3.4 The Derivative As A Rate of Change

* Instantaneous Rate of Change

DEFINITION: The instantaneous rate of change of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

provided the limit exists.

→ Instantaneous rates are limits of average rates.

→ When "rate of change" is mentioned, we mean "instantaneous rate of change."

3.4 The Derivative As A Rate Of Change

* Motion Along A Line: Displacement, Velocity, Speed, Acceleration, and Jerk

Suppose that an object (or body, considered as a whole mass) is moving along a coordinate line (an s-axis), usually horizontal or vertical, so that we know its position s on that line as a function of time t:

$$s = f(t)$$

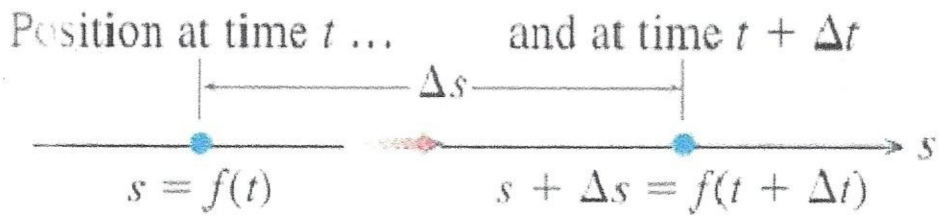


FIGURE 3.15 The positions of a body moving along a coordinate line at time t and shortly later at time $t + \Delta t$. Here the coordinate line is horizontal.

$$s = f(t)$$

The DISPLACEMENT of the object over the time interval from t to $t + \Delta t$ (Figure 3.15) is

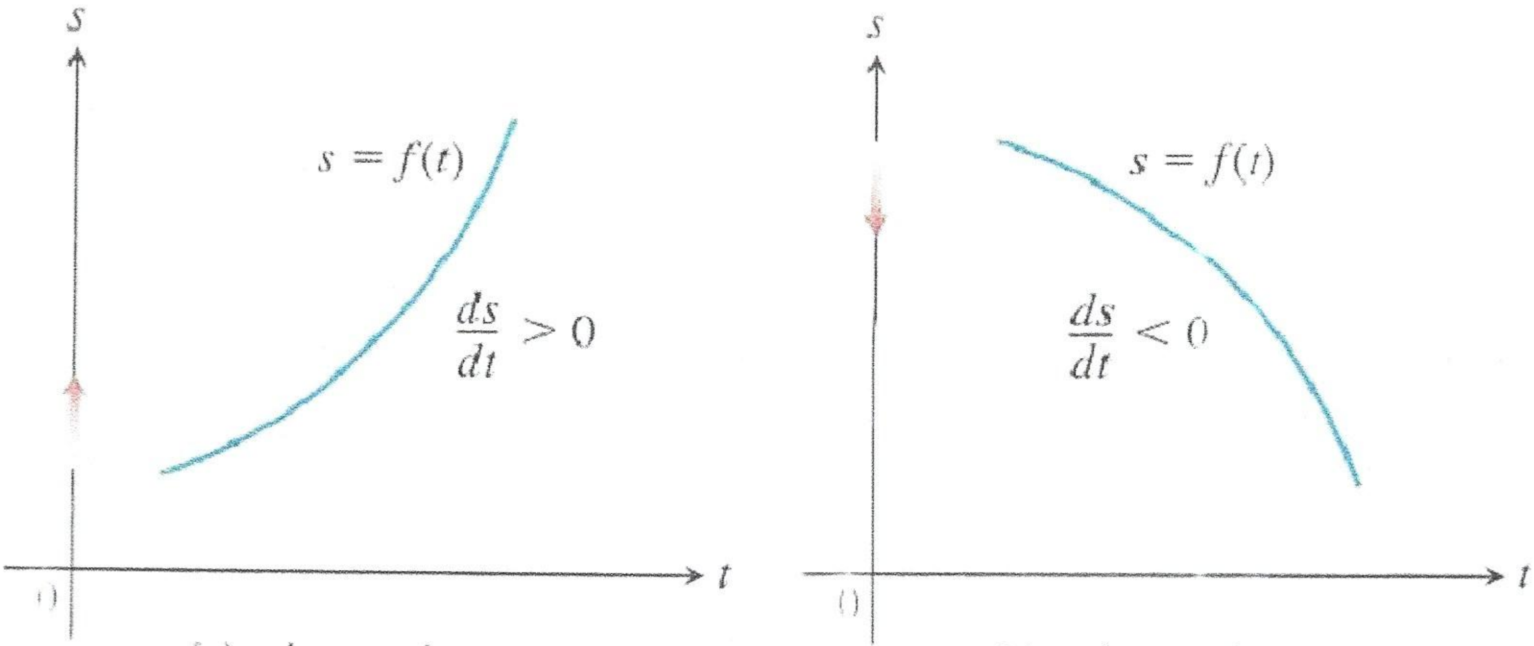
$$\Delta s = f(t + \Delta t) - f(t)$$

and the AVERAGE VELOCITY of the object over that time interval is

$$V_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

DEFINITION: VELOCITY (Instantaneous Velocity) is the derivative of position with respect to time. If a body's position at time t is $s=f(t)$, then the body's velocity at time t is

$$v(t)=\frac{ds}{dt}=\lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}$$



(a) s increasing: positive slope so moving upward

(b) s decreasing: negative slope so moving downward

FIGURE 3.16: For motion $s = f(t)$ along a straight line (the vertical axis), $v = \frac{ds}{dt}$ is

(a) positive when s increases and

(b) negative when s decreases.

* Velocity tells the direction of motion

→ when the object is moving forward (s increasing), the velocity is positive

→ when the object is moving backward (s decreasing), the velocity is negative

→ If the coordinate line is vertical, the object moves upward for positive velocity and downward for negative velocity.

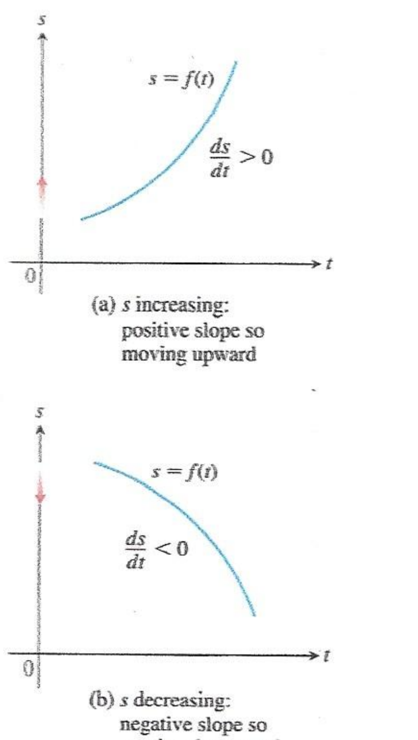


FIGURE 3.16 For motion $s = f(t)$ along a straight line (the vertical axis), $v = ds/dt$ is (a) positive when s increases and (b) negative when s decreases.

The blue curves in Figure 3.16 represent position along the line over time.

The blue curves do not portray the path of motion – which lies along the vertical s-axis.

* SPEED

speed is the absolute value of velocity

speed measures the rate of progress regardless of direction.

DEFINITION: SPEED is the absolute value of velocity.

$$\text{speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

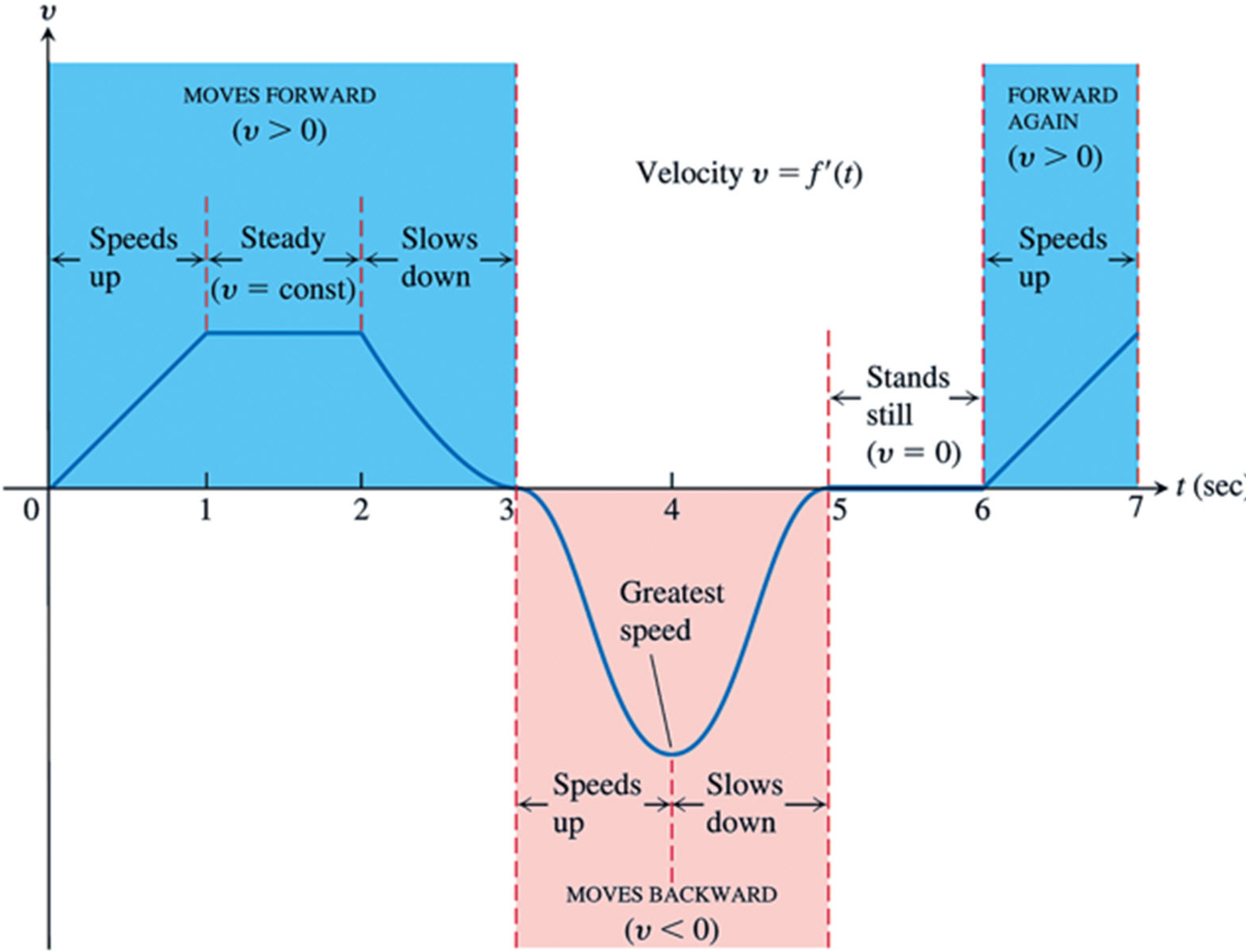


FIGURE 3.17 The velocity graph of a particle moving along a horizontal line, discussed in Example 2.

The rate at which a body's velocity changes is the body's acceleration. The acceleration measures how quickly the body picks up or loses speed.

A sudden change in acceleration is called a ***jerk***.

When a ride in a car or A bus is jerky, it is not that the accelerations involved are necessarily large, but that the changes in acceleration are abrupt.

DEFINITIONS **Acceleration** is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

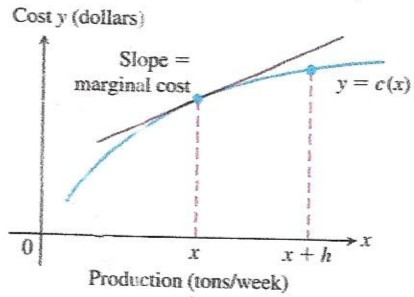


FIGURE 3.20 Weekly steel production: $c(x)$ is the cost of producing x tons per week. The cost of producing an additional h tons is $c(x + h) - c(x)$.

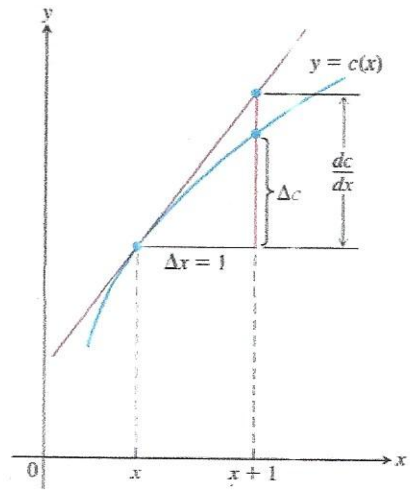


FIGURE 3.21 The marginal cost dc/dx is approximately the extra cost Δc of producing $\Delta x = 1$ more unit.

Suppose that $c(x)$ represents the dollars needed to produce x tons of steel in one week.

It costs more to produce $x+h$ tons per week, and the cost difference, divided by h, is the average cost of producing each additional ton:

$$\frac{c(x+h)-c(x)}{h} = \text{average cost of each of the additional } h \text{ tons of steel produced.}$$

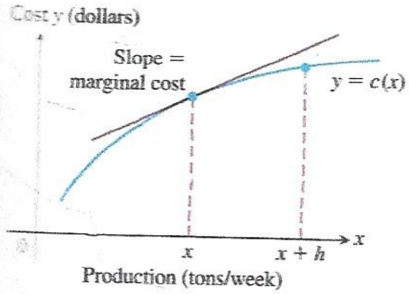


FIGURE 3.20 Weekly steel production: $c(x)$ is the cost of producing x tons per week. The cost of producing an additional h tons is $c(x+h) - c(x)$.

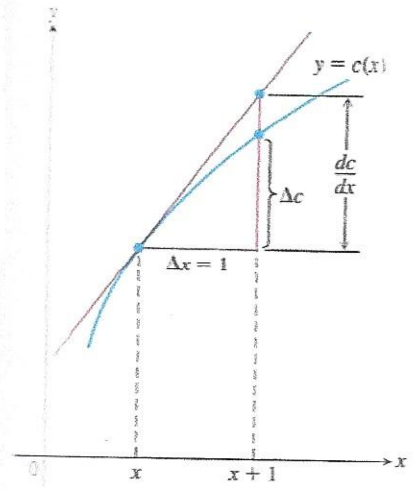


FIGURE 3.21 The marginal cost dc/dx is approximately the extra cost Δc of producing $\Delta x = 1$ more unit.

The limit of this ratio as $h \to 0$ is the MARGINAL COST of producing more steel per week when the current weekly production is x tons (Fig 3.20):

$$\frac{dc}{dx} = \lim_{h \to 0} \frac{c(x+h) - c(x)}{h} = \text{marginal cost of production}$$

Sometimes the marginal cost of production is loosely defined to be the extra cost of producing one additional unit:

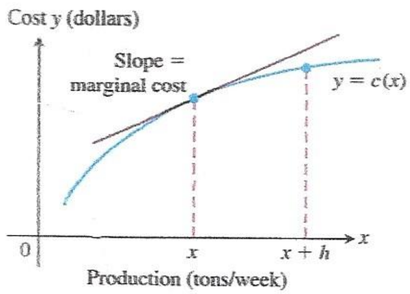


FIGURE 3.20 Weekly steel production: $c(x)$ is the cost of producing x tons per week. The cost of producing an additional h tons is $c(x + h) - c(x)$.

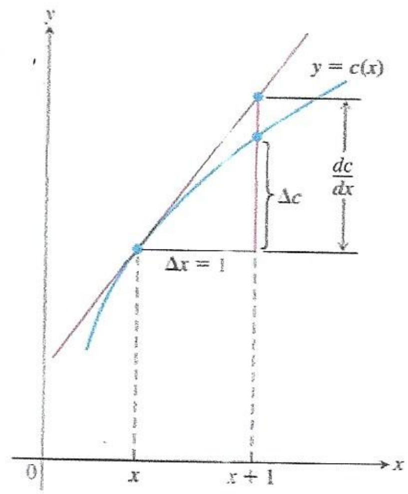


FIGURE 3.21 The marginal cost dc/dx is approximately the extra cost Δc of producing $\Delta x = 1$ more unit.

$$\frac{\Delta c}{\Delta x} = \frac{c(x+1) - c(x)}{1}$$

which is approximated by the value $\frac{dc}{dx}$ at x. This approximation is acceptable if the slope of the graph of c does not change quickly near x.

Then the difference quotient will be close to its limit dc/dx, which is the rise in the tangent line if $\Delta x = 1$. (Fig 3.21). The approximation often works well for large values of x.

Example: suppose the total cost in dollars per week by ABC Corporation for producing its best-selling product is given by

$$C(x) = 10,000 + 3000x - 0.4x^2$$

Find the actual cost of producing the 101^{st} item.

solution

$c(100) \Rightarrow$ the actual cost of producing 100 items

$$c(100) = 306000$$

$c(101) \Rightarrow$ the actual cost of producing 101 items

$$c(101) = 308919.6$$

The cost of producing the 101^{st} item can be found by computing the Average Rate of change.

$$\frac{C(x+h)-C(x)}{(x+h)-x} = \frac{C(101)-C(100)}{101-100}$$

$$= \frac{2919.6}{1}$$

$$= 2919.6$$

This will give us the actual cost of producing the 101^{st} item.

However, it is often inconvenient to use.

For this reason, Marginal Cost is usually approximated by the instantaneous rate of change of the total cost function evaluated at the specific point of interest.

Example 2: Suppose the total cost in dollars per week by ABC corporation for producing its best-selling product is given by

$$C(x) = -0.4x^2 + 3000x + 10,000$$

Find $c'(100)$ and interpret the results.

$$c'(x) = -0.8x + 3000$$

$c'(100) = 2920 \Rightarrow$ the cost of producing the 101^{st} item.

$$c'(100) - [c(101) - c(100)] = 2920 - 2919.6$$

$$= 0.4$$

observe that the answers for Example 1 and Example 2 are very close.

This shows you why we can work with the derivative of the cost function rather than the average rate of change.

Marginal Cost Function $\Rightarrow$ Derivative of the total cost function.