Introduction

To get the equation of the tangent line, we use the point slope formula

$$y - y_1 = m(x - x_1)$$

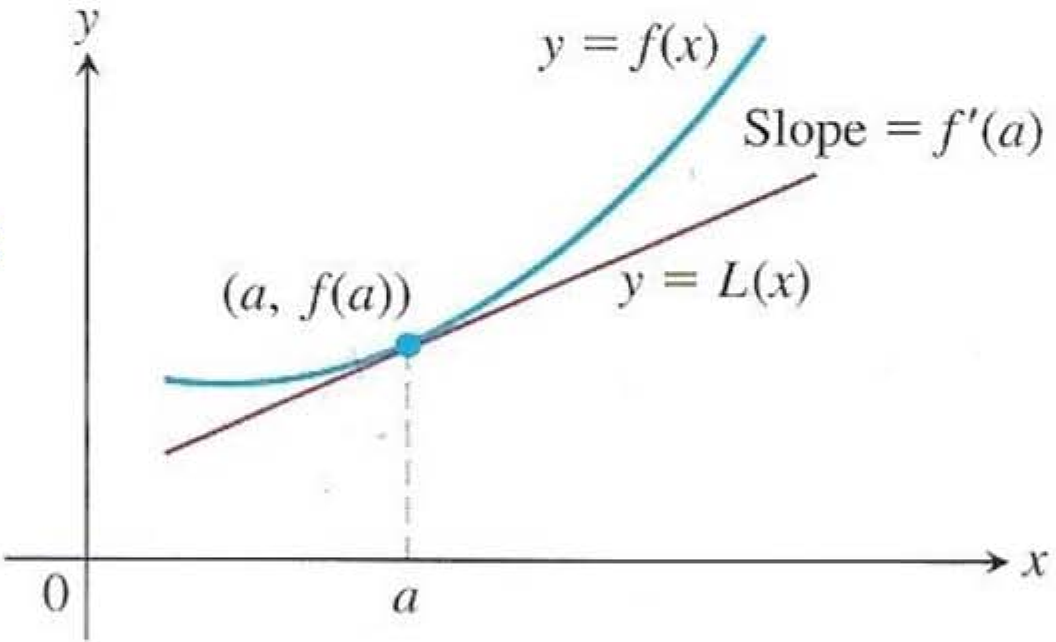


FIGURE 3.52 The tangent to the curve $y = f(x)$ at $x = a$ is the line $L(x) = f(a) + f'(a)(x - a)$.

Here

$$x_1 = a \,, \; y_1 = f(a) \,, \; m = f'(a)$$

We let

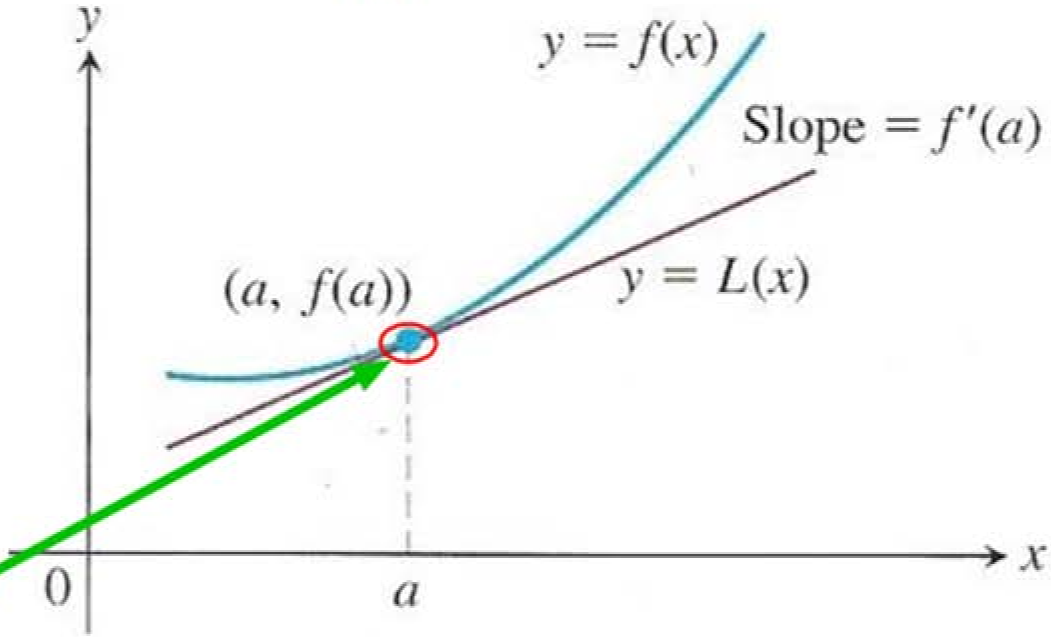
$$L(x) = y$$

$$L(x) = f(a) + f'(a)(x-a)$$

How closely does L(x) approximate the value of *f(x) when x=a ?*

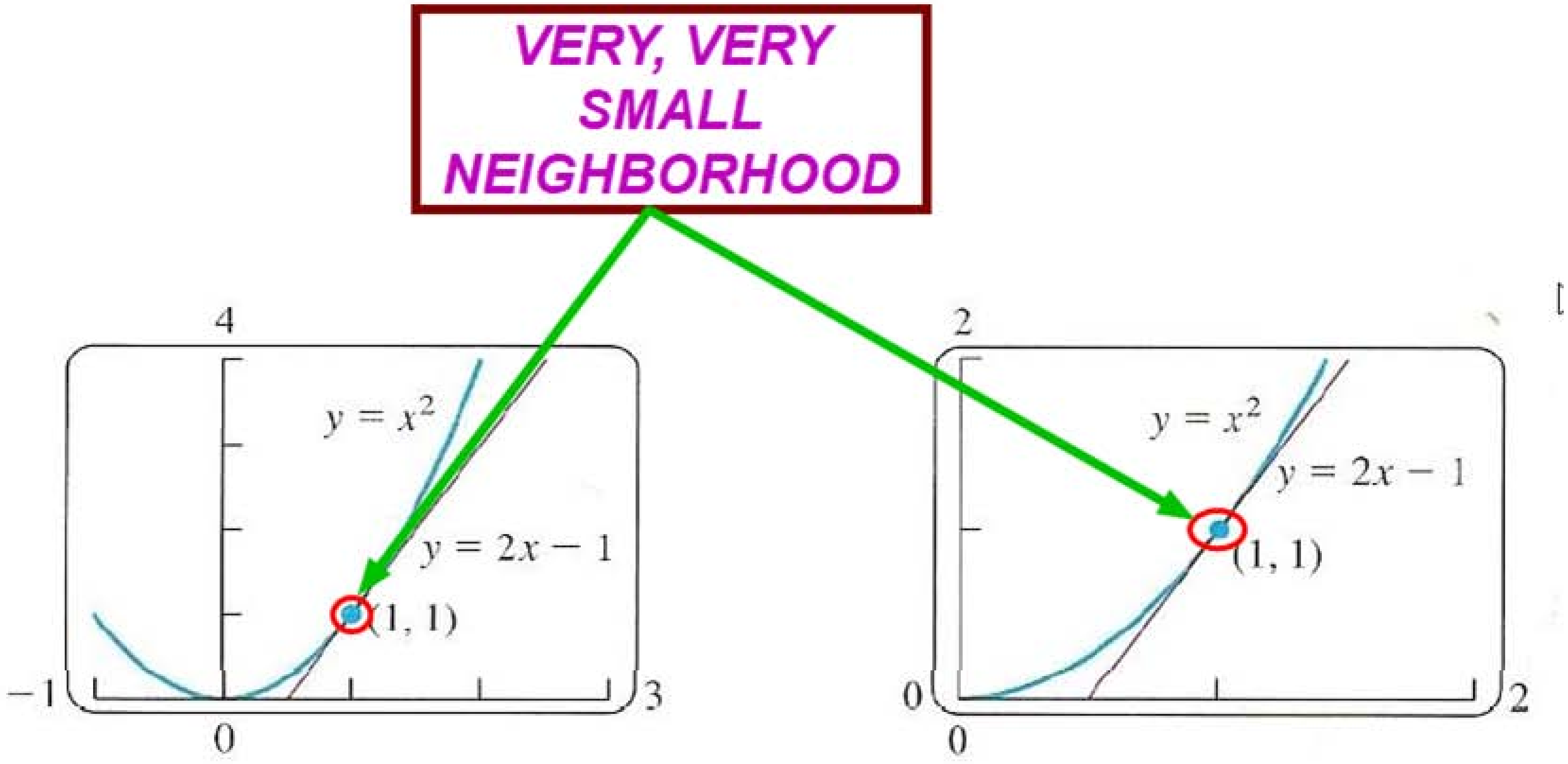
We want to look at the value of L(x) within a

Very Small neighbord surrounding *a*



We will exam the tangent line of $y=x^2$ at $(1,1)$

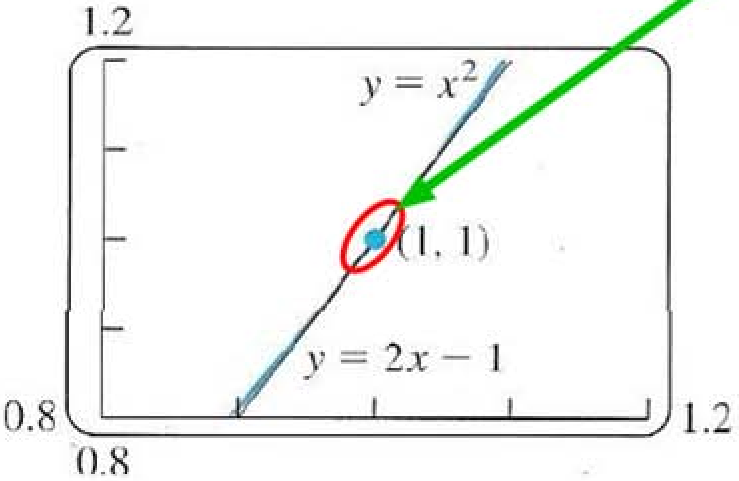
tangent: $y=2x-1$ at $(1,1)$



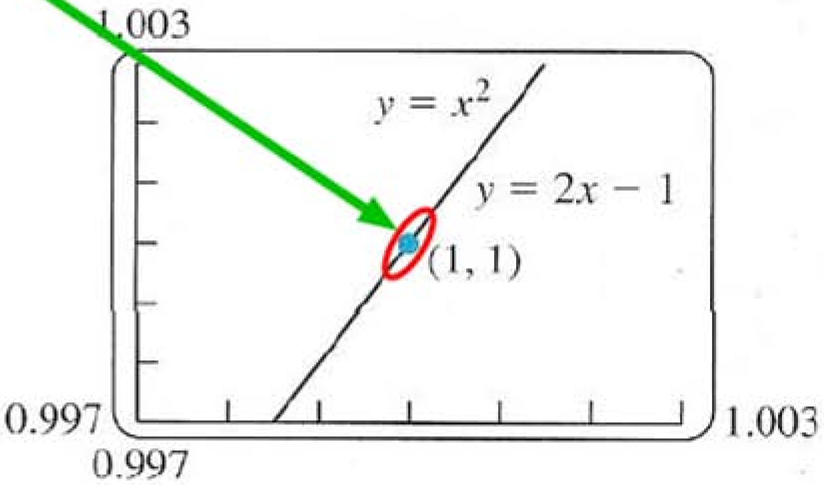
$y = x^2$ and its tangent $y = 2x - 1$ at (1, 1).

Tangent and curve very close near (1, 1).

A MUCH SMALLER NEIGHBORHOOD



Tangent and curve very close throughout entire x-interval shown.



Tangent and curve closer still. Computer screen cannot distinguish tangent from curve on this x-interval.

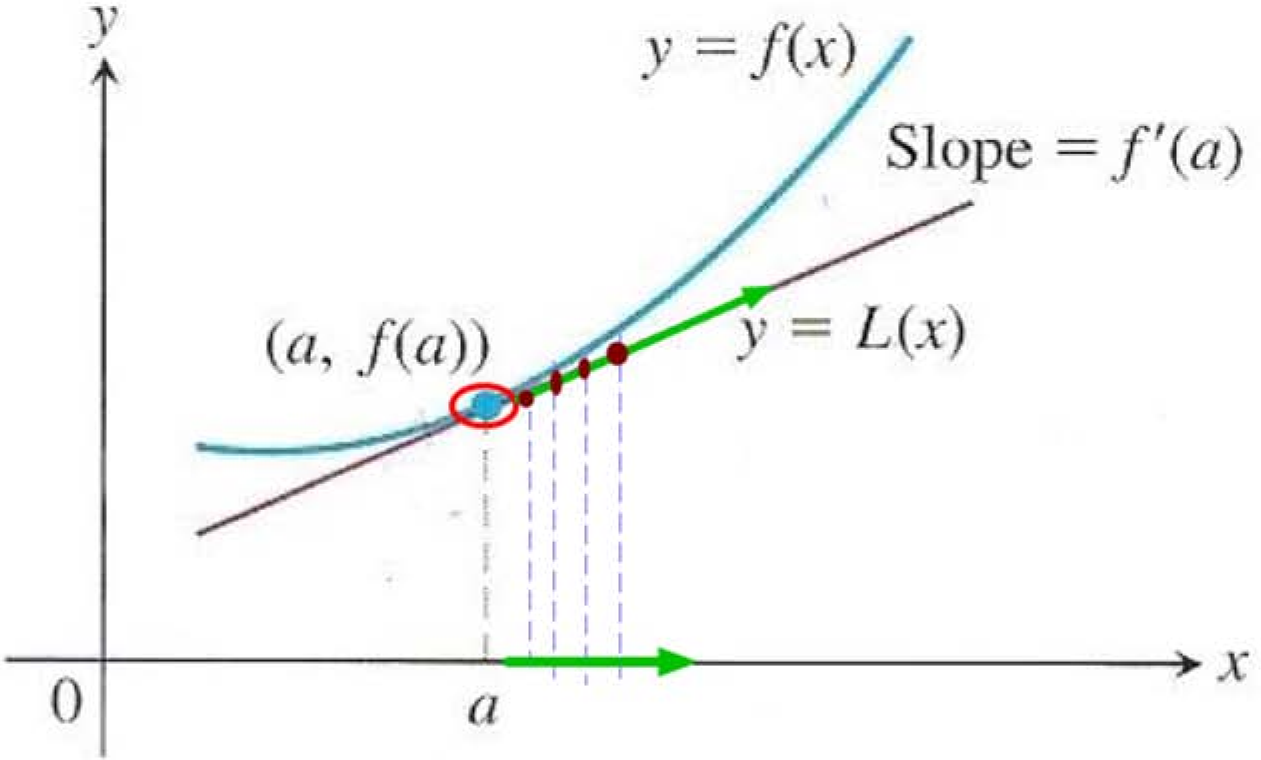
FIGURE 3.51 The more we magnify the graph of a function near a point where the function is differentiable, the flatter the graph becomes and the more it resembles its tangent.

As you can see above, as we make the interval smaller where the tangent at (1,1) is, the tangent line and the curve become indistinguishable. If the radius of the interval is less than 0.001, the tangent line at $x=a$

will be a very good approximation for the value

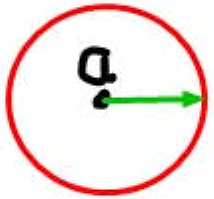
of $f(x)$ at $x=a$.

As we begin to travel along the tangent line AWAY from the



enclose neighborhood, the approximation

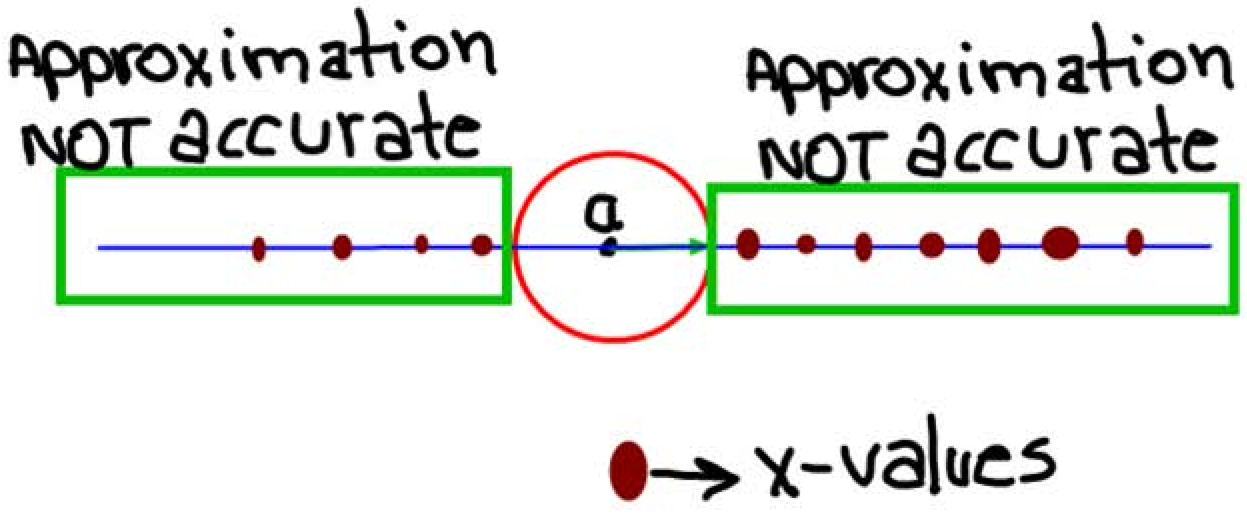
becomes less accurate and less reliable.



$x = a$ is the center of the approximation.

$$L(x) = f(a) + f'(a)(x-a)$$

Again, as we pick values of x that are outside the radius of approximation of $x=a$, the accuracy of the approximation about $x=a$ rapidly deteriorates.



DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

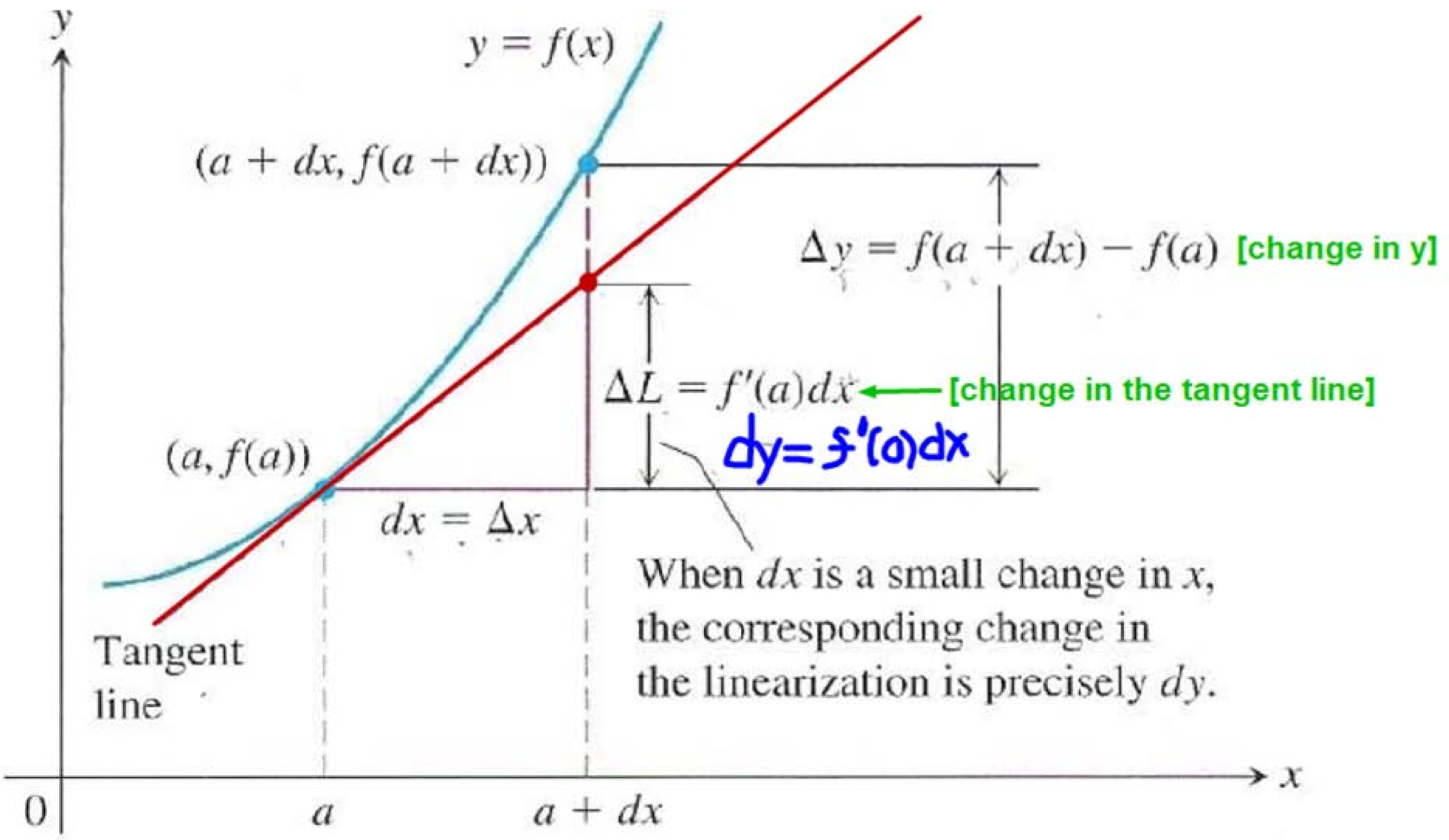
Differentials

We sometimes use the Leibniz notation dy/dx to represent the derivative of y with respect to x. Contrary to its appearance, it is not a ratio. We now introduce two new variables dx and dy with the property that when their ratio exists, it is equal to the derivative.

DEFINITION Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy** is

$$dy = f'(x)\,dx.$$

Unlike the independent variable dx, the variable dy is always a dependent variable. It depends on both x and dx. If dx is given a specific value and x is a particular number in the domain of the function f, then these values determine the numerical value of dy. Often the variable dx is chosen to be Δx, the change in x.



y
$y = f(x)$
$(a + dx, f(a + dx))$
$\Delta y = f(a + dx) - f(a)$ [change in y]
$\Delta L = f'(a)dx$ [change in the tangent line]
$dy = f'(a)dx$
$(a, f(a))$
$dx = \Delta x$
When dx is a small change in x, the corresponding change in the linearization is precisely dy.
Tangent line
0
a
$a + dx$
x

Geometric Meaning

Let $x = a$ and set $dx = \Delta x$ as long as Δx is very, very small.

The corresponding change in $y = f(x)$ is

$$\Delta y = f(a + dx) - f(a)$$

The corresponding change in

$$L(x) = f(a) + f'(a)(x-a)$$ is

$$L(a+dx) = f(a) + f'(a)[(a+dx)-a]$$

$$= f(a) + f'(a)(a-a+dx)$$

$$L(a+dx) = f(a) + f'(a)\,dx$$

$$L(a) = f(a) + f'(a)(a-a)$$

$$= f(a) + f'(a) \cdot 0$$

$$L(a) = f(a)$$

$$\Delta L = L(a+dx) - L(a)$$

$$\Delta L = f(a) + f'(a)dx - f(a)$$

$$\Delta L = f'(a)dx$$

The change in the linearization of f is precisely the value of the differential dy when $x = a$ and $dx = \Delta x$.

Therefore, dy represents the amount the tangent line rises or falls when x changes by an amount $dx = \Delta x$.