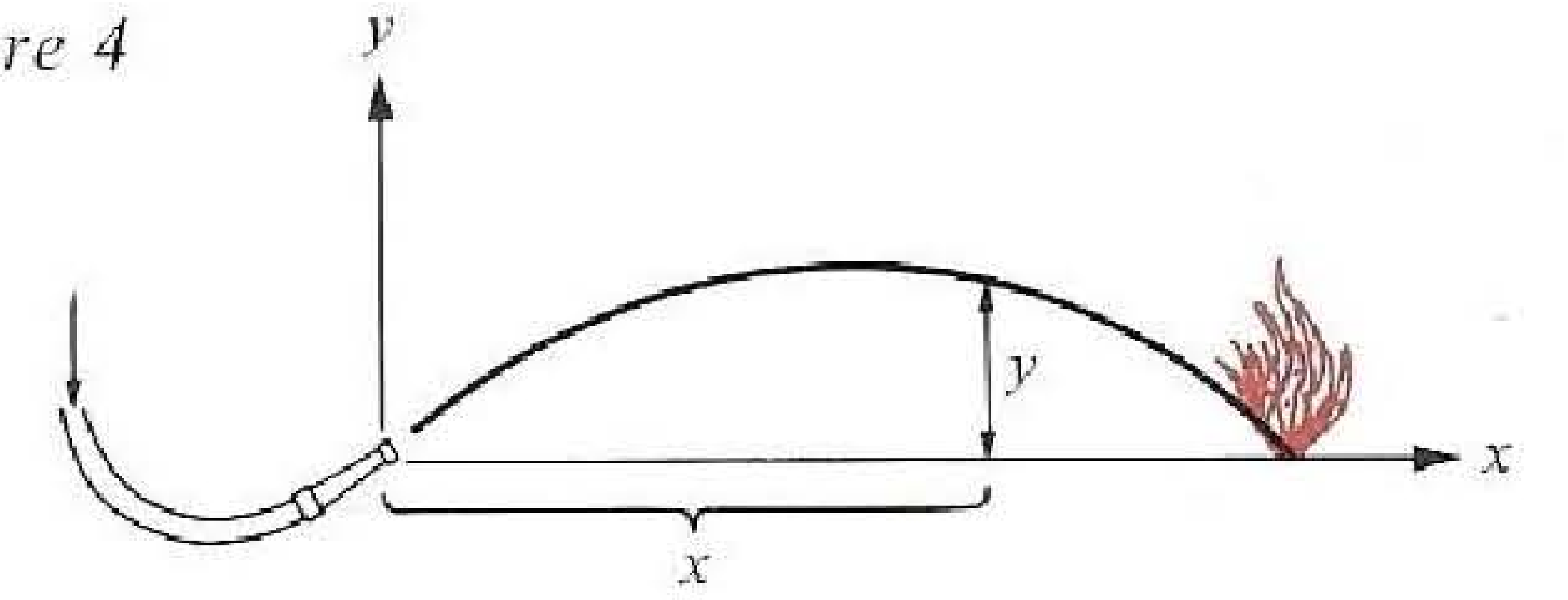
Example-1

2 Neglecting air resistance, the stream of water projected from a fire hose satisfies the equation

$$y = mx - 16(1 + m^2)\left(\frac{x}{v}\right)^2,$$

where m is the slope of the nozzle, v is the velocity of the stream at the nozzle in feet per second, and y is the height in feet of the stream x feet from the nozzle (Figure 4). Assume that v is a positive constant. Find (a) the value of x for which the height y of the stream is maximum for a fixed value of m, (b) the value of m for which the stream hits the ground at the greatest distance from the nozzle, and (c) the value of m for which the water reaches the greatest height on a vertical wall x feet from the nozzle.

Figure 4



SOLUTION

(a) Here m and v are both constant and we seek the value of x that makes $y = mx - 16(1 + m^2)(x/v)^2$ a maximum. Since

$$\frac{dy}{dx} = m - \frac{32(1 + m^2)}{v^2}x \quad \text{and} \quad \frac{d^2y}{dx^2} = -\frac{32(1 + m^2)}{v^2} < 0,$$

the critical value $x = \dfrac{mv^2}{32(1 + m^2)}$ gives the desired maximum height; hence, the maximum value of y is $\dfrac{m^2v^2}{64(1 + m^2)}$.

(b) For any given value of m, the stream hits the ground when $y = 0$; that is, when $mx = 16(1 + m^2)(x/v)^2$, $x > 0$. Solving for x, we obtain $x = \dfrac{mv^2}{16(1 + m^2)}$. [Comparing this with the result of part (a), we see that the stream reaches its maximum height halfway between the nozzle and the point where it hits the ground—all very reasonable.] Our problem here is to find the value of m that maximizes $\dfrac{mv^2}{16(1 + m^2)}$. To do this, set $D_m\left[\dfrac{mv^2}{16(1 + m^2)}\right] = 0$ and solve for the critical values m. Indeed,

$$D_m\left[\frac{mv^2}{16(1 + m^2)}\right] = \frac{v^2(1 - m^2)}{16(1 + m^2)^2},$$

so $m = \pm 1$ gives the critical values. We reject $m = -1$ on obvious physical grounds. The solution $m = 1$ would indicate that, to squirt the water a maximum distance, one holds the nozzle at a 45-degree angle. This seems so reasonable that only a dyed-in-the-wool skeptic would insist on completing the rigorous check to make certain that $m = 1$ gives an absolute maximum.

(c) Here, x and v are constants and y depends on the variable slope m according to $y = mx - 16(1 + m^2)(x/v)^2$. Thus,

$$\frac{dy}{dm} = x - 32\left(\frac{x}{v}\right)^2 m \quad \text{and} \quad \frac{d^2y}{dm^2} = -32\left(\frac{x}{v}\right)^2 < 0;$$

hence, the critical value $m = \dfrac{v^2}{32x}$ gives y the maximum value $\dfrac{v^2}{64} - \dfrac{16x^2}{v^2}$.

3.4, pg 150 to pg 154

#2) $s = 6t - t^2, \quad 0 \leq t \leq 6$

a) Displacement $= \Delta s$

$$\Delta s = s(6) - s(0) = 0 - 0 = 0$$

V_{AV} = Average Velocity

$$V_{AV} = \frac{\Delta s}{\Delta t} = \frac{0\,m}{6\,sec} = 0\,\frac{m}{sec}$$

b.) $$V = \frac{ds}{dt} = 6 - 2t$$

$$|v(0)| = |6| = 6\,\frac{m}{sec}$$

$$|v(6)| = |-6| = 6\,\frac{cm}{sec}$$

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = -2\,\frac{m}{sec^2}$$

c.) $v(t) = 0$

we have $6 - 2t = 0$

$t = 3$ sec

d.) For $0 \leq t \leq 3$, v is positive and s is increasing, whereas for $3 \leq t \leq 6$, v is negative and s is decreasing

$\therefore$ The body changes direction at $t = 3$

#4) $s = \frac{t^4}{4} - t^3 + t^2, \quad 0 \le t \le 2$

(a) $\Delta s = s(2) - s(0) = 0\,m$

$$V_{AV} = \frac{\Delta s}{\Delta t} = 0 \frac{m}{sec}$$

(b) $v = t^3 - 3t^2 + 2t$

$|v(0)| = 0 \frac{m}{sec}$ and $|v(2)| = 0 \frac{m}{sec}$

$a = 3t^2 - 6t + 2$

$a(0) = 2 \frac{m}{sec^2}$ & $a(2) = 2 \frac{m}{sec^2}$

(c) $v(t) = 0$

we have $t^3 - 3t^2 + 2t = 0$

$t(t-2)(t-1) = 0$

$t = 0, 1, 2$

$v(t) = t(t-2)(t-1)$

$v(t)$ is positive in the interval $0 < t < 1$

$v(t)$ is negative for $1 < t < 2$

$\therefore$ the body changes direction at $t = 1$

#6) $s = \dfrac{25}{t+5}, \quad -4 \leq t \leq 0$

(a) $\Delta s = s(0) - s(-4) = -20\,m$

$$V_{AV} = \frac{-20}{4} = -5\,\frac{m}{sec}$$

(b) $$V = \frac{-25}{(t+5)^2}$$

$$|v(-4)| = 25\,\frac{m}{sec} \quad \& \quad |v(0)| = 1\,\frac{m}{sec}$$

$$a = \frac{50}{(t+5)^3}$$

$$a(-4) = 50\,\frac{m}{sec^2} \quad \& \quad a(0) = \frac{2}{5}\,\frac{m}{sec^2}$$

(c) $v(t) = 0$

$$\frac{-25}{(t+5)^2} = 0$$

$v(t)$ is never zero

$\therefore$ The body never changes direction

#8) $v(t) = t^2 - 4t + 3$

$a = 2t - 4$

(a) $v(t) = 0$

$t^2 - 4t + 3 = 0$

$(t-1)(t-3) = 0$

$t = 1$ or $t = 3$

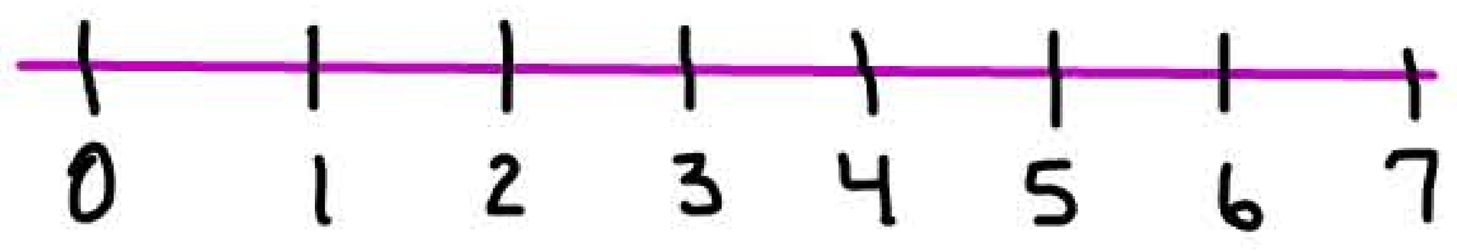
$a(1) = -2\frac{m}{sec^2}$ & $a(3) = 2\frac{m}{sec^2}$

(b) $v(t) > 0$

$(t-3)(t-1) > 0$

$t - 3 > 0$ or $t - 1 > 0$

$t > 3$ $\qquad$ $t > 1$



We have

$0 < t < 1$ or $t > 3$

The body is moving forward

$v(t) < 0$

$(t-3)(t-1) < 0$

$t < 1$ or $t < 3$

we have $1 < t < 3$

The body is moving backward

(C.) Velocity Increasing

$a(t) > 0$

$2t - 4 > 0$

$t > 2$

Velocity decreasing

$a(t) < 0$

$2t - 4 < 0$

$t < 2$

#9)

Mars

$S_m = 1.86t^2$

$V_m = 3.72t$

we want to find t

when $V_m = 27.8$

we have

$3.72t = 27.8$

$t = 7.5$ sec

Jupiter

$S_J = 11.44t^2$

$V_J = 22.88t$

we want to find t

when $V_J = 27.8$

we have

$22.88t = 27.8$

$t = 1.2$ sec

#10)

$$S = 24t - 0.8t^2$$

(a)

$v(t) = s'(t) = 24 - 1.6t$

$a(t) = v'(t) = s''(t)$

$a(t) = -1.6 \frac{m}{sec^2}$

(b) $v(t) = 0$

We have

$24 - 1.6t = 0$

$t = 15$ sec

(c)

$s(15) = 24(15) - 0.8(15)^2$

$s(15) = 180m$

(d)

$s(t) = 90$

$24t - 0.8t^2 = 90$

$0.8t^2 - 24t + 90 = 0$

$a = 0.8$, $b = -24$, $c = 90$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{24 \pm \sqrt{(-24)^2 - 4(0.8)(90)}}{2(0.8)}$$

$t = 4.39$ sec

Going Up

$t = 25.6$ sec

Going down

e.) Twice the time it took to reach its highest point or 30 sec

#11) $s = 15t - \frac{1}{2} g_s t^2$

$$\frac{ds}{dt} = v(t) = 15 - g_s t$$

$v(t) = 0$ at its maximum height

$$15 - g_s t = 0$$

$$t = \frac{15}{g_s}$$

$$t = 20$$

We have

$$\frac{15}{g_s} = 20$$

$$g_s = \frac{3}{4} \frac{m}{sec^2} = 0.75 \frac{m}{sec^2}$$

#12) The Moon

$s_m = 832t - 2.6t^2$

$s_m = 0$

solve for t to determine the time it takes to return to surface.

$832t - 2.6t^2 = 0$

$t(832 - 2.6t) = 0$

$t = 0$; $832 - 2.6t = 0$

$t = 320$ sec

The Earth

$s_e = 832t - 16t^2$

$s_e = 0$

solve for t to determine the time it takes to return to surface.

$832t - 16t^2 = 0$

$t(832 - 16t) = 0$

$t = 0$; $832 - 16t = 0$

$t = 52$ sec

#12) continued.

From Physics, we know that the travel time for a particle to reach its maximum height is equal to the time it takes for the particle to descend from its maximum height back to the surface.

When the bullet reaches its maximum height, its velocity is momentarily zero.

Velocity on the Moon

$S_m = 832t - 2.6t^2$

$\frac{dS_m}{dt} = V_m(t)$

$V_m(t) = 832 - 5.2t$

$V_m(t) = 0$ at its max height.

$832 - 5.2t = 0$

$5.2t = 832$

$t = 160$ sec

$S_m(160) =$

$832(160) - 2.6(160)^2$

$S_m(160) = 66{,}560$ ft

Height above the surface of the moon.

Velocity on the Earth

$S_e = 832t - 16t^2$

$\frac{dS_e}{dt} = V_e(t)$

$V_e(t) = 832 - 32t$

$V_e(t) = 0$ at its max height

$832 - 32t = 0$

$32t = 832$

$t = 26$ sec

$S_e(26) = 832(26) - 16(26)^2$

$S_e(26) = 10816$ ft

Height above the surface of the earth

#13)

(a)

$$s = 179 - 16t^2$$

$$v(t) = -32t$$

$$\text{speed} = |v(t)| = 32\,\frac{\text{ft}}{\text{sec}}$$

$$a(t) = -32\,\frac{\text{ft}}{\text{sec}^2}$$

(b)

$$s = 0$$

$$179 - 16t^2 = 0$$

$$16t^2 = 179$$

$$t = \sqrt{\frac{179}{16}}$$

$$t \approx 3.3 \text{ sec}$$

c.) When $t = \sqrt{\dfrac{179}{16}}$,

$$v = -32\sqrt{\frac{179}{16}} \approx 107\,\frac{\text{ft}}{\text{sec}}$$

#14)

a) $\lim_{\theta \to \frac{\pi}{2}} V = \lim_{\theta \to \frac{\pi}{2}} 9.8(\sin\theta)t = 9.8t$

We expect

$$V = 9.8t \frac{m}{sec} \quad \text{in free fall}$$

b.) $a(t) = \frac{dv}{dt} = 9.8 \frac{m}{sec^2}$

#16a)

(a) P is moving to the left when $2<t<3$

or $5<t<6$

P is moving to the right when

$0<t<1$

P is standing still when

$1<t<2$ or $3<t<5$

#16b)

(sec, cm)

O(0,0) ; A(1,2) ; B(2,2) ; C(3,-2) ; D(5,-2)

E(6,-4)

$$m_{\overline{OA}} = \frac{2-0}{1-0} = 2\,\frac{cm}{sec} \quad ; \quad [0,1]$$

$$m_{\overline{AB}} = \frac{2-2}{2-1} = 0\,\frac{cm}{sec} \quad ; \quad [1,2]$$

$$m_{\overline{BC}} = \frac{-2-2}{3-2} = -4\,\frac{cm}{sec} \quad ; \quad [2,3]$$

$$m_{\overline{CD}} = \frac{-2-(-2)}{5-3} = 0\,\frac{cm}{sec} \quad ; \quad [3,5]$$

$$m_{\overline{DE}} = \frac{-4+2}{6-5} = -2\,\frac{cm}{sec} \quad ; \quad [5,6]$$