Compute the derivative by definition: The four step procedure

Given a function $f(x)$, the definition of $f'(x)$, the **derivative** of $f(x)$, is

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$

provided the limit exists. The derivative function $f'(x)$ is sometimes also called a **slope-predictor function**. The following is a four-step process to compute $f'(x)$ by definition.
Input: a function $f(x)$
Step 1 Write $f(x+h)$ and $f(x)$.
Step 2 Compute $f(x+h)-f(x)$. Combine like terms. If h is a common factor of the terms, factor the expression by removing the common factor h.
Step 3 Simply $\frac{f(x+h)-f(x)}{h}$. As $h\to 0$ in the last step, we **must** cancel the zero factor h in the denominator in Step 3.
Step 4 Compute $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ by letting $h\to 0$ in the simplified expression.

Example 1 Let $f(x)=ax^2+bx+c$. Compute $f'(x)$ by the definition (that is, use the four step process).

Solution: Step 1, write

$$f(x+h)=a(x+h)^2+b(x+h)+c=a(x^2+2xh+h^2)+bx+bh+c=ax^2+2axh+ah^2+bx+bh+c.$$

Step 2: Use algebra to single out the factor h.

$$f(x+h)-f(x)=(ax^2+2axh+ah^2+bx+bh+c)-(ax^2+bx+c)=2axh+ah^2+bh=h(2ax+ah+b).$$

Step 3: Cancel the zero factor h is the most important thing in this step.

$$\frac{f(x+h)-f(x)}{h}=\frac{h(2ax+ah+b)}{h}=2ax+ah+b.$$

Step 4: Let $h\to 0$ in the resulted expression in Step 3.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}2ax+ah+b=2ax+0+b=2ax+b.$$

Example 2 Let $f(x)=\frac{1}{x+1}$. Compute $f'(x)$ by the definition (that is, use the four step process).

Solution: Step 1, write

$$f(x+h) = \frac{1}{(x+h)+1} = \frac{1}{x+h+1}.$$

Step 2: Use algebra to single out the factor h.

$$f(x+h) - f(x) = \frac{1}{x+h+1} - \frac{1}{x+1} = \frac{(x+1)-(x+h+1)}{(x+h+1)(x+1)} = \frac{h}{(x+h+1)(x+1)}.$$

Step 3: Cancel the zero factor h is the most important in this step.

$$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\left[\frac{1}{x+h+1} - \frac{1}{x+1}\right] = \frac{1}{h}\left[\frac{-h}{(x+h+1)(x+1)}\right] = \frac{-1}{(x+h+1)(x+1)}.$$

Step 4: Let $h \to 0$ in the resulted expression in Step 3.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{-1}{(x+h+1)(x+1)} = \frac{-1}{(x+0+1)(x+1)} = \frac{-1}{(x+1)^2}.$$

Example 3 Let $f(x) = \sqrt{2x+5}$. Compute $f'(x)$ by the definition (that is, use the four step process).

Solution: Step 1, write

$$f(x+h) = \sqrt{2(x+h)+5} = \sqrt{2x+2h+5}.$$

Step 2: Use algebra to single out the factor h. Here we need the identity $(A+B)(A-B) = A^2 - B^2$ to get rid of the square root so that h can be factored out.

$$f(x+h)-f(x) = \sqrt{2x+2h+5} - \sqrt{2x+5} = \frac{(2x+2h+5)-(2x+5)}{\sqrt{2x+2h+5}+\sqrt{2x+5}} = \frac{2h}{\sqrt{2x+2h+5}+\sqrt{2x+5}}.$$

Step 3: Cancel the zero factor h is the most important thing in this step.

$$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\left[\frac{2h}{\sqrt{2x+2h+5}+\sqrt{2x+5}}\right] = \frac{2}{\sqrt{2x+2h+5}+\sqrt{2x+5}}.$$

Step 4: Let $h \to 0$ in the resulted expression in Step 3.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{2}{\sqrt{2x+2h+5}+\sqrt{2x+5}} = \frac{2}{\sqrt{2x+0+5}+\sqrt{2x+5}} = \frac{1}{\sqrt{2x+5}}.$$

Find an equation of the tangent line (using the 4-step procedure to find slopes)

Given a curve $y = f(x)$ and a point (x_0, y_0) on it, an equation of the line tangent to the curve $y = f(x)$ at the point (x_0, y_0) is

$$y - y_0 = f'(x_0)(x - x_0),$$

provided the $f'(x_0)$ exists. (Therefore, $f'(x_0)$ is the slope of the tangent line at (x_0, y_0)).

Example 1 Let $f(x) = 4x^2 + 5x + 6$. Find an equation of the line tangent to the curve $y = f(x)$ at $(1, 15)$. Compute $f'(1)$ by the definition (that is, use the four step process).

Solution: Step 1, write

$$f(1 + h) = 4(1 + h)^2 + 5(1 + h) + 6 = 4(1 + 2h + h^2) + 5 + 5h + 6 = 15 + 13h + 4h^2.$$

Step 2: Use algebra to single out the factor h.

$$f(1 + h) - f(1) = (15 + 13h + 4h^2) - 15 = 13h + 4h^2 = h(13 + 4h).$$

Step 3: Cancel the zero factor h is the most important thing in this step.

$$\frac{f(x + h) - f(x)}{h} = \frac{h(13 + 4h)}{h} = 13 + 4h.$$

Step 4: Let $h \to 0$ in the resulted expression in Step 3.

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h\to 0} 13 + 4h = 13 + 0 = 13.$$

Therefore, the answer is

$$y - 15 = 13(x - 1).$$

Example 2 Let $f(x) = \frac{1}{x + 1}$. Find an equation of the line tangent to the curve $y = f(x)$ at the point where $x = 2$. Compute $f'(2)$ by the definition (that is, use the four step process).

Solution: Step 1, write

$$f(2 + h) = \frac{1}{(2 + h) + 1} = \frac{1}{2 + h + 1}.$$

Step 2: Use algebra to single out the factor h.

$$f(2+h) - f(2) = \frac{1}{2+h+1} - \frac{1}{2+1} = \frac{(2+1)-(2+h+1)}{(2+h+1)(2+1)} = \frac{-h}{(3+h)(3)}.$$

Step 3: Cancel the zero factor h is the most important in this step.

$$\frac{f(2+h)-f(2)}{h} = \frac{1}{h}\left[\frac{1}{3+h} - \frac{1}{3}\right] = \frac{1}{h}\left[\frac{-h}{(3+h)(3)}\right] = \frac{-1}{(3+h)(3)}.$$

Step 4: Let $h \to 0$ in the resulted expression in Step 3.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{-1}{(3+h)(3)} = \frac{-1}{(3+0)(3)} = \frac{-1}{9}.$$

Therefore, the slope $m = -\frac{1}{9}$. As $y_0 = f(2) = \frac{1}{3}$, the answer is

$$y - \frac{1}{3} = -\frac{1}{9}(x-2).$$

Example 3 Let $f(x) = \sqrt{2x+5}$. Find an equation of the line tangent to the curve $y = f(x)$ at the point where $x = 2$. Compute $f'(2)$ by the definition (that is, use the four step process).

Solution: Step 1, write

$$f(2+h) = \sqrt{2(2+h)+5} = \sqrt{4+2h+5} = \sqrt{9+2h}.$$

Step 2: Use algebra to single out the factor h. Here we need the identity $(A+B)(A-B) = A^2 - B^2$ to get rid of the square root so that h can be factored out.

$$f(x+h) - f(x) = \sqrt{9+2h} - \sqrt{9} = \frac{(9+2h)-3}{\sqrt{9+2h}+3} = \frac{2h}{\sqrt{9+2h}+3}.$$

Step 3: Cancel the zero factor h is the most important thing in this step.

$$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\left[\frac{2h}{\sqrt{9+2h}+3}\right] = \frac{2}{\sqrt{9+2h}+3}.$$

Step 4: Let $h \to 0$ in the resulted expression in Step 3.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{2}{\sqrt{9+2h}+3} = \frac{2}{\sqrt{9+0}+3} = \frac{2}{3+3} = \frac{1}{3}.$$

Therefore, the slope $m = \frac{1}{3}$. As $y_0 = f(2) = 3$, the answer is

$$y - 3 = \frac{1}{3}(x-2).$$

Example 4: Let $f(x) = \dfrac{2}{x-1}$ be given.
(a) Use definition of the derivative to find $f'(x)$.
(b) Find an equation of the line tangent to the curve $y = f(x)$ at the point where $x = 3$.

Solution: (a) Step 1: Compute

$$f(x+h) = \frac{2}{(x+h)-1} = \frac{2}{x+h-1}.$$

Step 2: Compute the difference $f(x+h) - f(x)$. (We must have h as a common factor in the result).

$$\begin{aligned} f(x+h) - f(x) &= \frac{2}{x+h-1} - \frac{2}{x-1} = 2\frac{x-1}{(x-1)(x+h-1)} - 2\frac{x+h-1}{(x-1)(x+h-1)} \\ &= 2\frac{(x-1)-(x+h-1)}{(x-1)(x+h-1)} = 2\frac{-h}{(x-1)(x+h-1)}. \end{aligned}$$

Step 3: Use the result in Step 2 to form and simplify the ratio (the denomination h must be cancelled with the numerator h).

$$\frac{f(x+h)-f(x)}{h} = 2\frac{-h}{(x-1)(x+h-1)}\frac{1}{h} = 2\frac{-1}{(x-1)(x+h-1)}.$$

Step 4: Find the answer by letting $h \to 0$:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} 2\frac{-1}{(x-1)(x+h-1)} = 2\frac{-1}{(x-1)(x+0-1)} = \frac{-2}{(x-1)^2}.$$

(b) First compute $f(3) = 2/(3-1) = 1$. At the point $(3, 1)$, the slope of tangent line is $f'(3) = (-2)/(3-1)^2 = -1/2$. Therefore an equation of the tangent line is

$$y - 1 = \frac{-1}{2}(x-3).$$

Example 5: Let $f(x) = \dfrac{1}{\sqrt{x+2}}$ be given.
(a) Use definition of the derivative to find $f'(x)$.
(b) Find an equation of the line tangent to the curve $y = f(x)$ at the point where $x = -1$.

Solution: (a) Step 1: Compute

$$f(x+h) = \frac{1}{\sqrt{(x+h)+2}} = \frac{1}{\sqrt{x+h+2}}.$$

Step 2: Compute the difference $f(x+h)-f(x)$. (We must have h as a common factor in the result). We shall utilizes the formula $(A+B)(A-B)=A^2-B^2$ (which, as we have seen, is a useful tool to deal with square roots).

$$\begin{aligned}
f(x+h)-f(x) &= \frac{1}{\sqrt{x+h+2}}-\frac{1}{\sqrt{x+2}}=\frac{\sqrt{x+2}}{\sqrt{x+h+2}\sqrt{x+2}}-\frac{\sqrt{x+h+2}}{\sqrt{x+h+2}\sqrt{x+2}}\\
&= \frac{\sqrt{x+2}-\sqrt{x+h+2}}{\sqrt{x+h+2}\sqrt{x+2}}=\frac{\sqrt{x+2}-\sqrt{x+h+2}}{\sqrt{x+h+2}\sqrt{x+2}}\frac{\sqrt{x+2}+\sqrt{x+h+2}}{\sqrt{x+2}+\sqrt{x+h+2}}\\
&= \frac{(x+2)-(x+h+2)}{\sqrt{x+h+2}\sqrt{x+2}(\sqrt{x+2}+\sqrt{x+h+2})}\\
&= \frac{-h}{\sqrt{x+h+2}\sqrt{x+2}(\sqrt{x+2}+\sqrt{x+h+2})}
\end{aligned}$$

Step 3: Use the result in Step 2 to form and simplify the ratio (the denomination h must be cancelled with the numerator h).

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{1}{h}\frac{-h}{\sqrt{x+h+2}\sqrt{x+2}(\sqrt{x+2}+\sqrt{x+h+2})}\\
&= \frac{-1}{\sqrt{x+h+2}\sqrt{x+2}(\sqrt{x+2}+\sqrt{x+h+2})}.
\end{aligned}$$

Step 4: Find the answer by letting $h\to 0$:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{-1}{\sqrt{x+h+2}\sqrt{x+2}(\sqrt{x+2}+\sqrt{x+h+2})}\\
&= \frac{-1}{\sqrt{x+0+2}\sqrt{x+2}(\sqrt{x+2}+\sqrt{x+0+2})}=\frac{-1}{2(x+2)\sqrt{x+2}}.
\end{aligned}$$

(b) First compute $f(-1)=\dfrac{1}{\sqrt{(-1)+2}}=1$. At the point $(-1,1)$, the slope of tangent line is $f'(-1)=\dfrac{-1}{2(-1+2)\sqrt{-1+2}}=\dfrac{-1}{2}$. Therefore an equation of the tangent line is

$$y-1=\frac{-1}{2}(x-(-1)).$$