Linearization & Differentials Notes
Prof Davis

DEFINITIONS: If f is differentiable at $x=a$, then the approximating function

$$L(x) = f(a) + f'(a)(x-a)$$

is the LINEARIZATION of f at a.

The approximation

$$f(x) \approx L(x)$$

of f by L is the *standard linear approximation* of f at a. The point $x=a$ is the *center* of the approximation.

As long as the tangent line at the point $(a, f(a))$ to the graph of $f(x)$ remains CLOSE to the graph of $f(x)$ as we move off the point of tangency, $L(x)$ gives a good approximation to $f(x)$.

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⇒ VERY IMPORTANT APPROXIMATIONS

① An important linear approximation for roots and powers:

$$(1+x)^k \approx 1+kx$$

NOTE: x is extremely close to 0, k is any number

② $\sqrt{1+x} \approx 1+\frac{1}{2}x$ | $k=\frac{1}{2}$

③ $\frac{1}{1-x}=(1-x)^{-1}\approx 1+(-1)(-x)=1+x$ | $k=-1$; replace x by $-x$

④ $\sqrt[3]{1+5x^4}=(1+5x^4)^{1/3}\approx 1+(\frac{1}{3})(5x^4)=1+\frac{5}{3}x^4$ | $k=\frac{1}{3}$; replace x by $5x^4$

⑤ $\frac{1}{\sqrt{1-x^2}}=(1-x^2)^{-1/2}\approx 1+(\frac{-1}{2})(-x^2)=1+\frac{1}{2}x^2$ | $k=\frac{-1}{2}$; replace x by $-x^2$

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* DIFFERENTIALS

When the ratio $\frac{dx}{dy}$ exists, it is equal to the derivative

DEFINITION: Let $y = f(x)$ be a differentiable function. The *differential* dx is an independent variable

The *differential* dy is

$$dy = f'(x)dx$$

unlike the independent variable **dx**, the variable **dy** is always a dependent variable. It depends on both x and dx. If dx is given a specific value and x is a particular number in the domain of the function f, then these values determine the numerical value of dy. Often the variable dx is chosen to be Δx, the change in x

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The geometric meaning of differentials is shown in the figure below. Let $x = a$ and set $dx = \Delta x$.
The corresponding change in $y = f(x)$ is

$$\Delta y = f(a + dx) - f(a)$$

The corresponding change in the tangent line L is

$$\Delta L = L(a + dx) - L(a)$$

$$= \underbrace{f(a) + f'(a)[(a+dx) - a]}_{L(a+dx)} - \underbrace{f(a)}_{L(a)}$$

$$= f'(a)\,dx$$

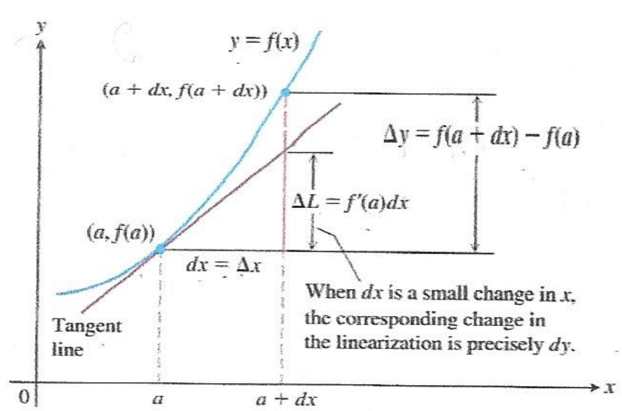

FIGURE 3.56 Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.

The change in the linearization of f is precisely the value of the differential dy when $x = a$ and $dx = \Delta x$. Therefore, dy represents the amount the tangent line rises or falls when x changes by an amount $dx = \Delta x$.

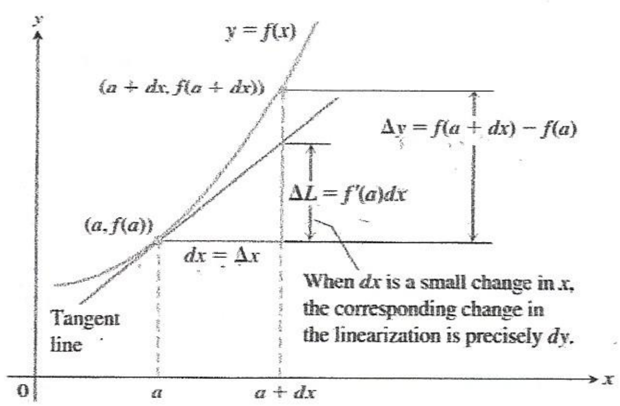

FIGURE 3.56 Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.

For the following, find the linearization $L(x)$ of $f(x)$ at $x=a$

#1) $f(x) = x^3 - x$ at $x = 1$

$f'(x) = 3x^2 - 1$

$f(1) = 0$

$f'(1) = 3 - 1 = 2$

$L(x) = f(a) + f'(a)(x-a)$

$L(x) = f(1) + f'(1)(x-1)$

$L(x) = 0 + 2(x-1)$

$L(x) = 2(x-1)$

#2) $f(x) = x^3 - 2x + 3$ at $x = 2$

$f'(x) = 3x^2 - 2$

$f(2) = 8 - 4 + 3 = 7$

$f'(2) = 12 - 2 = 10$

$L(x) = f(2) + f'(2)(x-2)$

$= 7 + 10(x-2)$

$= 7 + 10x - 20$

$L(x) = 10x - 13$

#3) $f(x) = \sqrt{x}$ at $x = 4$

$f(x) = x^{1/2}$

$f(4) = \sqrt{4} = 2$

$f'(x) = \frac{1}{2\sqrt{x}}$

$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$L(x) = f(a) + f'(a)(x-a)$

$L(x) = f(4) + f'(4)(x-4)$

$L(x) = 2 + \frac{1}{4}(x-4)$

$L(x) = 2 + \frac{1}{4}x - 1$

$L(x) = \frac{1}{4}x + 1$

#4) $f(x) = \sqrt{x^2+9}$ at $x = -4$

$f(x) = (x^2+9)^{1/2}$ $f(4) = \sqrt{25} = 5$

$f'(x) = \frac{1}{2}(x^2+9)^{-1/2}(2x)$

$f'(x) = \frac{x}{\sqrt{x^2+9}}$ $f'(4) = \frac{-4}{\sqrt{25}} = \frac{-4}{5}$

$L(x) = f(a) + f'(a)(x-a)$

$L(x) = f(-4) + f'(-4)(x+4)$

$L(x) = 5 + \frac{-4}{5}(x+4)$

$L(x) = 5 + \frac{-4}{5}(x+4)$

$L(x) = \frac{-4}{5}x + \frac{9}{5} = \frac{1}{5}(-4x+9)$

#5) $f(x) = e^{-x}$, $a = -0.1$ use $a = 0$

$f'(x) = -e^{-x}$

$f(0) = 1$

$f'(0) = -1$

$L(x) = f(a) + f'(a)(x-a)$

$L(x) = f(0) + f'(0)x$

$L(x) = 1 - x$

#6) $f(x) = \sin^{-1} x$, $a = \frac{\pi}{12} = 0.262$

use $a = \frac{1}{2}$

$$f\left(\tfrac{1}{2}\right) = \sin^{-1}\tfrac{1}{2} = \frac{\pi}{6}$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f'\left(\tfrac{1}{2}\right) = \frac{1}{\sqrt{1-\frac{1}{4}}} = \frac{1}{\sqrt{\frac{3}{4}}} = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{2}{\sqrt{3}}$$

$$L(x) = f\left(\tfrac{1}{2}\right) + f'\left(\tfrac{1}{2}\right)\left(x - \tfrac{1}{2}\right)$$

$$L(x) = \frac{\pi}{6} + \frac{2}{\sqrt{3}}\left(x - \frac{1}{2}\right)$$

#7)

$$f(x) = (1+x)^{k}$$

$$L(x) = f(a) + f'(a)(x-a) \;;\; a=0$$

$$f'(x) = k(1+x)^{k-1}$$

$$f(0) = 1$$

$$f'(0) = k$$

$$L(x) = f(a) + f'(a)(x-a)$$

$$L(x) = 1 + kx$$

#8) $L(x) = 1 + kx$ where the value is very close to zero.

(a) $f(x) = (1-x)^6 = [1+(-x)]^6$

$k=6$, Replace x with $-x$

$L(x) = 1-6x$

b.) $f(x) = \dfrac{2}{1-x}$

$f(x) = 2[1+(-x)]^{-1}$

$k = -1$, Replace x with $-x$

$L(x) = 2[1+(-1)(-x)]$

$L(x) = 2+2x$

#8)

c.) $$f(x) = \frac{1}{\sqrt{1+x}} = (1+x)^{\frac{-1}{2}}$$

$k = \frac{-1}{2}$

$$L(x) = 1 - \frac{1}{2}x$$

d.) $$f(x) = \sqrt{2+x^2} = (2+x^2)^{1/2}$$

$k = \frac{1}{2}$ Replace x with x^2

$$L(x) = 1 + \frac{1}{2}x^2$$

#8)

e.)

$$f(x) = (4+3x)^{1/3} = \left[4\left(1+\tfrac{3}{4}x\right)\right]^{1/3}$$

$$f(x) = 4^{1/3}\left(1+\tfrac{3}{4}x\right)^{1/3}$$

$K = \frac{1}{3}$, Replace x with $\frac{3}{4}x$

$$L(x) = 4^{1/3}\left[1+\left(\tfrac{1}{3}\right)\left(\tfrac{3}{4}x\right)\right]$$

$$L(x) = 4^{1/3}\left(1+\tfrac{1}{4}x\right)$$

#8)

f.) $$f(x) = \sqrt[3]{\left(1 - \frac{x}{2+x}\right)^2}$$

$$f(x) = \left[1 + \left(\frac{-x}{2+x}\right)\right]^{2/3}$$

$K = \frac{2}{3}$; Replace x with $\frac{-x}{2+x}$

$$L(x) = 1 + kx$$

$$L(x) = 1 + \frac{2}{3}\left(\frac{-x}{2+x}\right)$$

$$L(x) = 1 - \frac{2x}{6+3x}$$

#9) Find the linearization of $f(x) = \sqrt{x+1} + \sin x$ at $x = 0$. How is it related to the individual linearizations of $\sqrt{x+1}$ and $\sin x$ at $x = 0$

$$L(x) = f(a) + f'(a)(x-a)$$

$$f(x) = \sqrt{x+1} + \sin x$$

$$f(0) = \sqrt{0+1} + \sin 0$$

$$f(0) = \sqrt{1}$$

$$f(0) = 1$$

$$f'(x) = \frac{1}{2\sqrt{x+1}} + \cos x$$

$$f'(0) = \frac{1}{2} + 1 = \frac{3}{2}$$

#9)

$$L(x) = 1 + \frac{3}{2}x$$

Let $g(x) = (x+1)^{1/2}$; $g'(x) = \frac{1}{2\sqrt{x+1}}$

$g(0) = 1$; $g'(0) = \frac{1}{2}$

$$L_g(x) = g(a) + g'(a)(x-a)$$

$$L_g(x) = g(0) + g'(0)(x-0)$$

$$L_g(x) = 1 + \frac{1}{2}x$$

#9) Let $h(x) = \sin x$ $\quad$ $h'(x) = \cos x$

$h(0) = 0$ $\quad$ $h'(0) = 1$

$$L_h(x) = h(a) + h'(a)(x-a)$$

$$L_h(x) = h(0) + h'(0)(x-0)$$

$$L_h(x) = 0 + 1x = x$$

The linearization of

$$L_f(x) = L_g(x) + L_h(x)$$

This implies that the linearization of a sum is equal to the sum of the linearizations.

For problems 10 to 14, find dy

#10) Given $y = x\sqrt{1-x^2}$

$$y = x(1-x^2)^{1/2}$$

$$\frac{dy}{dx} = (1-x^2)^{1/2} + x\left\{\tfrac{1}{2}(1-x^2)^{-1/2}(-2x)\right\}$$

$$\frac{dy}{dx} = (1-x^2)^{1/2} - x^2(1-x^2)^{-1/2}$$

$$\frac{dy}{dx} = (1-x^2)^{-1/2}\left[(1-x^2)-x^2\right]$$

$$\frac{dy}{dx} = (1-x^2)^{-1/2}(1-2x^2)$$

#10)

$$\frac{dy}{dx} = \frac{1-2x^2}{(1-x^2)^{1/2}}$$

$$\frac{dy}{dx} = \frac{(1-2x^2)}{(1-x^2)^{1/2}}$$

$$dy = \left[\frac{(1-2x^2)}{(1-x^2)^{1/2}}\right]dx$$

#11) $$xy^2 - 4x^{3/2} - y = 0$$

$$y^2 + 2xy\frac{dy}{dx} - 4\left(\frac{3}{2}\right)x^{1/2} - \frac{dy}{dx} = 0$$

$$y^2 + 2xy\frac{dy}{dx} - 6x^{1/2} - \frac{dy}{dx} = 0$$

$$\left\{2xy\frac{dy}{dx} - \frac{dy}{dx}\right\} + \left(y^2 - 6x^{1/2}\right) = 0$$

$$(2xy-1)\frac{dy}{dx} = 6x^{1/2} - y^2$$

$$\frac{dy}{dx} = \frac{6x^{1/2} - y^2}{2xy - 1}$$

$$dy = \left[\frac{6x^{1/2} - y^2}{2xy-1}\right]dx$$

#12) $y = \cos(x^2)$

$$\frac{dy}{dx} = -\sin(x^2)\cdot(2x)$$

$$\frac{dy}{dx} = -2x\sin(x^2)$$

$$dy = -2x\sin(x^2)\,dx$$

#13)

$$y = \ln\left(\frac{x+1}{\sqrt{x-1}}\right)$$

$$y = \ln\left(\frac{x+1}{(x-1)^{1/2}}\right)$$

Method #1

$$\frac{dy}{dx} = \frac{1}{\left[\frac{x+1}{(x-1)^{1/2}}\right]} \cdot \frac{d}{dx}\left[\frac{x+1}{(x-1)^{1/2}}\right]$$

$$= \frac{(x-1)^{1/2}}{(x+1)} \cdot \left\{\frac{(x-1)^{1/2}(1) - (x+1)\left(\frac{1}{2}\right)(x-1)^{-1/2}}{(x-1)}\right\}$$

$$\frac{dy}{dx} = \frac{(x-1) - \frac{1}{2}(x+1)(x-1)^{-1/2}}{(x+1)\,(x-1)^{1/2}}$$

#13)

$$\frac{dy}{dx} = \frac{(x-1)^{-1/2}\left[(x-1) - \frac{1}{2}(x+1)\right\}}{(x+1)(x-1)^{1/2}}$$

$$\frac{dy}{dx} = \frac{(x-1) - \frac{1}{2}x - \frac{1}{2}}{(x+1)(x-1)}$$

$$\frac{dy}{dx} = \frac{\left(x - \frac{1}{2}x\right) + \left(-1 - \frac{1}{2}\right)}{x^2 - 1}$$

$$\frac{dy}{dx} = \frac{\left(\frac{1}{2}x - \frac{3}{2}\right)}{x^2 - 1}$$

$$dy = \left[\frac{x-3}{2(x^2-1)}\right] dx$$

#13)

Method #2

$$y = \ln\left[\frac{x+1}{(x-1)^{1/2}}\right]$$

$$y = \ln(x+1) - \ln(x-1)^{1/2}$$

$$y = \ln(x+1) - \frac{1}{2}\ln(x-1)$$

$$dy = \left[\frac{1}{x+1} - \frac{1}{2(x-1)}\right]dx$$

$$dy = \left[\frac{2(x-1)-(x+1)}{2(x+1)(x-1)}\right]dx$$

#13) Method #2

$$dy = \left[\frac{2x-2-x-1}{2(x^2-1)}\right]dx$$

$$dy = \left[\frac{x-3}{2(x^2-1)}\right]dx$$

#14)

$$y = e^{\tan^{-1}\sqrt{x^2+1}}$$

$$\frac{dy}{dx} = e^{\tan^{-1}(x^2+1)^{1/2}} \cdot \frac{d}{dx}\left[\tan^{-1}(x^2+1)^{1/2}\right]$$

$$\frac{dy}{dx} = e^{\tan^{-1}(x^2+1)^{1/2}} \cdot \frac{1}{1+\left[(x^2+1)^{1/2}\right]^2} \cdot \frac{d}{dx}\left[(x^2+1)^{1/2}\right]$$

$$\frac{dy}{dx} = \frac{e^{\tan^{-1}(x^2+1)^{1/2}}}{1+(x^2+1)} \left\{\frac{1}{2}(x^2+1)^{-1/2}(2x)\right\}$$

#14)

$$\frac{dy}{dx} = \frac{x\,e^{\tan^{-1}(x^2+1)^{1/2}}}{(x^2+1)^{1/2}(x^2+2)}$$

$$dy = \left[\frac{x\,e^{\tan^{-1}(x^2+1)^{1/2}}}{(x^2+1)^{1/2}(x^2+2)}\right]dx$$