Week 7.

Linear Approximations and Differentials

Now we will see two applications of the derivative. First, let's go back to tangent lines, the concept that helped ease us into a discussion of limits and derivatives in the first place.

7.1. Linear Approximations

We now know how to calculate the tangent line at a point, $(a, f(a))$, very easily by taking the derivative of the function as the slope. Linear approximation is exactly what it sounds like; now we use that tangent line to approximate the curve $f(x)$ when x is close to a.

Definition 7.1. Let $f(x)$ be a function. Then $y = f(a) + f'(a)(x - a)$ is the equation of the tangent line at $x = a$. The approximation

$$L(x) \approx f(a) + f'(a)(x - a)$$

is called the linear approximation of f at a.

Example 7.1. What is the linear approximation for $f(\theta) = \sin\theta$ at $\theta = 0$?

Solution. Compute the tangent line at $\theta = 0$.

$$f'(\theta) = \cos\theta$$

$$f'(a) = \cos(0) = 1$$

Also, $f(a) = \sin(0) = 0$, so the tangent line goes through the point $(0, 0)$. Now calculate the linear approximation:

$$L(x) = f(a) + f'(a)(\theta - a) = f(0) + f'(0)(\theta - 0) = 0 + 1(\theta - 0) = \theta$$

This means that for very small values of θ, then $\sin\theta \approx \theta$. This approximation is very useful in physics.

Your Turn. Find the linear approximation for the function $f(x) = \dfrac{1}{\sqrt{x+3}}$ at $a = 0$.

Example 7.2. Approximate $\sqrt[3]{64.2}$.

Solution. Let $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$. Find the linear approximation at $x = 64$.

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$

$$f'(64) = \frac{1}{3}(64)^{-\frac{2}{3}} = \frac{1}{48}$$

And $f(64) = 4$ So

$$L(x) = 4 + \frac{1}{48}(x - 64) = \frac{1}{48}x + \frac{8}{3}$$

Now plug in 64.2 into the linear approximation function L:

$$L(64.2) \approx 4.0041667$$

Which seems reasonable, since it's slightly more than 4.

Your Turn. Approximate $\sqrt{16.05}$.

7.2. Differentials

Definition 7.2. Given a function $f(x)$, we call dy and dx differentials, where

$$\frac{dy}{dx} \approx f'(x)$$

Example 7.3. Compute the differential of $y = t^2 + 5$

Solution.

$$dy = 2tdt$$

It's really that simple. Just take the derivative, change y to dy and add a dt to the end of the equation.

Your Turn. Compute the differential of

$$A = x\sin(3x)$$

Example 7.4. Compute Δy when $x = 2$ and $\Delta x = .05$ for the function

$$y = \cos(x^2 + 1)$$

Solution. Calculate the change in y for x starting at 2 and ending at 2.05.

$$\Delta y = f(2.05) - f(2) = \cos((2.05)^2 + 1) - \cos(2^2 + 1) \approx 0.1871$$

Just to see how close (or far) we are, let's calculate dy precisely. Using the relationship that $dy = f'(x)dx$, we get:

$$dy = -2x\sin(x^2 + 1)dx$$

And plugging in $x = 2$, $dx = .05$:

$$dy = -2(2)\sin(2^2 + 1)(.05) \approx 0.1918$$

Your Turn. Compute Δy and dy when $x = 1$ and $\Delta x = .01$ for the function

$$y = 2x^3 - x^2 + 1$$

Example 7.5.† A sphere was measured and its radius was found to be 45 inches with a possible error of no more that 0.01 inches. What is the maximum possible error in the volume if we use this value of the radius?

Another variation in wording of this problem goes like this:

Example 7.6. Approximately how much paint do you need to paint a sphere with radius of 45 inches, with a coat of paint 0.01 inches thick?

Solution. The volume of a sphere is given by

$$V = \frac{3}{4}\pi r^3$$

The differential is:

$$dV = \frac{9}{4}\pi r^2 dr$$

We are going to calculate

$$\Delta V = \frac{9}{4}\pi r^2 \Delta r$$

Plug in $r = 45$ and $\Delta r = 0.01$:

$$dV \approx \Delta V = \frac{9}{4}\pi 45^2(.01) = 45.5625\pi$$

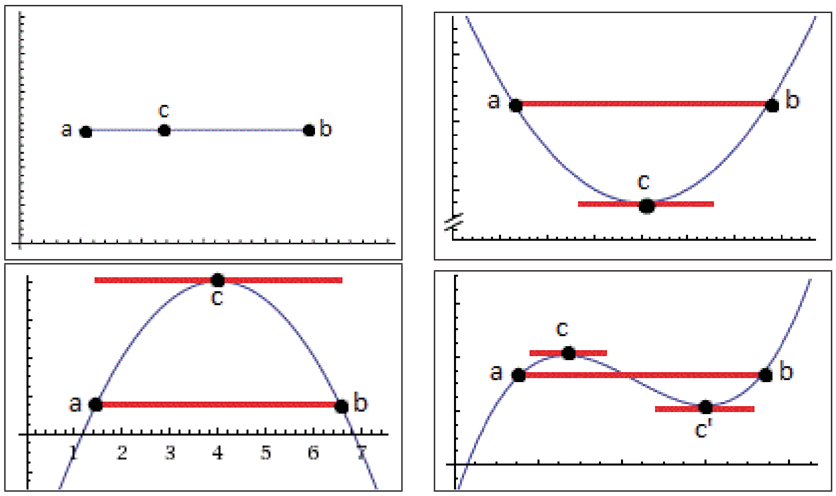
Your Turn. Use differentials to find a formula for the approximate volume of a thin cylindrical shell with height h, inner radius r and thickness Δr. What is the maximum error?

7.3. Mean Value Theorem for Derivatives

This theorem will probably seem strange to you at first. It's called Rolle's Theorem.

Theorem 7.1. *Let $f(x)$ be a continuous function with two points, a and b, such that $f(a) = f(b)$. Graphically, this means that a and b lie in the same horizontal line. Then at some point, call it c between a and b, the slope of the graph is completely flat. In other words, the derivative at that point is zero.*

Example 7.7. Explain how Rolle's theorem applies to the following graphs. Find the point(s), c, that satisfy the theorem.

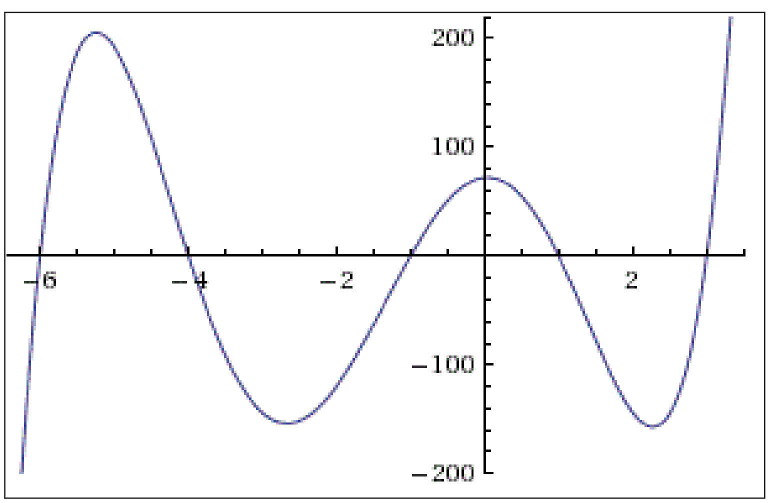


Solution. In the first graph, the function is flat everywhere, or constant, between a and b. That also means that the slope of the graph is flat at any point c that I pick between a and b. So there are actually infinitely many points between a and b that satisfy Rolle's theorem.

In both the parabolic graphs, there is only the point c that satisfies Rolle's theorem. The red line between a and b shows that the two points lie horizontally in the same line, which is the same as saying that $f(a) = f(b)$. The shorter red line represents the zero slope at the point c.

In the fourth graph, there are exactly two points that satisfy Rolle's theorem. Do you see why?

Your Turn. Using $a = -6$ and $b = 3$ (marked with heavy dots on the graph) as your interval, find all the points the satisfy Rolle's Theorem. Draw the tangent lines at those points.



The next theorem is a more general case of Rolle's Theorem. We can roughly re-word Rolle's Theorem this way: there exists at least one point, c, between a and b where the slope of the tangent line at $(c, f(c))$ is parallel to the horizontal line between $f(a)$ and $f(b)$. Now take out the word "horizontal," and you have the Mean Value Theorem:

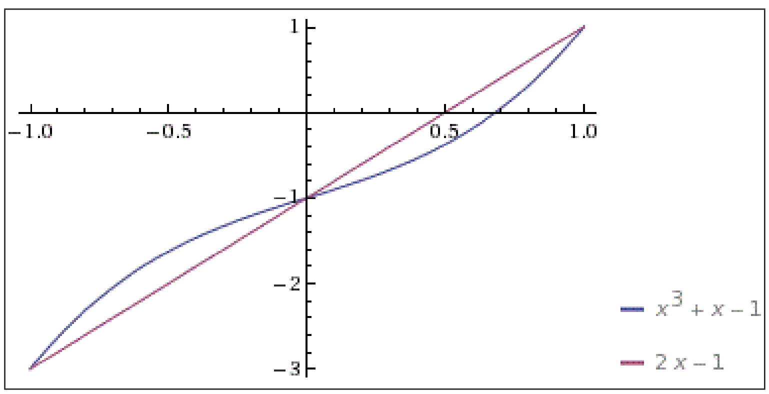
Theorem 7.2. *[In words:] Let $f(x)$ be a continuous function. Take any two points on the graph of the function, $(a, f(a))$ and $(b, f(b))$. Then at some point, call it c, between a and b, the slope of the tangent line at $(c, f(c))$ is the same as the slope of the secant line between $f(a)$ and $f(b)$.*

More precisely, If $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on its interior (a, b), then there exists at least one number c in (a, b) where

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Example 7.8. Find all the points where $f(x) = x^3 + x - 1$ satisifies the Mean Value Theorem on the interval $[-1, 1]$.

Solution. Because f is a polynomial, it is continuous everywhere. Therefore we can use the Mean Value Theorem with the given interval.



The picture shows the secant line through $(1, f(1))$ and $(-1, f(-1))$. First, we need to find the slope of that line, so that we can find other points on the graph of f that have the same slope. To do this, use the definition of the slope:

$$m = \frac{f(-1) - f(1)}{-1 - 1} = \frac{-3 - 1}{-1 - 1} = \frac{-4}{-2} = 2$$

Now we look for a point (or points) c so that $f'(c) = 2$.

$$\begin{aligned} f'(c) &= 3c^2 + 1 \\ \implies 2 &= 3c^2 + 1 \\ \implies \pm\sqrt{\frac{1}{3}} &= c \end{aligned}$$

Draw on the graph the two tangent lines that are parallel to the secant line shown.

Your Turn. Find all the points where $f(x) = \sqrt[3]{x}$ satisifies the Mean Value Theorem on the interval $[0, 1]$.