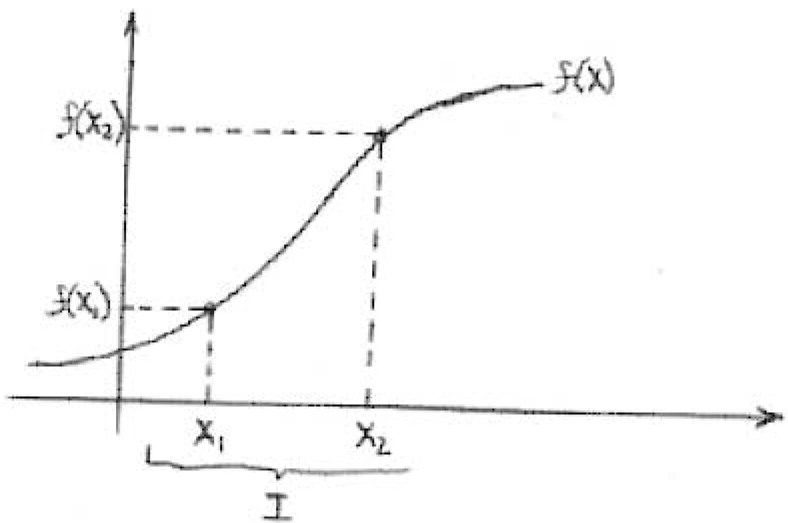
The Derivative As A Function

DEFINITION→ We say that the function f is *increasing on an interval* I if

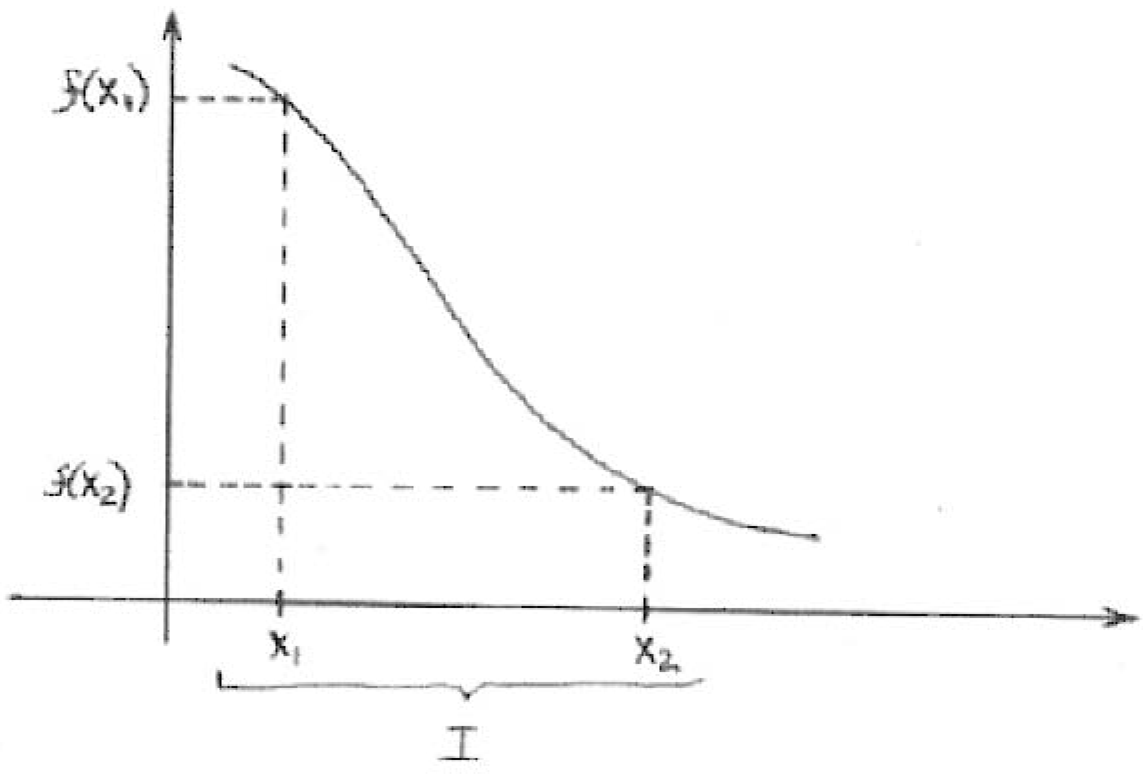
$$x_1 < x_2 \text{ in } I \text{ implies } f(x_1) \leq f(x_2)$$



Increasing on I

we say f is *decreasing* on I if

$$x_1 < x_2 \text{ in } I \text{ implies } f(x_1) \geq f(x_2)$$



Decreasing on I

we use the words *strictly increasing* or *strictly decreasing*

when strict inequalities hold in these statements.

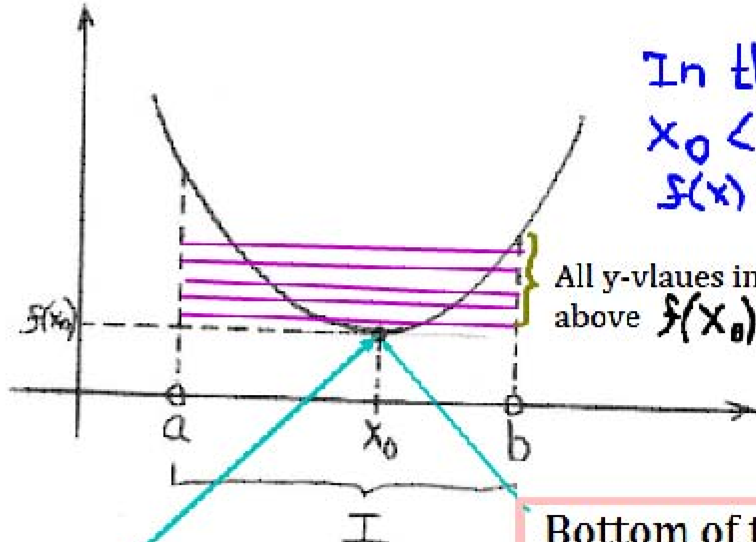
These are "left-to-right" definitions in that f is strictly increasing if it goes up as you move from left to right, and strictly decreasing if it goes down as you move from left to right.

DEFINITION: We say that the function f *has a relative minimum at* x_0 if $f(x_0)$ is the minimum value of f on some open interval (a,b) containing x_0. That is

$$f(x_0) \leq f(x) \quad \text{for all } x \text{ in } (a,b)$$

In the interval $x_0 < x < b$ $f(x)$ is strictly increasing

In the interval $a < x < x_0$ $f(x)$ is strictly decreasing



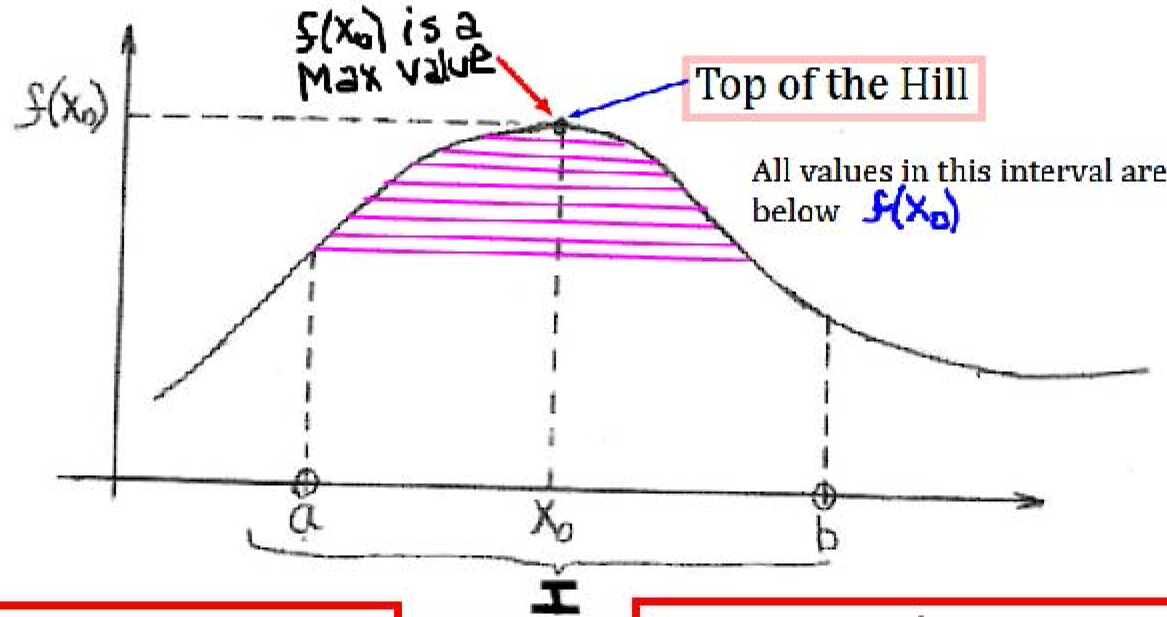
All y-vlaues in this interval are above $f(x_0)$

Bottom of the Hill.

min value $f(x_0)$

Similarly, we say that f *has a relative maximum at* x_0 if $f(x_0)$ is the maximum value of f on some open interval (a, b) containing x_0. That is

$$f(x_0) \geq f(x) \quad \text{for all } x \text{ in } (a, b)$$



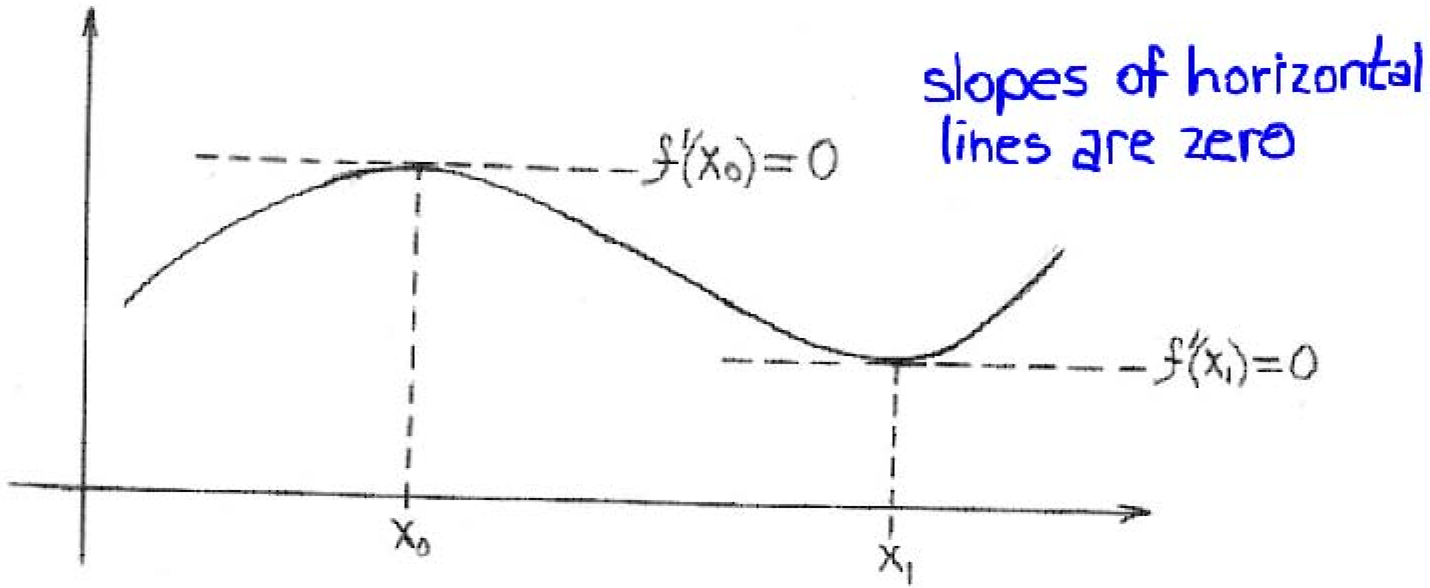
In the interval $a < x < x_0$, $f(x)$ is strictly increasing.

In the interval $x_0 < x < b$, $f(x)$ is strictly decreasing

THEOREM: If x_0 is a relative extreme of f and $f'(x_0)$ exists, then

$$f'(x_0) = 0$$

That is, the tangent line at x_0 is horizontal.



If $f'(x_0)$ does not exist, f may still have a relative extreme in any of several ways:

Fig #1) The left-hand and right-hand limits may exist but fail to be equal, as indicated by the dashed one-sided tangents in the figure. Under such circumstances we have a "CORNER."

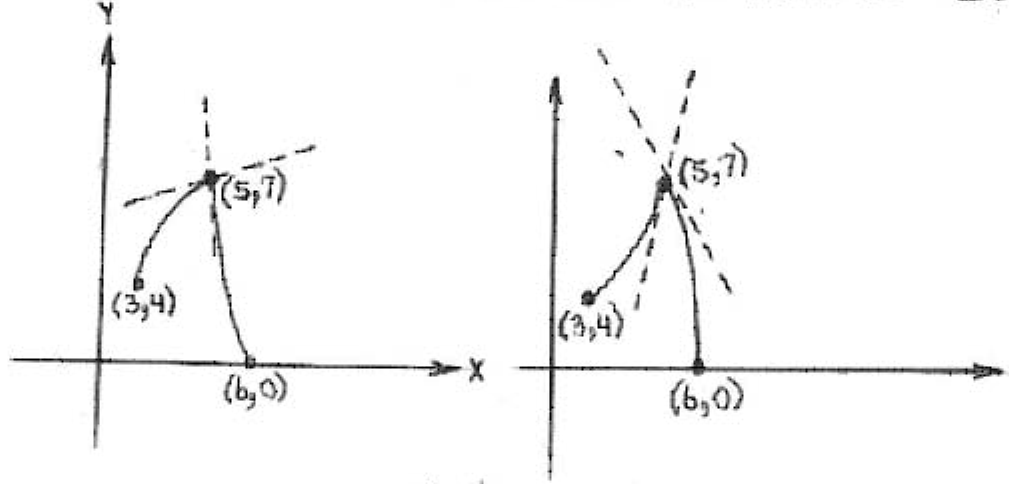
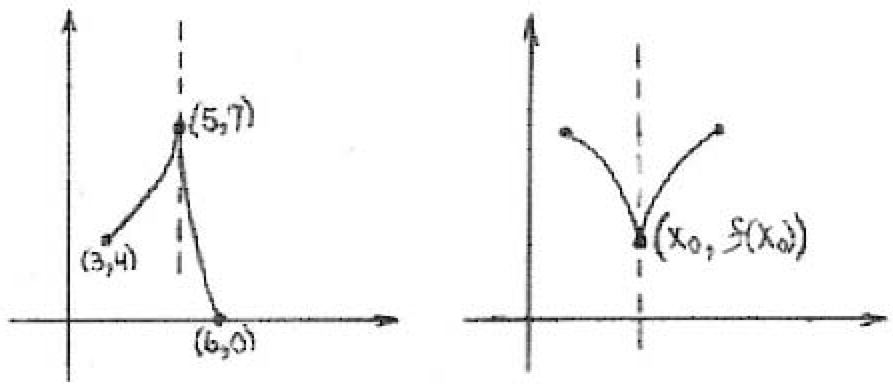


Fig #1

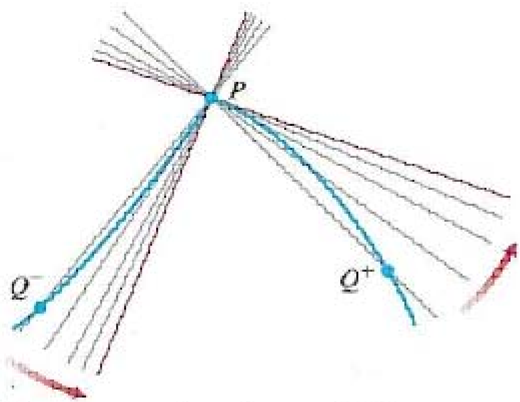
Fig[#]2) The left and right limits may grow or decrease without bound, yielding vertical tangents and "CUSPS" as indicated in the figure, Combinations of Fig[#]1 and Fig[#]2 may also occur.



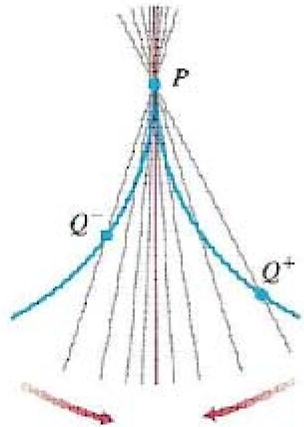
Fig[#]2

When Does a Function *Not* Have a Derivative at a Point?

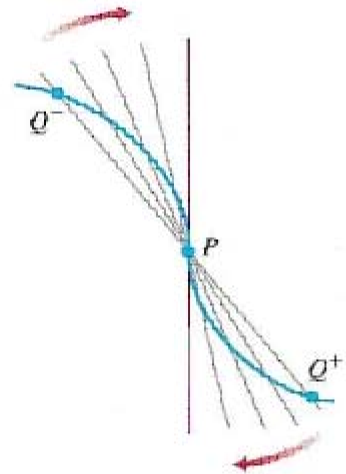
A function has a derivative at a point x_0 if the slopes of the secant lines through $P(x_0, f(x_0))$ and a nearby point Q on the graph approach a finite limit as Q approaches P. Thus differentiability is a "smoothness" condition on the graph of f. A function can fail to have a derivative at a point for many reasons, including the existence of points where the graph has



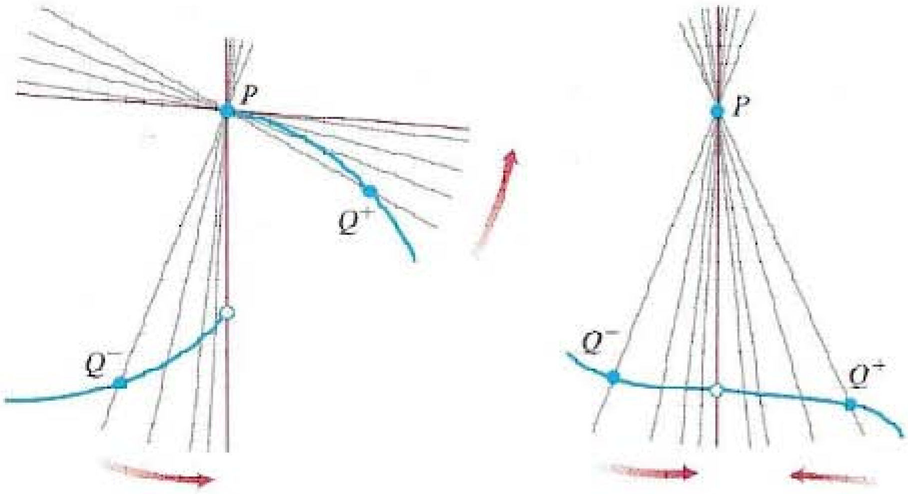
1. a *corner*, where the one-sided derivatives differ



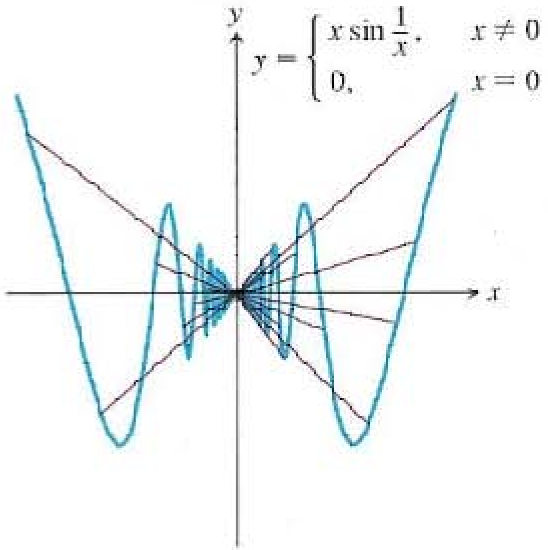
2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other



3. a *vertical tangent line*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$)

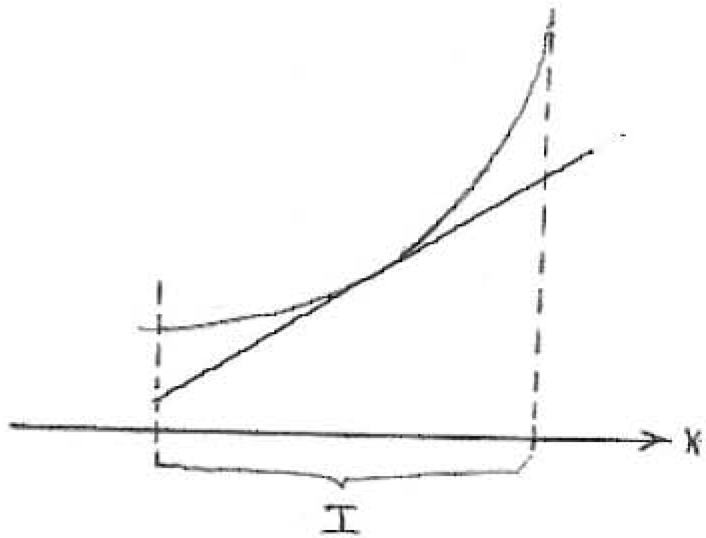


4. a *discontinuity* (two examples shown)

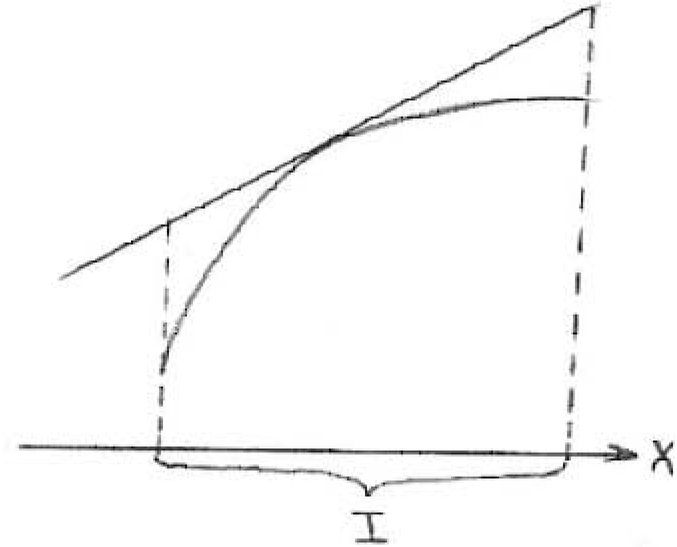


5. wild oscillation

The last example shows a function that is continuous at $x = 0$, but whose graph oscillates wildly up and down as it approaches $x = 0$. The slopes of the secant lines through 0 oscillate between -1 and 1 as x approaches 0, and do not have a limit at $x = 0$.



Concave up

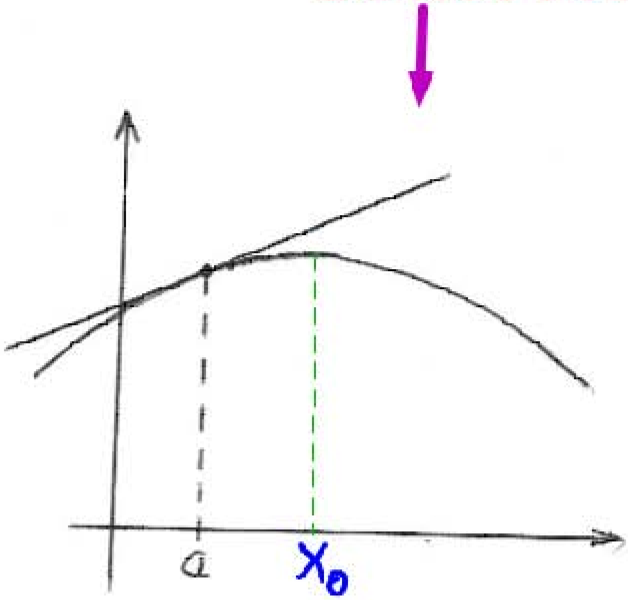


Concave Down

DEFINITION: We say that the function f is *concave up* on the interval I if the graph of f is above its tangent line at each point of I. Similarly, it is *concave down* on I if its graph lies below its tangent line at each point of I

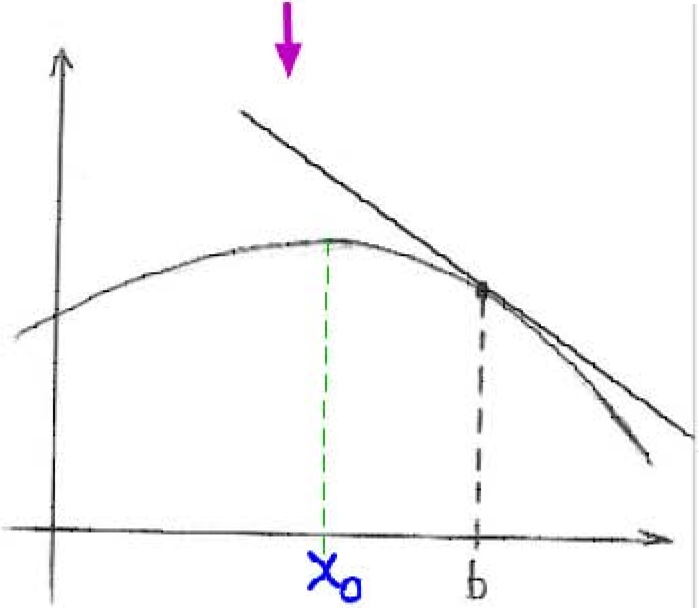
The concept below is very, very important.
We will use this to prove various theorems.

DO NOT FORGET THIS CONCEPT !!!



Positive slope

$f'(x) > 0$



Negative slope

$f'(x) < 0$

1) If $f'(x) > 0$ on an interval I, then f is increasing

2) If $f'(x) < 0$ on an interval I, then f is decreasing

Alternate Formula For The Derivative

Differentiability and Continuity

The following alternative limit form of the derivative is useful in investigating the relationship between differentiability and continuity. The derivative of f at c is

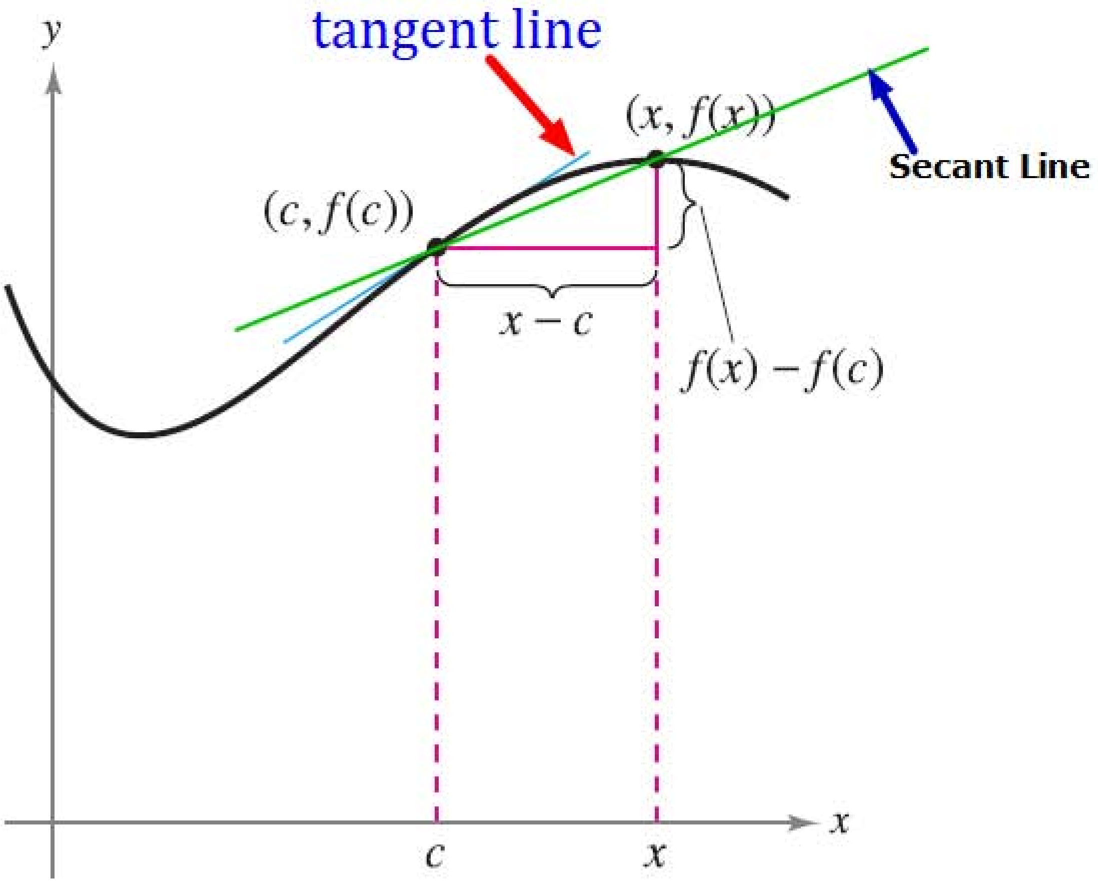
$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Alternative form of derivative

provided this limit exists.

(x_1, y_1)
$\downarrow$ $\downarrow$
$(c, f(c))$

(x_2, y_2)
$\downarrow$ $\downarrow$
$(x, f(x))$



As x approaches c, the secant line approaches the tangent line.

Note that the existence of the limit in this alternative form requires that the one-sided limits

we are still using

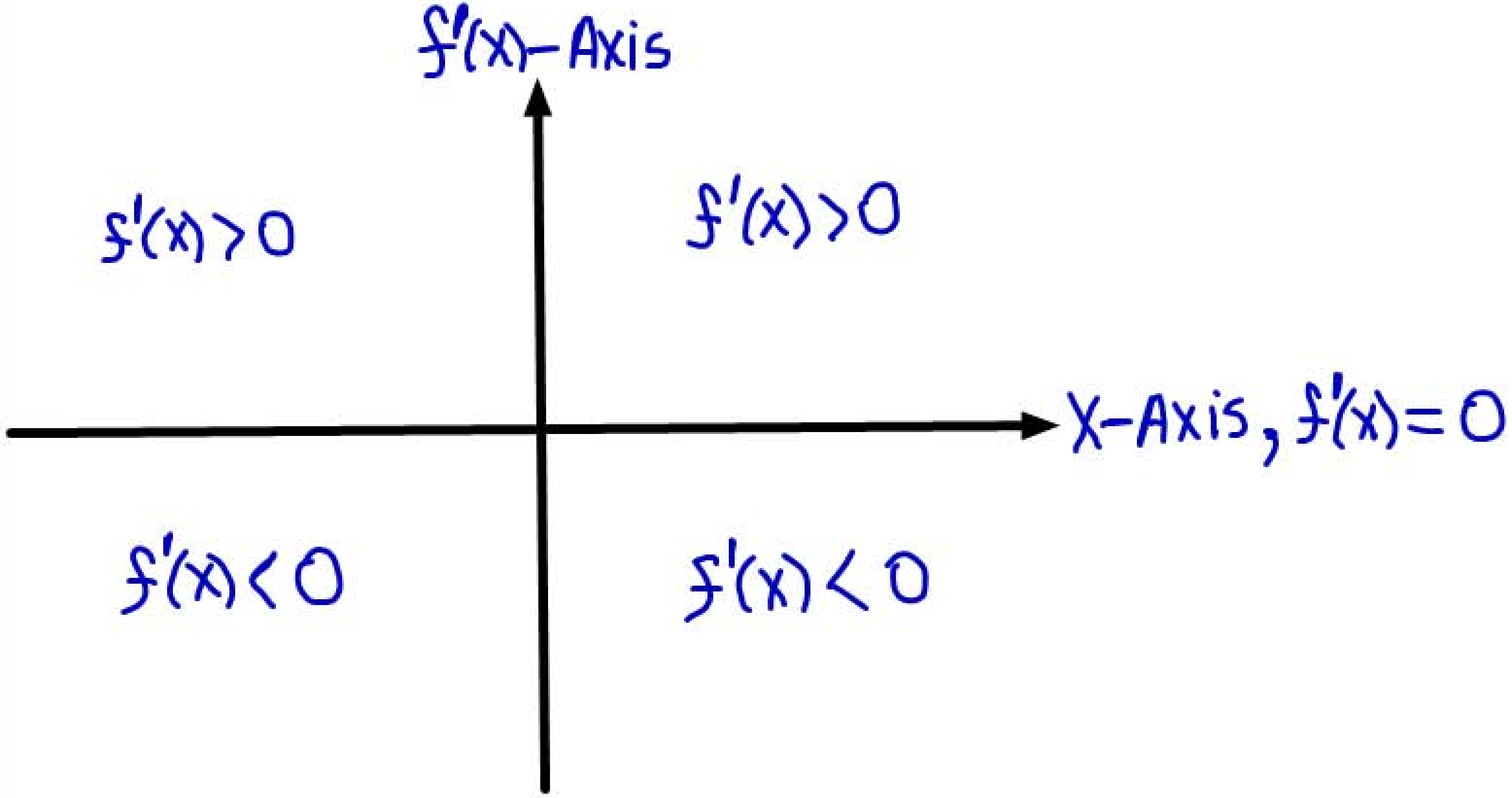
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \quad \text{and} \quad \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

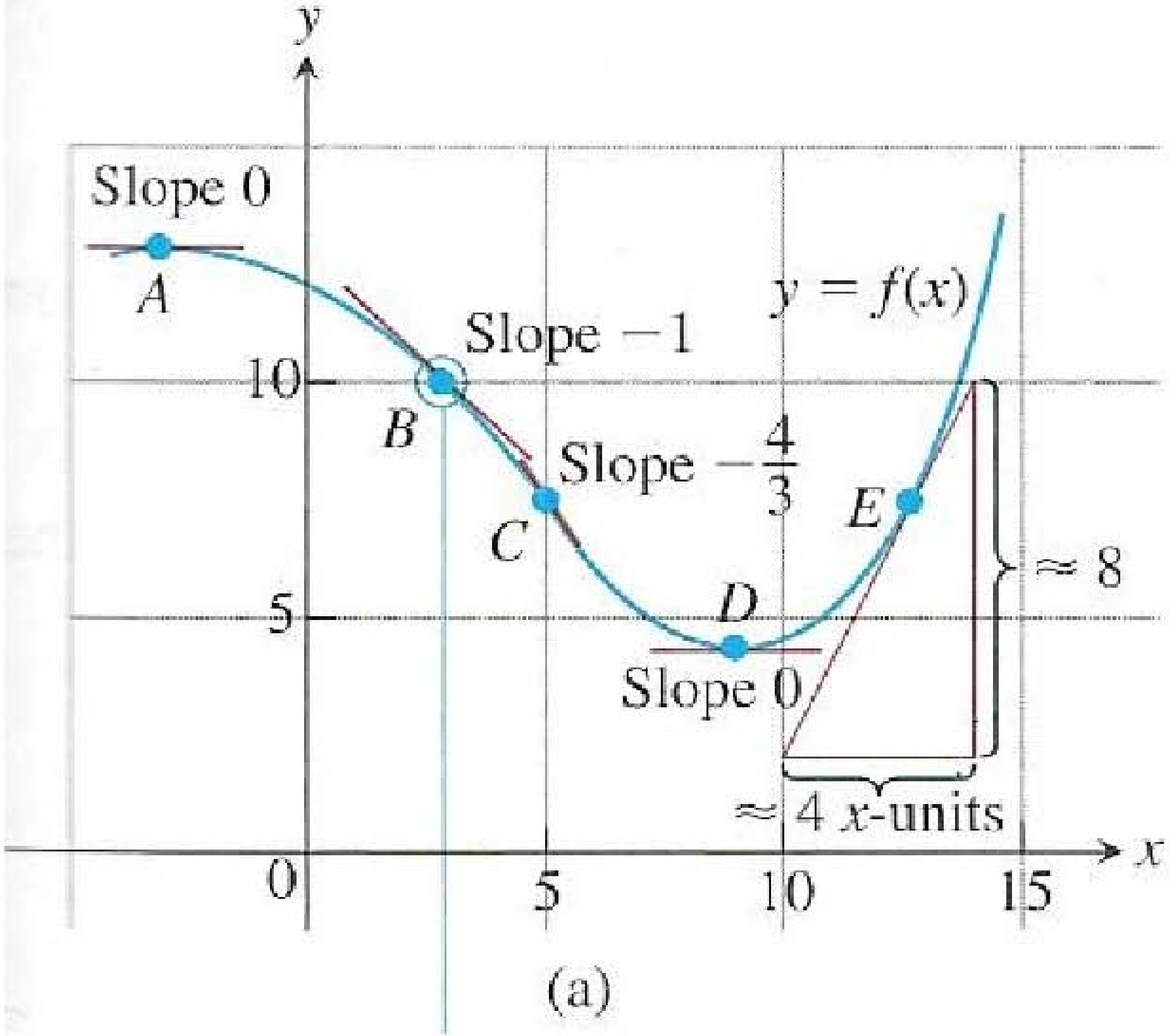
exist and are equal. These one-sided limits are called the **derivatives from the left and from the right,** respectively. It follows that f is **differentiable on the closed interval $[a, b]$** if it is differentiable on (a, b) and if the derivative from the right at a and the derivative from the left at b both exist.

THEOREM 2.1 DIFFERENTIABILITY IMPLIES CONTINUITY

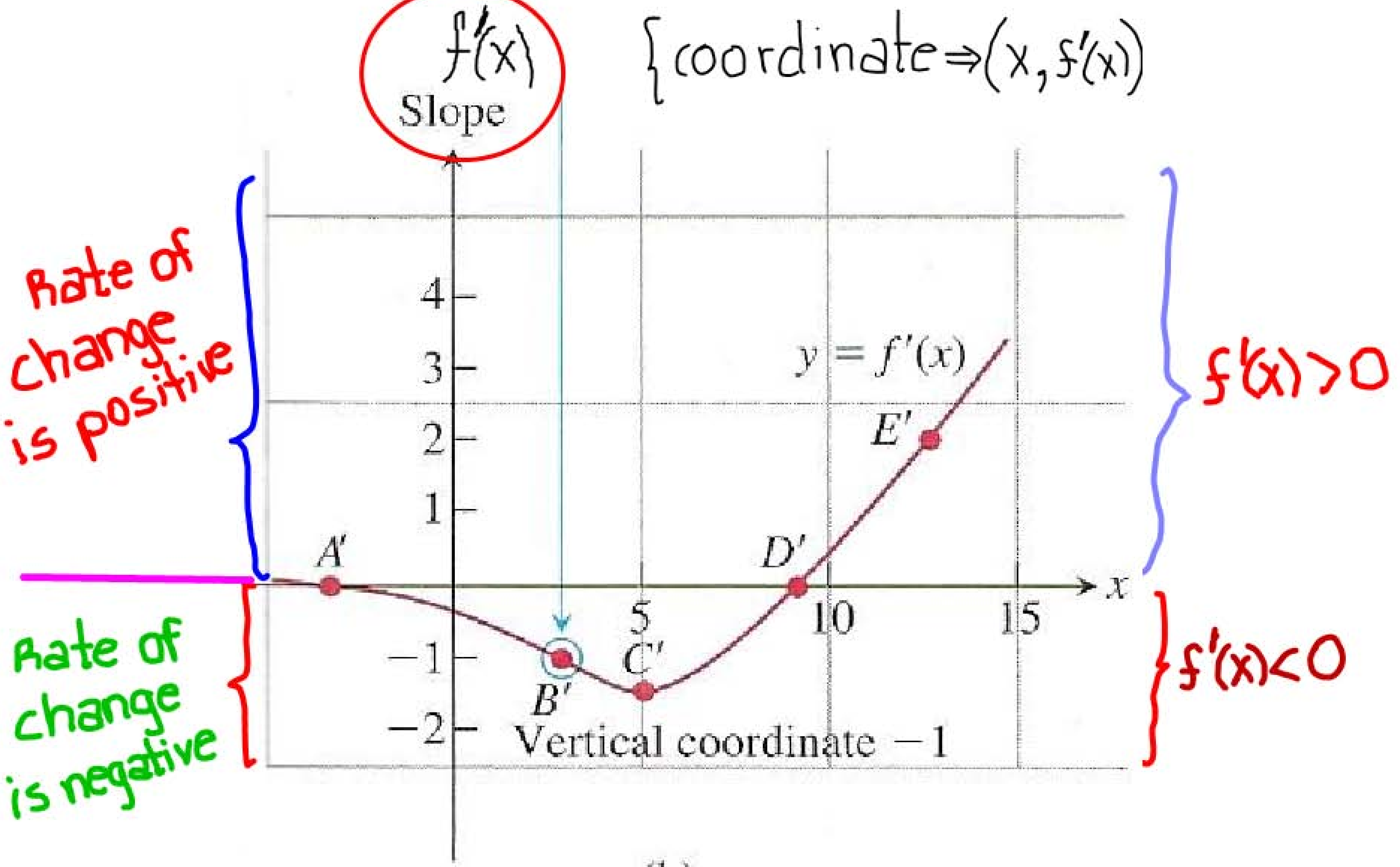
If f is differentiable at $x = c$, then f is continuous at $x = c$.



Derivative As A Function


y
Slope 0
A
y = f(x)
Slope −1
10
B
Slope $-\frac{4}{3}$
E
C
≈ 8
D
5
Slope 0
≈ 4 x-units
x
0
5
10
15

(a)

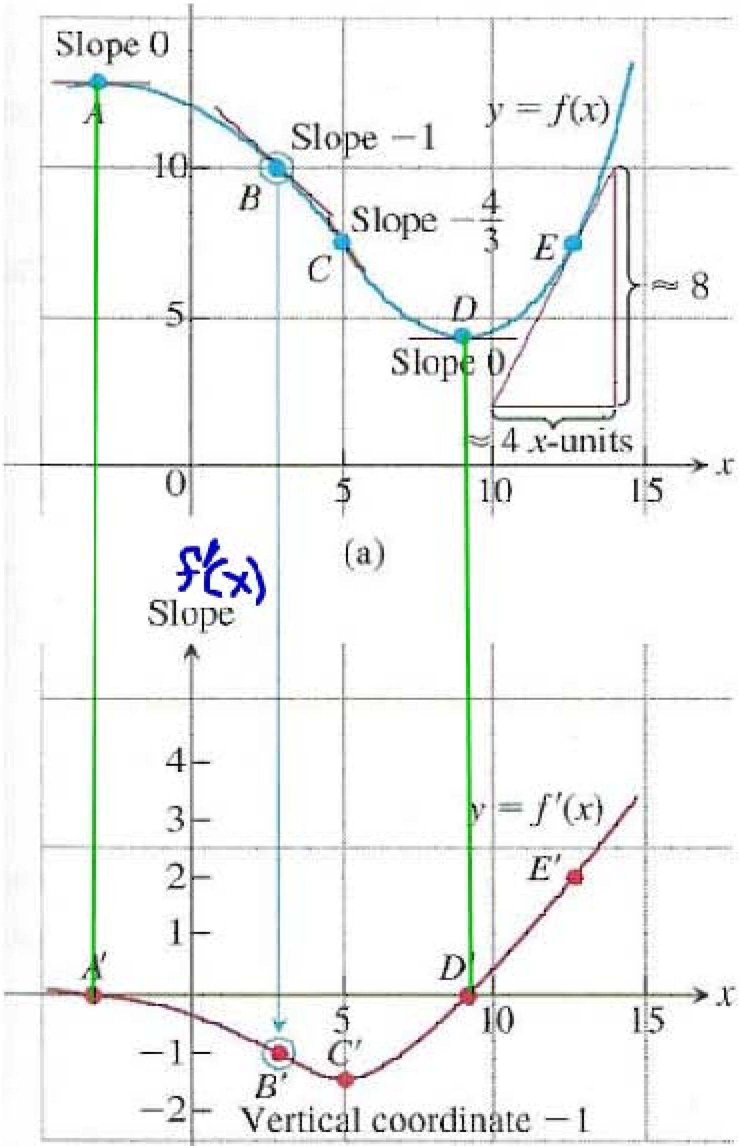

f'(x)
{coordinate ⇒ (x, f'(x))
Slope
Rate of change is positive
f'(x) > 0
y = f'(x)
E'
D'
A'
x
5
10
15
4
3
2
1
−1
−2
B'
C'
Rate of change is negative
f'(x) < 0
Vertical coordinate −1

(b)

FIGURE 3.6

We made the graph of $y=f'(x)$ in (b) by plotting slopes from the graph of $y=f(x)$ in (a). The vertical coordinate of B' is the slope at B and so on.

The slope at E is approximately $8/4 = 2$. In (b), we see that the rate of change of f is negative for x between A' and D'; the rate of change is positive for x to the right of D'.



Important

Point A → $f'(x) = 0$, A' is the x-intercept of $f'(x)$

Point B, $f'(x) = -1$, below the x-Axis of x and $f'(x)$ coordinate grid.

Point C → $f'(x) = -\frac{4}{3}$ below the x-Axis of the x and $f'(x)$ coordinate grid.

Point D → $f'(x) = 0$, D' is the x-intercept of the x and $f'(x)$ Axis

Point E → $f'(x) = 2$, E' is above the x and $f'(x)$ coordinate system.

Observe points A' and D' for the graph of $y = f'(x)$. A' and D' are x-intercepts of $y=f'(x)$ because at those points $f'(x)=0$

when $y=f'(x)$ is graphed as function, then the values where $f'(x)=0$ are the $\boxed{\text{x-intercepts}}$ of $f'(x)$.

What can we learn from the graph of $y = f'(x)$? At a glance we can see

1. where the rate of change of f is positive, negative, or zero;
2. the rough size of the growth rate at any x and its size in relation to the size of $f(x)$;
3. where the rate of change itself is increasing or decreasing.

we must remember this because the above information will be essential to working with proofs involving Rolle's Theorem and the Mean Value Theorem.