Calculus I - Lecture 15
Linear Approximation & Differentials

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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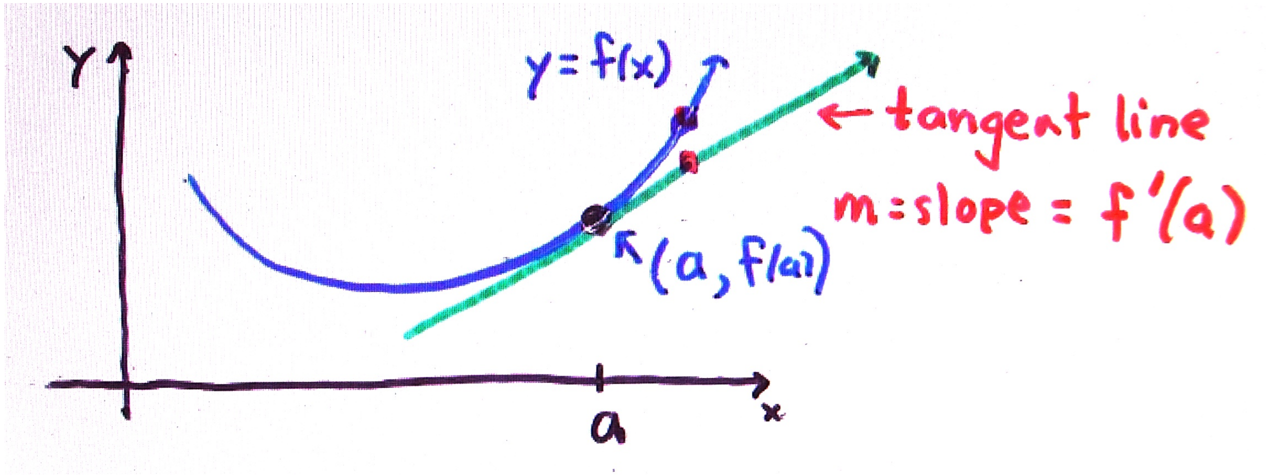
March 11, 2014

Equation of Tangent Line

Recall the equation of the tangent line of a curve $y = f(x)$ at the point $x = a$.



The **general equation of the tangent line** is

$$y = L_a(x) := f(a) + f'(a)(x - a).$$

That is the point-slope form of a line through the point $(a, f(a))$ with slope $f'(a)$.

Linear Approximation

It follows from the geometric picture as well as the equation

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

which means that $\frac{f(x)-f(a)}{x-a} \approx f'(a)$ or

$$f(x) \approx f(a) + f'(a)(x - a) = L_a(x)$$

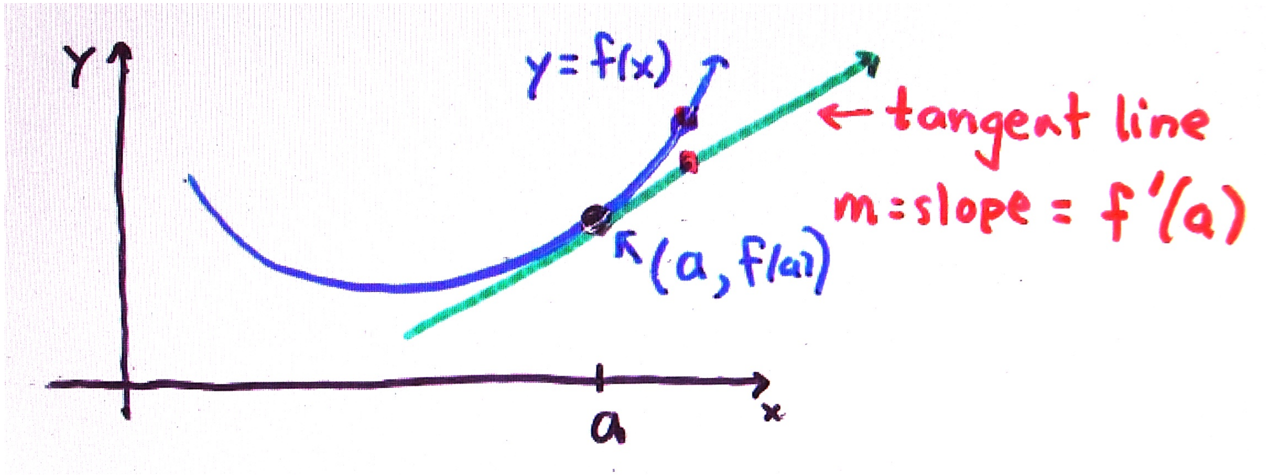
for x close to a. Thus $L_a(x)$ is a good **approximation** of $f(x)$ for x near a.

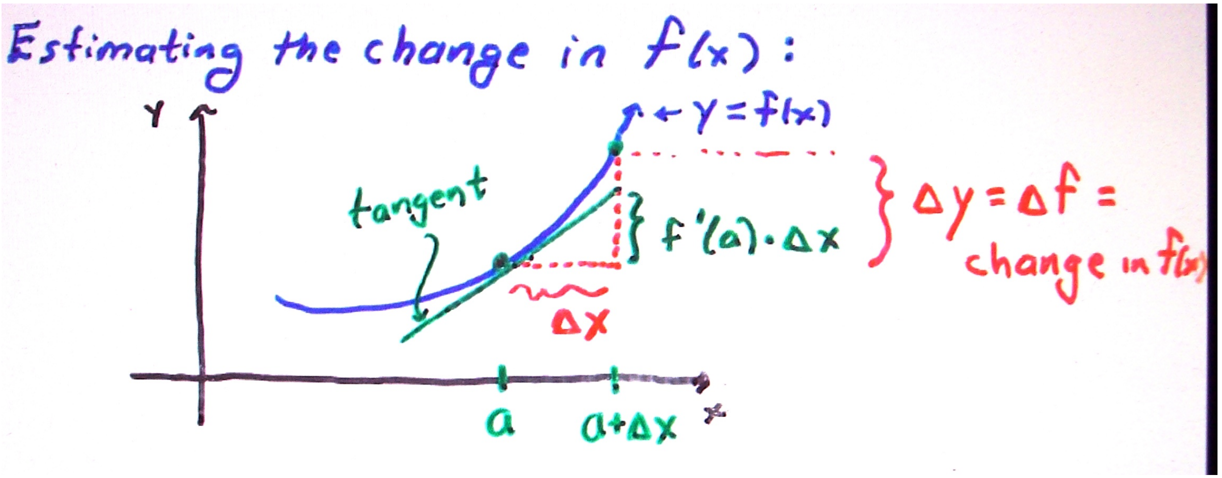
If we write $x = a + \Delta x$ and let Δx be sufficiently small this becomes $f(a + \Delta x) - f(a) \approx f'(a)\Delta x$. Writing also $\Delta y = \Delta f := f(a + \Delta x) - f(a)$ this becomes

$$\Delta y = \Delta f \approx f'(a)\Delta x$$

In words: for small Δx the **change** Δy in y if one goes from x to $x + \Delta x$ is approximately equal to $f'(a)\Delta x$.

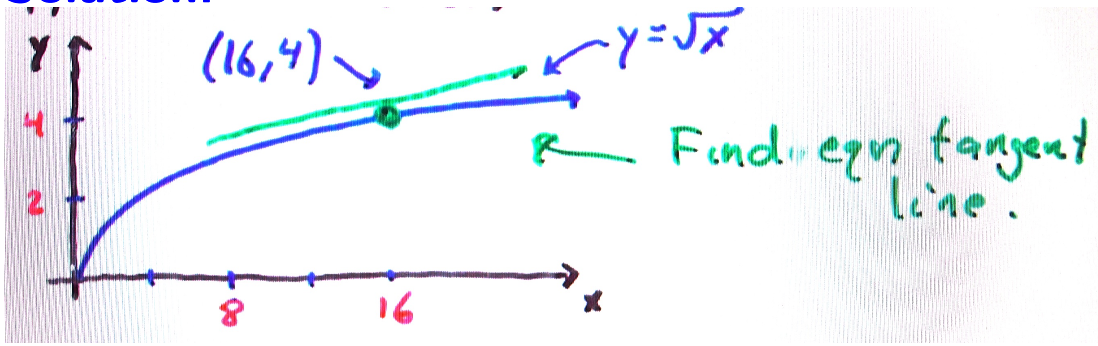
Visualization of Linear Approximation





Example: a) Find the linear approximation of $f(x) = \sqrt{x}$ at $x = 16$.
b) Use it to approximate $\sqrt{15.9}$.

Solution:



a) We have to compute the equation of the tangent line at $x = 16$.

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\, x^{1/2} = \frac{1}{2} x^{-1/2}$$

$$f'(16) = \tfrac{1}{2} 16^{-1/2} = \tfrac{1}{2} \cdot \tfrac{1}{\sqrt{16}} = \tfrac{1}{8}$$

$$L(x) = f'(a)(x - a) + f(a)$$

$$= \tfrac{1}{8}(x - 16) + \sqrt{16} = \tfrac{1}{8}x - 2 + 4 = \tfrac{1}{8}x + 2$$

b) $\sqrt{15.9} = f(15.9)$

$$\approx L(15.9) = \tfrac{1}{8} \cdot 15.9 + 2 = \tfrac{1}{8}(16 - .1) + 2 = 4 - \tfrac{1}{80} = \tfrac{319}{80}.$$

Example: Estimate $\cos(\frac{\pi}{4} + 0.01) - \cos(\frac{\pi}{4})$.

Solution:

Let $f(x) = \cos(x)$. Then we have to find $\Delta f = f(a + \Delta x) - f(a)$ for $a = \frac{\pi}{4}$ and $\Delta x = .01$ (which is small).

Using linear approximation we have:

$$\begin{aligned}
\Delta f &\approx f'(a) \cdot \Delta x \\
&= -\sin(\frac{\pi}{4}) \cdot .01 \qquad \text{(since } f'(x) = -\sin x\text{)} \\
&= -\frac{\sqrt{2}}{2} \cdot \frac{1}{100} = -\frac{\sqrt{2}}{200}
\end{aligned}$$

Example: The radius of a sphere is increased from 10 cm to 10.1 cm. Estimate the change in volume.

Solution:

$$V = \tfrac{4}{3}\pi r^3 \qquad \text{(volume of a sphere)}$$

$$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi \cdot 3r^2 = 4\pi r^2$$

$$\Delta V \approx \frac{\mathrm{d}V}{\mathrm{d}r} \cdot \Delta r = 4\pi r^2 \cdot \Delta r$$

$$= 4\pi \cdot 10^2 \cdot (10.1 - 10) = 400\pi \cdot \frac{1}{10} = 40\pi$$

The volume of the sphere is increased by $40\pi\ \mathrm{cm}^3$.

Example: The radius of a disk is measured to be $10 \pm .1$ cm (error estimate). Estimate the maximum error in the approximate area of the disk.

Solution:

$$A = \pi r^2 \qquad \text{(area of a disk)}$$

$$\frac{\mathrm{d}A}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}(\pi \cdot r^2) = 2\pi r$$

$$\Delta A \approx \frac{\mathrm{d}A}{\mathrm{d}r} \cdot \Delta r = 2\pi r \cdot \Delta r$$

$$= 2\pi \cdot 10 \cdot (\pm 0.1) = \pm 20\pi \cdot \frac{1}{10} = \pm 2\pi$$

The area of the disk has approximately a maximal error of $2\pi \ \mathrm{cm}^2$.

Example: The dimensions of a rectangle are measured to be 10 ± 0.1 by 5 ± 0.2 inches.
What is the approximate uncertainty in the area measured?

Solution: We have $A = xy$ with $x = 10 \pm 0.1$ and $y = 5 \pm 0.2$.

We estimate the measurement error $\Delta_x A$ with respect to the variable x and the error $\Delta_y A$ with respect to the variable y.

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(xy) = y$$

$$\frac{\mathrm{d}A}{\mathrm{d}y} = \frac{\mathrm{d}}{\mathrm{d}y}(xy) = x$$

$$\Delta_x A \approx \frac{\mathrm{d}A}{\mathrm{d}x} \cdot \Delta x = y \cdot \Delta x$$

$$\Delta_y A \approx \frac{\mathrm{d}A}{\mathrm{d}y} \cdot \Delta y = x \cdot \Delta y$$

The **total** estimated uncertainty is:

$$\Delta A = \Delta_x A + \Delta_y A = y \cdot \Delta x + x \cdot \Delta y$$

$$\Delta A = 5 \cdot (\pm .1) + 10 \cdot (\pm .2) = \pm .5 + \pm 2.0 = \pm 2.5$$

The uncertainty in the area is approximately of 2.5 inches^2.

Differentials

Those are a the most murky objects in Calculus I. The way they are usually defined in in calculus books is difficult to understand for a Mathematician and maybe for students, too.

Remember that we have

$$\frac{\Delta y}{\Delta x} \approx \frac{\mathrm{d}y}{\mathrm{d}x} = f'(x).$$

The idea is to consider $\mathrm{d}y$ and $\mathrm{d}x$ as infinitesimal small numbers such that $\frac{\mathrm{d}y}{\mathrm{d}x}$ is not just an approximation but equals $f'(a)$ and one rewrites this as

$$\mathrm{d}y = f'(x) \cdot \mathrm{d}x.$$

This obviously makes no sense since the only “infinitesimal small number” which I know in calculus is 0 which gives the true but useless equation $0 = f'(x) \cdot 0$ and the nonsense equation $\frac{0}{0} = f'(x)$.

So the official explanation is that

$$\mathrm{d}y = f'(x) \cdot \mathrm{d}x$$

describes the linear approximation for the **tangent line** to $f(x)$ at the point x which gives indeed this equation. Then $\mathrm{d}x$ and $\mathrm{d}y$ are numbers satisfying this equation. One problem is that one does not like to keep x fixed and $f'(x)$ varies with x. But how to understand the dependence of $\mathrm{d}x$ and $\mathrm{d}y$ on x?

The symbolic explanation is that

$$\mathrm{d}y = f'(x) \cdot \mathrm{d}x$$

is an equation between the old variable x and the new variables $\mathrm{d}x$ and $\mathrm{d}y$ but we **never will plug in numbers** for $\mathrm{d}x$ and $\mathrm{d}y$.

Note that $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$ makes sense with both interpretations for $\mathrm{d}x \neq 0$.

For applications (substitution in integrals) we will usually need the second interpretation

Example: Express $\mathrm{d}x$ in terms of $\mathrm{d}y$ for the function $y = e^{x^2}$.

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x^2} \cdot 2x$$

$$\mathrm{d}y = e^{x^2} \cdot 2x \cdot \mathrm{d}x$$

$$\mathrm{d}x = \frac{1}{e^{x^2} \cdot 2x} \cdot \mathrm{d}y = \frac{\mathrm{d}y}{2xy}$$