Differentials

For a while now, we have been using the notation

$$\frac{dy}{dx}$$

to mean the derivative of y with respect to x. Here x is any variable, and y is a variable whose value depends on x.

One of the reasons that we like this notation is that it suggests the meaning of the derivative. The quantities dx and dy are called **differentials**, and represent very small changes in the values of x and y. Specifically, if we change x by a small amount dx, then y will change by a small amount dy, and the ratio dy/dx is the derivative.

It has been a while since we discussed these ideas. The following example should help you to remember:

EXAMPLE 1 At a certain instant, the value of x is 3, and the value of y is 5. A short time later, the value of x has increased to 3.01, and the value of y has increased to 5.04. Estimate dy/dx.

SOLUTION The small increase in x is

$$dx = 3.01 - 3 = 0.01,$$

and the corresponding increase in y is

$$dy = 5.04 - 5 = 0.04.$$

Therefore

$$\frac{dy}{dx} \approx \frac{0.04}{0.01} = 4.$$

■

Now, the calculation above isn't quite right, because it doesn't really make sense to write $dx = 0.01$. The idea of the differential dx is that it's *infinitesimal*—infinitely small but still nonzero. Since 0.01 isn't infinitesimal, it can't really be the value of dx.

However, the calculation above does make sense as an approximation. Even though 0.01 isn't really infinitesimal, it is very small, so treating it as infinitesimal ought to yield answers that are approximately right. If we want the approximation to be more accurate, we would need to use a smaller change in x, such as 0.001 or 0.0001.

We discussed all of these ideas once before. The idea of better and better approximations leads naturally to the idea of as limit. Indeed, it is possible to define the derivative entirely using limits:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

This modern "limit definition" of the derivative eschews differentials and infinitesimals, relying instead on the more concrete notion of a limit. Before the limit definition of the derivative, mathematicians spent a century and a half arguing about the legitimacy of infinite and infinitesimal numbers, and the legitimacy of calculus itself. The limit definition puts those objections to rest, and provides a solid foundation for modern calculus.

However, the idea of infinitesimal numbers remains. Though calculus is now based on limits, many problems are most easily solved using infinitesimals, and reasoning using infinitesimals can be a very powerful technique. Ultimately, such reasoning needs to be justified using limits, but in practice infinitesimals are often easier to understand and easier to use.

Equations Involving Differentials

A **differential** is a variable whose value is infinitesimal. By convention, all differentials are preceded by the letter "*d*", which usually means something like "little bit of" or "little change in".

Any equation that involves derivatives can also be written as an equation involving differentials. For example, if a square has side length x, then the area of the square is given by the formula

$$A = x^2.$$

Taking the derivative with respect to x yields the equation

$$\frac{dA}{dx} = 2x.$$

This equation involves a derivative, which is really a ratio of two differentials. If we multiply through by dx, we get an equation relating the two differentials:

$$dA = 2x\,dx.$$

Note that both sides of this equation are infinitesimal. This equation tells us how much the area of the square will change if we increase the side length by a small amount. For example, if x is 4 and we increase x by 0.003, then the change in A will be approximately

$$2(4)(0.003) = 0.024.$$

Here is a summary of this technique:

Formula for dy in terms of x and dx.

Let x and y be variables, where $y = f(x)$. To find a formula for dy in terms of x and dx, start by taking the derivative with respect to x:

$$\frac{dy}{dx} = f'(x).$$

Next, multiply through by dx to obtain the desired formula:

$$dy = f'(x)\,dx.$$

EXAMPLE 2 Suppose that $y = x^3 + 4x$.

(a) Find a formula for dy in terms of x and dx.

(b) Suppose we increase x from 3 to 3.02. Use differentials to estimate the corresponding increase in the value of y.

SOLUTION Taking the derivative of the given formula yields

$$\frac{dy}{dx} = 3x^2 + 4.$$

We can now multiply through by dx to get

$$dy = (3x^2 + 4)\,dx.$$

This answers part (a). For part (b), we substitute $x = 3$ and $dx = 0.02$ to get

$$dy = \big(3(3)^2 + 4\big)(0.02) = 0.62. \qquad \blacksquare$$

In science, differentials are often used to estimate the possible error in the value of a variable obtained through calculation. The following example illustrates this technique:

EXAMPLE 3 The energy stored in a certain capacitor obeys the formula

$$E = \frac{1}{2}CV^2$$

where V is the voltage difference across the leads, and $C = 0.15$ Joules/volt^2.

(a) Find a formula for dE in terms of V and dV.

(b) An engineer measures the voltage across the leads as 2.8 volts, with an error of ± 0.05 volts. Find the energy stored in the capacitor, and estimate the error in your answer.

SOLUTION Taking the derivative with respect to V yields

$$\frac{dE}{dV} = CV.$$

We can now multiply through by dV to get

$$dE = CV\,dV.$$

This answers part (a). For part (b), the energy stored in the capacitor is

$$E = \frac{1}{2}CV^2 = \frac{1}{2}(0.15)(2.8)^2 = 0.588 \text{ Joules}.$$

To estimate the error, we imagine what happens if we change V by $dV = \pm 0.05$ volts. Using our differentials formula,

$$dE = CV\,dV = (0.15)(2.8)(\pm 0.05) = \pm 0.021 \text{ Joules},$$

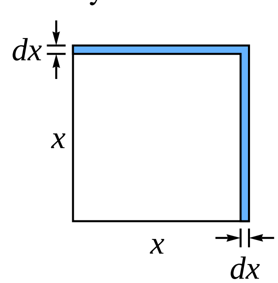
This is roughly the error in the value of the energy. ■

Differential Area

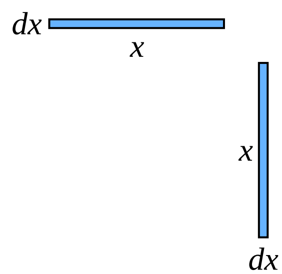
In geometry, we can sometimes use differentials to represent very small amounts of length or area. The following example illustrates this idea.

EXAMPLE 4 Consider again a square with area $A = x^2$, where x is the side length. As we saw before, if we increase x by a small amount dx, then the area increases by $dA = 2x\,dx$.

We can see this formula geometrically in the following figure:

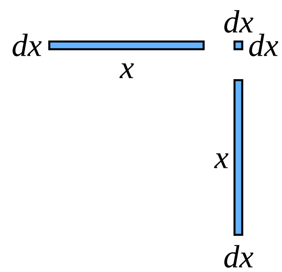


The added area dA has been shaded in blue. The shaded region has the shape of an L, which can be cut into two rectangles:



Each rectangle has an area of $x\,dx$, which explains the formula $dA = 2x\,dx$. ■

In the previous example, you might object to the breaking of the L shape into two rectangles. If you think about it carefully, there will be a small leftover square with area $(dx)^2$:



The area of this small square is a **second order infinitesimal**—an infinitesimal portion of an infinitesimal. Such a quantity arises whenever an infinitesimal is squared, or when two infinitesimals are multiplied together. When we are computing with infinitesimals, we usually ignore second-order infinitesimals, in the same way that we usually ignore infinitesimals when we are computing with real numbers.

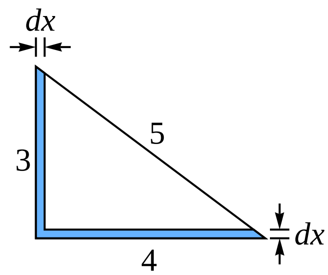
Second-Order Infinitesimals

A *second-order infinitesimal* is a quantity obtained by squaring an infinitesimal or by multiplying two infinitesimals together:

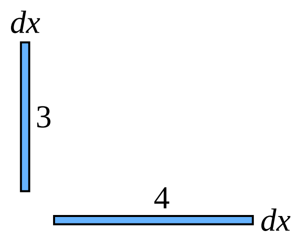
$$(dx)^2 \qquad \text{or} \qquad (dx)(dy)$$

When computing with infinitesimals, second-order infinitesimals can usually be ignored.

EXAMPLE 5 Find the shaded area in the following figure:



SOLUTION The shaded area can be broken up into two rectangles, one with an area of $3\,dx$, and the other with an area of $4\,dx$.



Again, this neglects a small square in the corner, as well as some small triangles on the ends. The areas of these neglected pieces are all second-order infinitesimals, so they can safely be ignored. Therefore, the total area is

$$dA = 3\,dx + 4\,dx = 7\,dx.$$

In preparation for the next example, consider the formulas for the area and circumference of a circle:

$$A = \pi r^2 \qquad \text{and} \qquad C = 2\pi r.$$

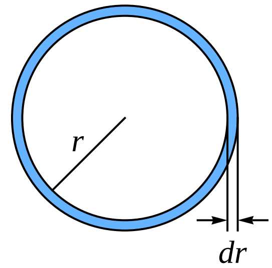
Curiously, the formula for the circumference is the derivative of the formula for the area:

$$\frac{dA}{dr} = \frac{d}{dr}\left[\pi r^2\right] = 2\pi r = C.$$

The following example explains this intriguing observation.

EXAMPLE 6 A circle has a radius of r. If the radius is increased by a small amount dr, how much does the area increase?

SOLUTION The following picture shows the increase in the area:



The added area dA has been shaded in blue. We can unbend the shaded area into the shape of a rectangle:

The length of this rectangle is the same as the circumference of the circle, namely $2\pi r$. Thus

$$dA = 2\pi r\,dr.$$

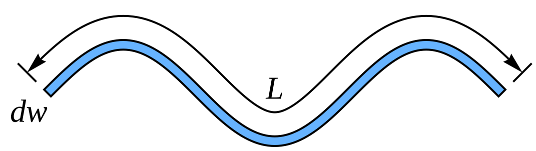
In particular, dA/dr is the equal to the circumference $2\pi r$. ■

The "bending" that we used in the last example slightly decreased the area of the blue region. To be precise, we can calculate the area of the blue region precisely using the difference of the areas of two circles:

$$\begin{aligned} dA &= \pi(r+dr)^2 - \pi r^2 \\ &= \pi r^2 + 2\pi r\,dr + \pi(dr)^2 - \pi r^2 \\ &= 2\pi r\,dr + \pi(dr)^2 \end{aligned}$$

As you can see, the bending decreased the area of the blue region by $\pi(dr)^2$, which is a second-order infinitesimal. Thus the bending did not change the area by a significant amount.

In general, it is possible to bend a region with infinitesimal thickness without changing the area significantly. For example, consider a curved strip with length L and width dw:



Such a strip can be bent into the shape of a rectangle with length L and width dw. Thus the area of the strip is the product $L\,dw$.

Area of a Narrow Strip

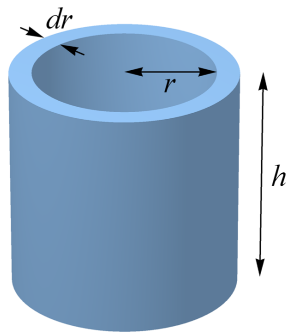
The area of a narrow strip with length L and width dw is given by the formula

$$dA = L\,dw.$$

Differential Volume

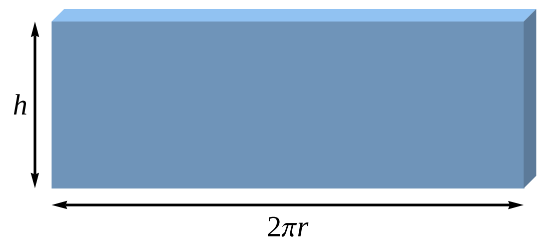
We can use similar methods to find the differential volume of shapes in three dimensions.

EXAMPLE 7 A cylindrical shell has height h, radius r, and thickness dr.



Find a formula for the volume of the shell in terms of h, r, and dr.

SOLUTION If we make a vertical incision along the side of the cylindrical shell, we can unroll it into the shape of a thin rectangular sheet:



This rectangle has a height of h, a width of $2\pi r$ (the circumference of the shell), and a thickness of dr. Therefore, the total volume is

$$dV = (2\pi r)(h)(dr) = 2\pi rh\, dr.$$

■

Note that the volume in this example was equal to the area A of the side of the cylinder multiplied by the thickness dr. This works for any thin sheet of material:

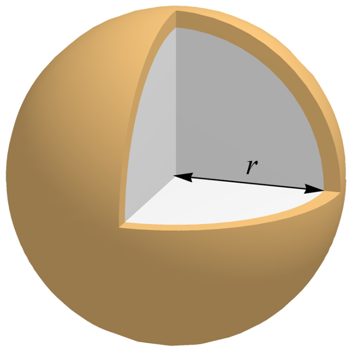
Volume of a Thin Sheet

The volume of a thin sheet with area A and thickness ds is given by the formula

$$dV = A\, ds.$$

EXAMPLE 8 A sphere has a radius of r. If the radius is increased by a small amount dr, how much does the volume increase?

SOLUTION The following picture shows a cutaway of the sphere, with the new volume highlighted in orange.



The orange shape is known as a ***spherical shell***. Its volume is equal to the surface area of the sphere ($4\pi r^2$) multiplied by the thickness of the shell:

$$dV = 4\pi r^2\, dr.$$

■

As with the circle, we can divide through by dr in the formula above to obtain

$$\frac{dV}{dr} = 4\pi r^2.$$

This explains why the formula for the surface area of a sphere ($4\pi r^2$) is the derivative of the formula for the volume of a sphere ($\frac{4}{3}\pi r^3$).

All of the examples so far have involved actual differentials, which are always infinitesimal in size. However, it is possible to use these ideas to estimate the volumes of real objects. The following examples illustrate this principle.

EXAMPLE 9 A cube with a side length of 25 cm is covered with paint having a thickness of 0.5 mm. Estimate the amount of paint used.

SOLUTION The paint forms a thin sheet around the outside of the cube. The area covered by the paint is

$$A = 6(25\text{ cm})^2 = 3750\text{ cm}^2.$$

The thickness is $ds = 0.5\text{ mm} = 0.05\text{ cm}$. Therefore, the total volume of paint is

$$dV \approx A\, ds = (3750\text{ cm}^2)(0.05\text{ cm}) = 187.5\text{ cm}^3.$$

Note that this is only an estimate, since the thickness ds of the paint is not a true infinitesimal. In particular, this calculation does not take into account the paint required to cover the edges and the corners of the cube. This hardly affects the answer: a more careful calculation shows that 0.751 cm^3 additional paint are required for the edges and corners, which is less than 1% of the paint used. ■

EXAMPLE 10 A hemispherical dome is constructed from concrete. The dome is 60 ft high, and the concrete used to make the dome is 9 in thick. Estimate the total volume of concrete used to construct the dome.

SOLUTION The surface area of the dome is half of the surface area of a sphere:

$$A = \frac{1}{2}(4\pi r^2) = \frac{1}{2}(4\pi)(60\text{ ft})^2 = 22{,}619.47\text{ ft}^2$$

The thickness is $ds = 9\text{ in} = 0.75\text{ ft}$, so the total volume is

$$dV \approx A\,ds = (22{,}619.47\text{ ft}^2)(0.75\text{ ft}) = 16{,}964.6\text{ ft}^3.$$

Again, this is only an estimate, since the thickness of the concrete is not really infinitesimal. ■