SECTION 3.5: DIFFERENTIALS and LINEARIZATION OF FUNCTIONS

LEARNING OBJECTIVES

- Use differential notation to express the approximate change in a function on a small interval.
- Find linear approximations of function values.
- Analyze how errors can be propagated through functional relationships.

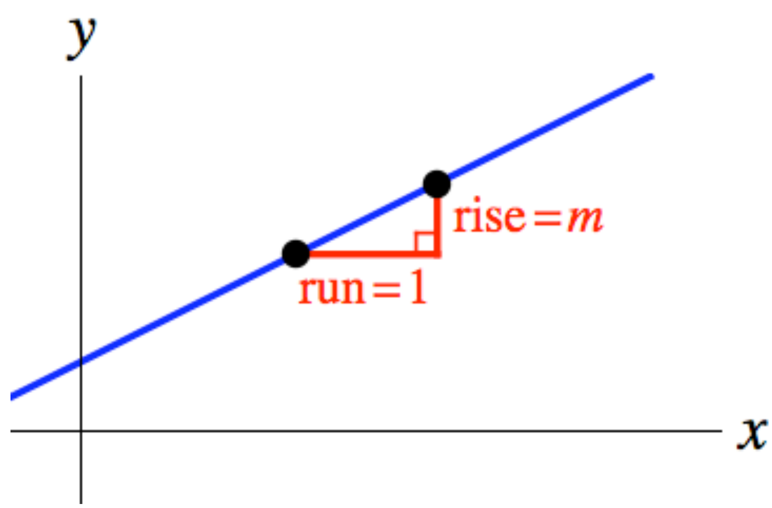
PART A: INTERPRETING SLOPE AS MARGINAL CHANGE

Our story begins with **lines**. We know lines well, and we will use them to (locally) model graphs we don't know as well.

Interpretation of Slope m as Marginal Change

For every unit increase in x along a line, y changes by m.

- If $m < 0$, then y **drops** in value.



If $\text{run} = 1$, then slope $m = \dfrac{\text{rise}}{\text{run}} = \dfrac{\text{rise}}{1} = \text{rise}$.

PART B: DIFFERENTIALS and CHANGES ALONG A LINE

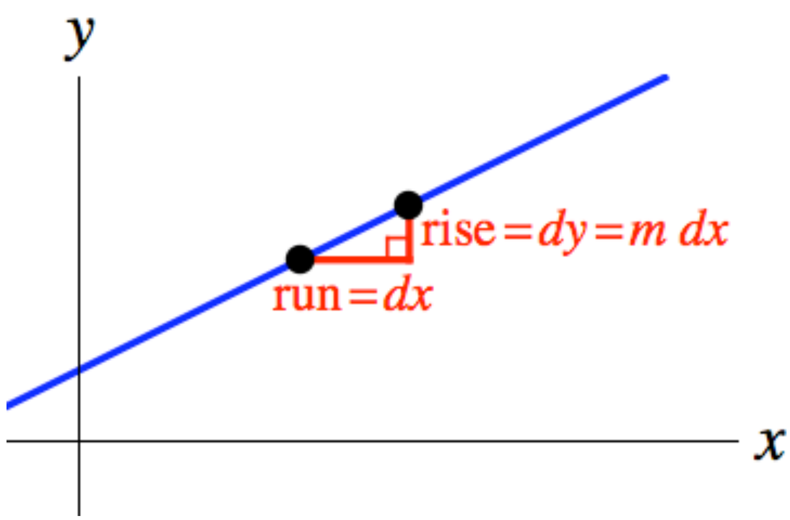
dx and dy are the differentials of x and y, respectively.
They correspond to "small" changes in x and y along a **tangent line**.
We associate dy with "rise" and dx with "run."

- If $dy < 0$, we move **down** along the line.
- If $dx < 0$, we move **left** along the line.

The following is key to this section:

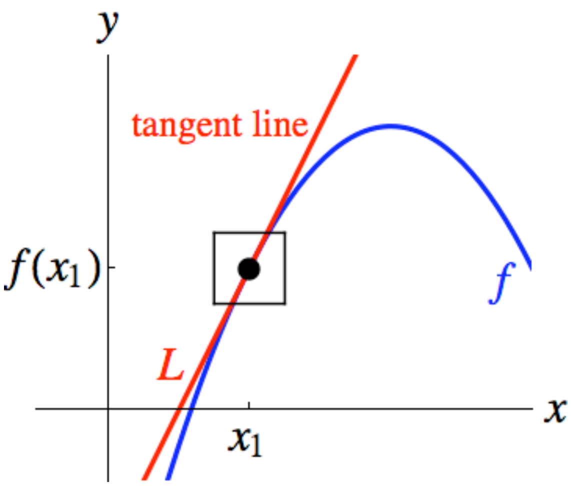
Rise Along a Line

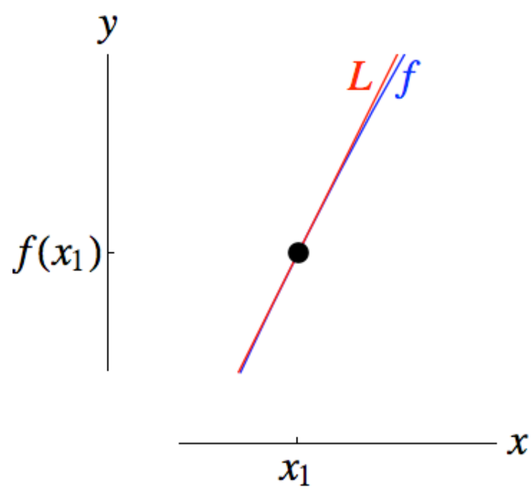
$\text{slope} = \dfrac{\text{rise}}{\text{run}}$, so:	$m = \dfrac{dy}{dx}$ as a quotient of differentials, so:
$(\text{rise}) = (\text{slope})(\text{run})$	$dy = m\,dx$



PART C: LINEARIZATION OF FUNCTIONS

Remember the **Principle of Local Linearity** from Section 3.1. Assume that a function f is differentiable at x_1, which we will call the "**seed**." Then, the **tangent line** to the graph of f at the point $\left(x_1, f\left(x_1\right)\right)$ represents the function L, the **best local linear approximation** to f close to x_1. L models (or "linearizes") f locally on a small interval containing x_1.



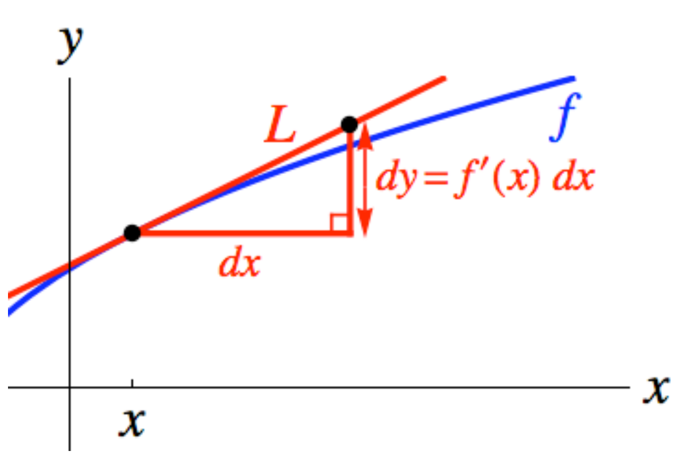


The **slope** of the tangent line is given by $f'\left(x_1\right)$ or, more generically, by $f'(x)$, so changes along the tangent line are related by the following formulas:

$$dy = m\,dx$$
$$dy = f'(x)\,dx$$

• This formula, written in differential form, is used to relate dx and dy as **variables**.

• In Leibniz notation, this can be written as $dy = \frac{dy}{dx}\,dx$, though those who see $\frac{dy}{dx}$ as an inseparable entity may object to the appearance of "cancellation."



Δx and Δy are the increments of x and y, respectively.
They represent **actual changes** in x and y along the graph of f.

• If x changes Δx units from, say, x_1 ("**seed, old *x***") to $x_1 + \Delta x$ ("**new *x***"), then y (or f) changes Δy units from $f(x_1)$ to $f(x_1) + \Delta y$, or $f(x_1 + \Delta x)$.

The "new" **function value** $f(x_1 + \Delta x) = f(x_1) + \Delta y$.

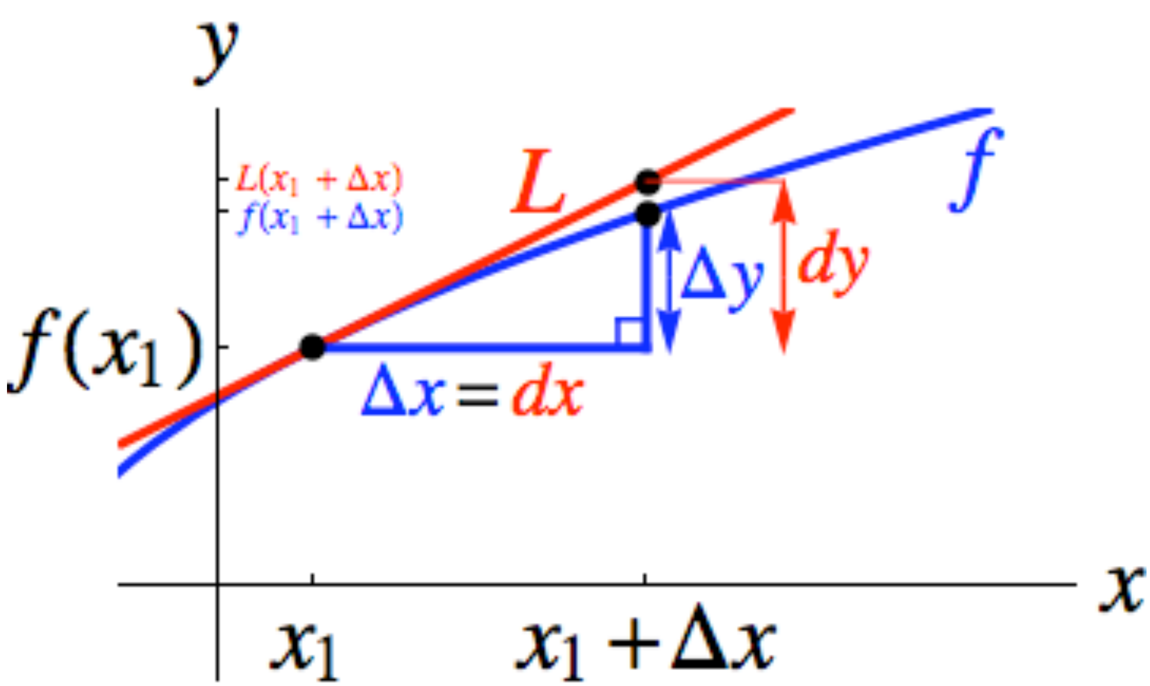
Informally, $f(\text{new } x) = f(\text{old } x) + (\text{actual rise } \Delta y)$.

$L(x_1 + \Delta x)$, our **linear approximation** of $f(x_1 + \Delta x)$, is given by:

$$L(x_1 + \Delta x) = f(x_1) + dy$$
$$L(x_1 + \Delta x) = f(x_1) + [f'(x_1)][dx]$$

Informally, $L(\text{new } x) = f(\text{old } x) + (\text{tangent rise } dy)$
$L(\text{new } x) = f(\text{old } x) + [\text{slope}][\text{run } dx]$

When finding $L(x_1 + \Delta x)$, we set $dx = \Delta x$, and then we hope that $dy \approx \Delta y$.
Then, $L(x_1 + \Delta x) \approx f(x_1 + \Delta x)$.



PART D: EXAMPLES

Example 1 (Linear Approximation of a Function Value)

Find a **linear approximation** of $\sqrt{9.1}$ by using the value of $\sqrt{9}$.
Give the exact value of the linear approximation, and also give a decimal approximation rounded off to six significant digits.

In other words: Let $f(x) = \sqrt{x}$. Find a linear approximation $L(9.1)$ for $f(9.1)$ if x changes from 9 to 9.1.

§ Solution Method 1 (Using Differentials)

• f is **differentiable** on $(0, \infty)$, which includes both 9 and 9.1, so this method is appropriate.

• We know that $f(9) = \sqrt{9} = 3$ **exactly**, so 9 is a reasonable choice for the "seed" x_1.

• Find $f'(9)$, the **slope** of the tangent line at the "seed point" $(9, 3)$.

$$f(x) = \sqrt{x} = x^{1/2} \quad \Rightarrow$$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \quad \Rightarrow$$

$$f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}$$

• Find the **run** dx (or Δx).

$$\begin{aligned} \text{run } dx &= \text{"new } x\text{"} - \text{"old } x\text{"} \\ &= 9.1 - 9 \\ &= 0.1 \end{aligned}$$

• Find *dy*, the **rise** along the tangent line.

$$\begin{aligned} \text{rise } dy &= (\text{slope}) \cdot (\text{run}) \\ &= \left[f'(9) \right] \cdot \left[dx \right] \\ &= \left[\frac{1}{6} \right] \left[0.1 \right] \\ &= \frac{1}{60} \end{aligned}$$

• Find $L(9.1)$, our **linear approximation** of $f(9.1)$.

$$\begin{aligned} L(9.1) &= f(9) + dy \\ &= 3 + \frac{1}{60} \\ &= \frac{181}{60} \quad (\text{exact value}) \\ &\approx 3.01667 \end{aligned}$$

WARNING 1: Many students would forget to add $f(9)$ and simply give *dy* as the approximation of $\sqrt{9.1}$. This mistake can be avoided by observing that $\sqrt{9.1}$ should be very different from $\frac{1}{60}$.

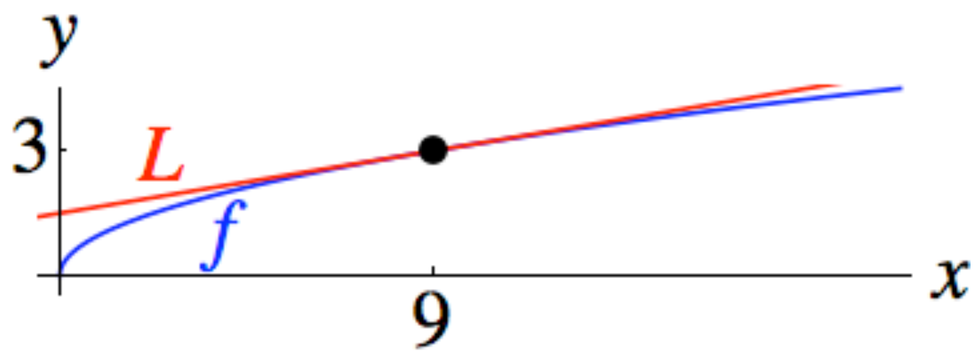
• In fact, $f(9.1) = \sqrt{9.1} \approx 3.01662$, so our approximation was accurate to five significant digits. Also, the **actual change in *y*** is given by:

$$\begin{aligned} \Delta y &= f(9.1) - f(9) \\ &= \sqrt{9.1} - \sqrt{9} \\ &\approx 3.01662 - 3 \\ &\approx 0.01662 \end{aligned}$$

This was approximated by $dy = \frac{1}{60} \approx 0.01667$.

The **error** is given by: $dy - \Delta y \approx 0.01667 - 0.01662 \approx 0.00005$.
(See Example 5 for details on relative error, or percent error.)

• Our approximation $L(9.1)$ was an **overestimate** of $f(9.1)$, because the graph of f curves downward (it is "concave down"; see Chapter 4).



• Differentials can be used to quickly find **other** linear approximations close to $x = 9$. (The approximations may become unreliable for values of x far from 9.)

Find a linear approximation of $f(9+\Delta x)$	**run** $dx = \Delta x$	**rise** $dy = \left(\frac{1}{6}\right) dx$	**Linear approximation,** $L(9+\Delta x)$
$f(8.8) = \sqrt{8.8}$	-0.2	$-\frac{1}{30} \approx -0.03333$	$L(8.8) = 3 - \frac{1}{30}$ ≈ 2.96667
$f(8.9) = \sqrt{8.9}$	-0.1	$-\frac{1}{60} \approx -0.01667$	$L(8.9) = 3 - \frac{1}{60}$ ≈ 2.98333
$f(9.1) = \sqrt{9.1}$ (We just did this.)	0.1	$\frac{1}{60} \approx 0.01667$	$L(9.1) = 3 + \frac{1}{60}$ ≈ 3.01667
$f(9.2) = \sqrt{9.2}$	0.2	$\frac{1}{30} \approx 0.03333$	$L(9.2) = 3 + \frac{1}{30}$ ≈ 3.03333

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§ Solution Method 2 (Finding an Equation of the Tangent Line First: y = L(x))

• Although this method may seem easier to many students, it does not stress the idea of **marginal change** the way that Method 1 does. Your instructor may demand Method 1.

• The **tangent line** at the "seed point" $(9,3)$ has **slope** $f'(9)=\frac{1}{6}$, as we saw in Method 1. Its equation is given by:

$$y-y_1=m(x-x_1)$$

$$y-3=\frac{1}{6}(x-9)$$

$L(x)$, or $y=3+\frac{1}{6}(x-9)$, which takes the form (with variable dx):

$$L(x)=f(9)+\underbrace{[f'(9)][dx]}_{dy}$$

Also, $L(x)=3+\frac{1}{6}(x-9)$ simplifies as: $L(x)=\frac{1}{6}x+\frac{3}{2}$.

• In particular, $L(9.1)=3+\frac{1}{6}(9.1-9)$ or $\frac{1}{6}(9.1)+\frac{3}{2}\approx 3.01667$, as before. §

Example 2 (Linear Approximation of a Trigonometric Function Value)

Find a **linear approximation** of $\tan\left(42^\circ\right)$ by using the value of $\tan\left(45^\circ\right)$. Give the exact value of the linear approximation, and also give a decimal approximation rounded off to six significant digits.

§ Solution Method 1 (Using Differentials)

Let $f(x) = \tan x$.

WARNING 2: Convert to radians. In this example, we need to compute the **run** *dx* using radians. If $f(x)$ or $f'(x)$ were $x \tan x$, for example, then we would also need to convert to radians when evaluating **function values** and **derivatives**. (Also, a Footnote in Section 3.6 will discuss why the differentiation rules for trigonometric functions given in Section 3.4 do **not** apply if *x* is measured in degrees.)

• Converting to **radians**, $45^\circ = \dfrac{\pi}{4}$ and $42^\circ = \dfrac{7\pi}{30}$. (As a practical matter, it turns out that we don't have to do these conversions, as long as we know what the **run** *dx* is in radians.) We want to find a linear approximation $L\left(\dfrac{7\pi}{30}\right)$ for $f\left(\dfrac{7\pi}{30}\right)$ if *x* changes from $\dfrac{\pi}{4}$ to $\dfrac{7\pi}{30}$.

• f is **differentiable** on, among other intervals, $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, which includes both $\dfrac{\pi}{4}$ and $\dfrac{7\pi}{30}$, so this method is appropriate.

• We know that $f\left(\dfrac{\pi}{4}\right) = \tan\left(\dfrac{\pi}{4}\right) = 1$ **exactly**.

• Find $f'\left(\dfrac{\pi}{4}\right)$, the **slope** of the tangent line at the "seed point" $\left(\dfrac{\pi}{4}, 1\right)$.

$$f(x) = \tan x \quad \Rightarrow$$

$$f'(x) = \sec^2 x \quad \Rightarrow$$

$$f'\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = 2$$

• Find the **run** dx (or Δx). It is usually easier to subtract the degree measures before converting to **radians**. Since it turns out $dx < 0$ here, we run **left** (as opposed to right) along the tangent line.

$$\begin{aligned}\text{run } dx &= \text{"new } x\text{"} - \text{"old } x\text{"}\\ &= 42^\circ - 45^\circ\\ &= -3^\circ\\ &= \left(-3^\circ\right)\left(\frac{\pi}{180^\circ}\right) \quad \text{(Converting to radians)}\\ &= -\frac{\pi}{60}\end{aligned}$$

• Find dy, the **rise** along the tangent line. Actually, since it turns out $dy < 0$ here, it really corresponds to a **drop**.

$$\begin{aligned}\text{rise } dy &= (\text{slope})\cdot(\text{run})\\ &= \left[f'\left(\frac{\pi}{4}\right)\right]\cdot\left[dx\right]\\ &= [2]\left[-\frac{\pi}{60}\right]\\ &= -\frac{\pi}{30}\end{aligned}$$

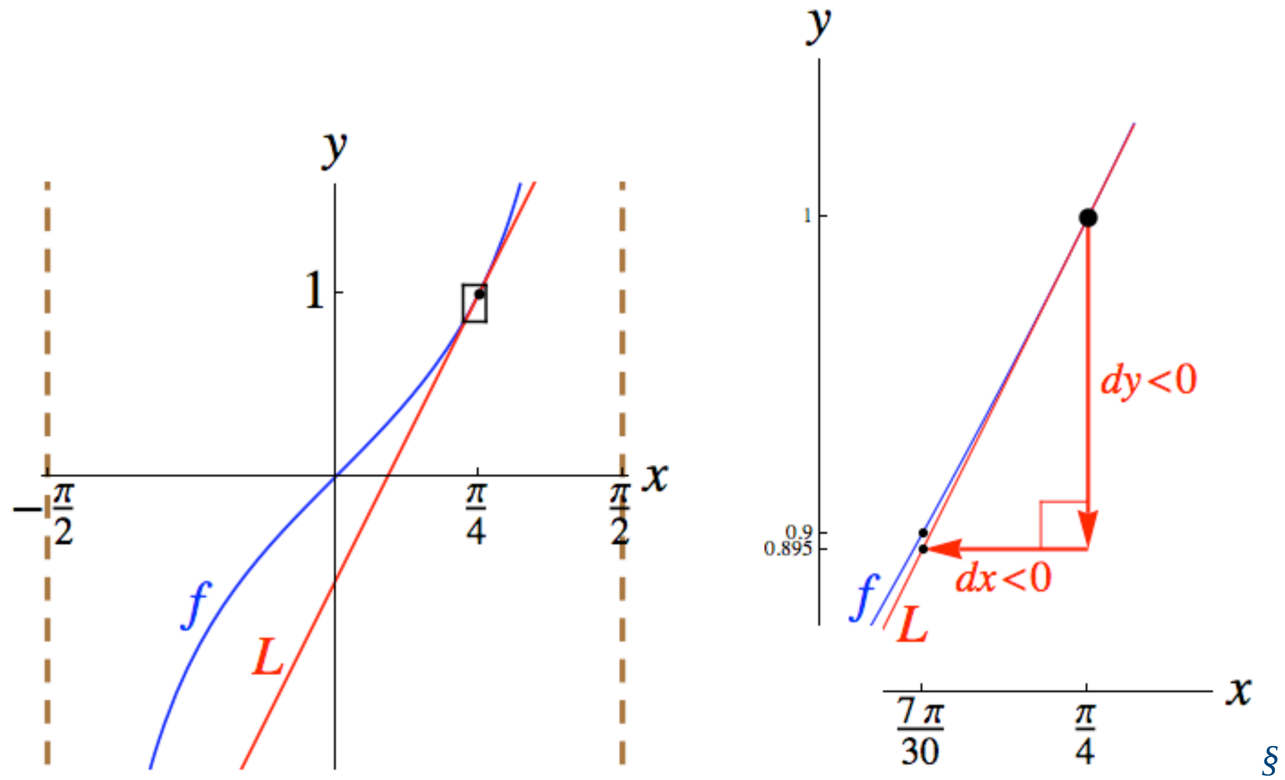
• Find $L\left(\frac{7\pi}{30}\right)$, our **linear approximation** of $\tan\left(42^\circ\right)$, or $f\left(\frac{7\pi}{30}\right)$.

$$\begin{aligned}L\left(\frac{7\pi}{30}\right) &= f\left(\frac{\pi}{4}\right) + dy\\ &= 1 + \left(-\frac{\pi}{30}\right)\\ &= \frac{30-\pi}{30} \quad \text{(exact value)}\\ &\approx 0.895280\end{aligned}$$

• In fact, $f\left(\frac{7\pi}{30}\right) = \tan\left(42^\circ\right) \approx 0.900404$.

• Our approximation $L\left(\frac{7\pi}{30}\right)$ was an **underestimate** of $f\left(\frac{7\pi}{30}\right)$.

See the figures below.



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<u>*§ Solution Method 2 (Finding an Equation of the Tangent Line First: y = L(x))*</u>

• The **tangent line** at the "seed point" $\left(\frac{\pi}{4},1\right)$ has **slope** $f'\left(\frac{\pi}{4}\right)=2$, as we saw in Method 1. Its equation is given by:

$$y-y_1=m\left(x-x_1\right)$$

$$y-1=2\left(x-\frac{\pi}{4}\right)$$

$L(x)$, or $y=1+2\left(x-\frac{\pi}{4}\right)$, which takes the form $\left(\text{with variable } dx\right)$:

$$L(x)=f\left(\frac{\pi}{4}\right)+\underbrace{\left[f'\left(\frac{\pi}{4}\right)\right]\left[dx\right]}_{dy}$$. Simplified, $L(x)=2x+\frac{2-\pi}{2}$.

• In particular, $L\left(\frac{7\pi}{30}\right)=1+2\left(\frac{7\pi}{30}-\frac{\pi}{4}\right)=1+2\left(-\frac{\pi}{60}\right)\approx 0.895280$, or

$L\left(\frac{7\pi}{30}\right)=2\left(\frac{7\pi}{30}\right)+\frac{2-\pi}{2}\approx 0.895280$, as before. §

PART E: APPLICATIONS

Since a typical calculator has a square root button and a tangent button, the previous examples might not seem useful.

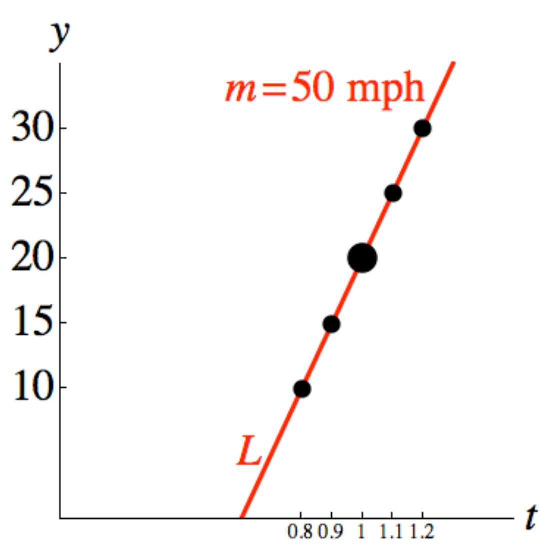
The table on Page 3.5.7 demonstrates how differentials can be used to quickly find **multiple approximations** of function values on a **small interval** around the "seed."

Differentials can be used even if we **do not have a formula** for the function we are approximating, as we now demonstrate.

Example 3 (Approximating Position Values in the Absence of Formulas)

A car is moving on a coordinate line. Let $y = s(t)$, the **position** of the car (in miles) t hours after noon. We are given that $s(1) = 20 \text{ miles}$ and $v(1) = s'(1) = 50 \text{ mph}$.

Find a linear approximation of $s(1+\Delta t)$	**run** $dt = \Delta t$	**rise** $dy = 50\ dt$	**Linear approximation,** $L(1+\Delta t)$
$s(0.8)$	-0.2	-10	$L(0.8) = 20 - 10$ $= 10 \text{ mi}$
$s(0.9)$	-0.1	-5	$L(0.9) = 20 - 5$ $= 15 \text{ mi}$
$s(1.1)$	0.1	5	$L(1.1) = 20 + 5$ $= 25 \text{ mi}$
$s(1.2)$	0.2	10	$L(1.2) = 20 + 10$ $= 30 \text{ mi}$



(Axes are scaled differently.) §

PART F: MEASUREMENT ERROR and PROPAGATED ERROR

Let x be the **actual** (or **exact**) length, weight, etc. that we are trying to measure.

Let Δx (or dx) be the **measurement error**. The error could be due to a poorly calibrated instrument or merely random chance.

Example 4 (Measurement Error)

The radius of a circle (unbeknownst to us) is 10.7 inches (the **actual value**), but we measure it as 10.5 inches (the **measured value**). Sources differ on the definition of **measurement error**.

1) If we let measurement error = (measured value) – (actual value), then the "seed" $x = 10.7$ inches, and $\Delta x = 10.5 - 10.7 = -0.2$ inches.

   This approach is more consistent with our usual notion of "error."

2) If we let measurement error = (actual value) – (measured value), then the "seed" $x = 10.5$ inches, and $\Delta x = 10.7 - 10.5 = 0.2$ inches.

   We will adopt this approach, because we know the measured value but not the actual value in an exercise such as Example 5. This suggests that the **measured value** is more appropriate than the actual value as the "**seed**."

(See Footnote 1.) §

If $y = f(x)$ for some function f, then an error in measuring x may lead to **propagated error** in y, denoted by Δy. As before, we approximate Δy by dy for convenience.

Example 5 (Propagated Error)

Let x be the radius of a circle, and let y be its area. Then, y, or $f(x) = \pi x^2$. We measure the **radius** using an instrument that may be "off" by as much as 0.5 inches; more precisely, it has a maximum possible absolute value of **measurement error** of 0.5 inches. We use the instrument to obtain a **measured value** of 10.5 inches. **Estimate** the maximum possible absolute value of the **propagated error** that we will obtain for the **area** of the circle. Use differentials and give an approximation written out to five significant digits.

§ Solution

$$\begin{aligned} dy &= f'(x)\,dx \\ &= 2\pi x\,dx \end{aligned}$$

We use the "seed" $x_1 = 10.5$ inches, the **measured value**, for x.

Let the **run** dx be the maximum possible absolute value of the **measurement error**, which is 0.5 inches. (Really, $-0.5 \text{ in} \le dx \le 0.5 \text{ in}$.)

Then, $|dy|$, our approximation for the maximum possible absolute value of the **propagated error** in y, is given by:

$$\begin{aligned} |dy| &= |2\pi x\, dx| \\ &\le |2\pi(10.5)(0.5)| \\ &\le 32.987 \text{ in}^2 \end{aligned}$$

That is, we estimate that the **propagated error** in the **area** of the circle will be "off" by no more than 32.987 in^2 in either direction (high or low).

• **Estimates of the area.** If we take the measured value of 10.5 inches for the radius, we will obtain the following **measured value** for the area:

$$\begin{aligned} y, \text{ or } f(10.5) &= \pi(10.5)^2 \\ &\approx 346.361 \text{ in}^2 \end{aligned}$$

Since we estimate that we are "off" by no more than 32.987 in^2, we estimate that the **actual value** of the area is between 313.374 in^2 and 379.348 in^2.

Without differentials, we would say that the actual value of the area is between $f(10.0)$ and $f(11.0)$, or between 314.159 in^2 and 380.133 in^2. (The benefit of using differentials is more apparent when the function involved is more complicated.)

• **Relative error and percent error.** Is 32.987 in^2 "bad" or "good"? The relative error and the percent error give us some context to decide.

$$\begin{aligned} \text{Relative error} &= \frac{dy}{y} \quad \left(\text{really, } \frac{\text{maximum } |dy|}{\text{measured } y}\right) \\ &\approx \frac{32.987}{346.361} \\ &\approx 0.095239, \text{ or } 9.5239\% \ (\text{percent error}) \end{aligned}$$

(If the measured area had been something like $y = 1{,}000{,}000 \text{ in}^2$, then 32.987 in^2 probably wouldn't be a big deal.)

• **Actual propagated error.** If the actual value of the **radius** is 10.7 inches, then the actual value of the **area** is $f(10.7) = \pi(10.7)^2 \approx 359.681 \text{ in}^2$, and the actual propagated error is given by:

$$\begin{aligned}\Delta y &= f(10.7) - f(10.5) \\ &= \pi(10.7)^2 - \pi(10.5)^2 \\ &\approx 13.320 \text{ in}^2\end{aligned}$$

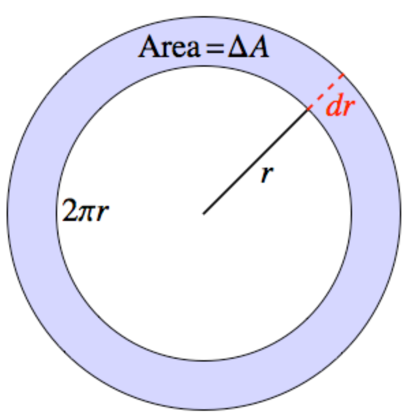
• **Notation.** If we let r = the radius and A = the area, then we obtain the more familiar formula $A = \pi r^2$. Also, $dA = 2\pi r\, dr$.

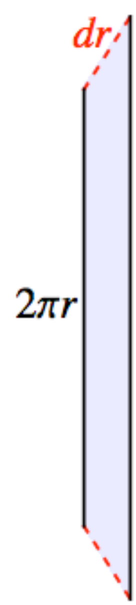
• **Geometric approach.** Observe that:

dA = (circumference of circle of radius r) (thickness of ring)

This approximates ΔA, the actual propagated error in the area. The figures below demonstrate why this approximation makes sense.

Imagine cutting the shaded ring below along the dashed slit and straightening it out. We obtain a trapezoid that is approximately a rectangle with dimensions $2\pi r$ and dr. The area of both shaded regions is ΔA, and we approximate it by dA, where $dA = 2\pi r\, dr$.





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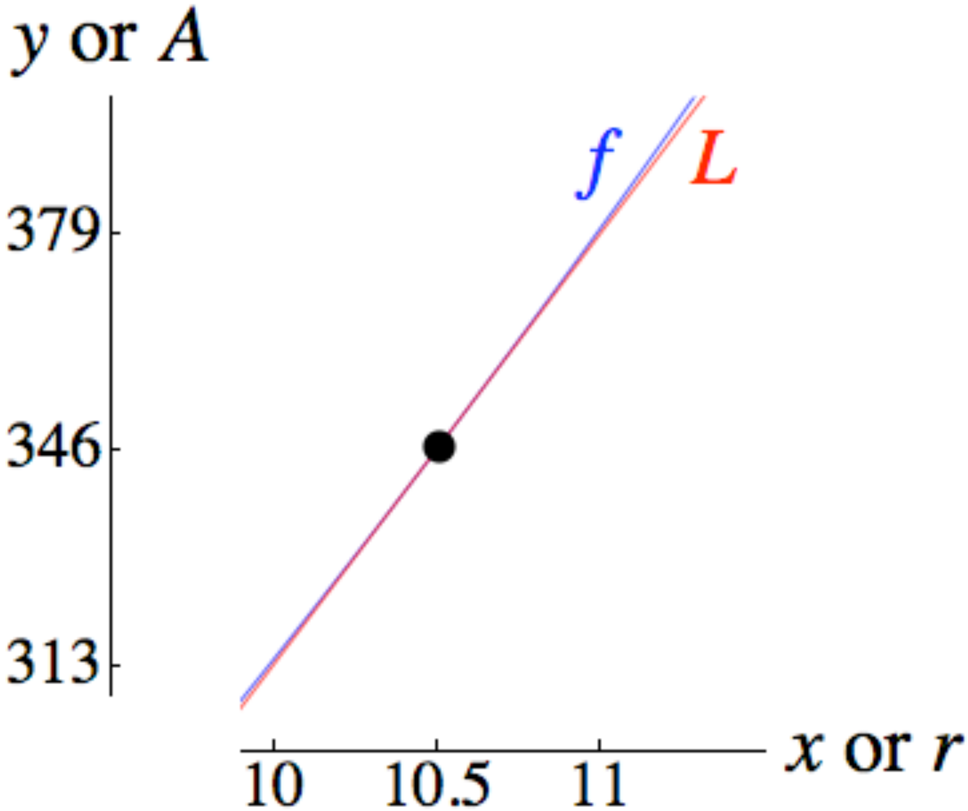

(Axes are scaled differently.) §

FOOTNOTES

1. **Defining measurement (or absolute) error.** Definition 1) in Example 4 is used in *Wolfram MathWorld*, namely that measurement error = (measured value) – (actual value). Sometimes, absolute value is taken. See "Absolute Error," *Wolfram Mathworld*, Web, 25 July 2011, <http://mathworld.wolfram.com/AbsoluteError.html>.
   • Larson in his calculus text (9th ed.) uses Definition 2):
   measurement error = (actual value) – (measured value). On top of the rationale given in Example 4, this is also more consistent with the notion of error when studying confidence intervals and regression analysis in statistics.