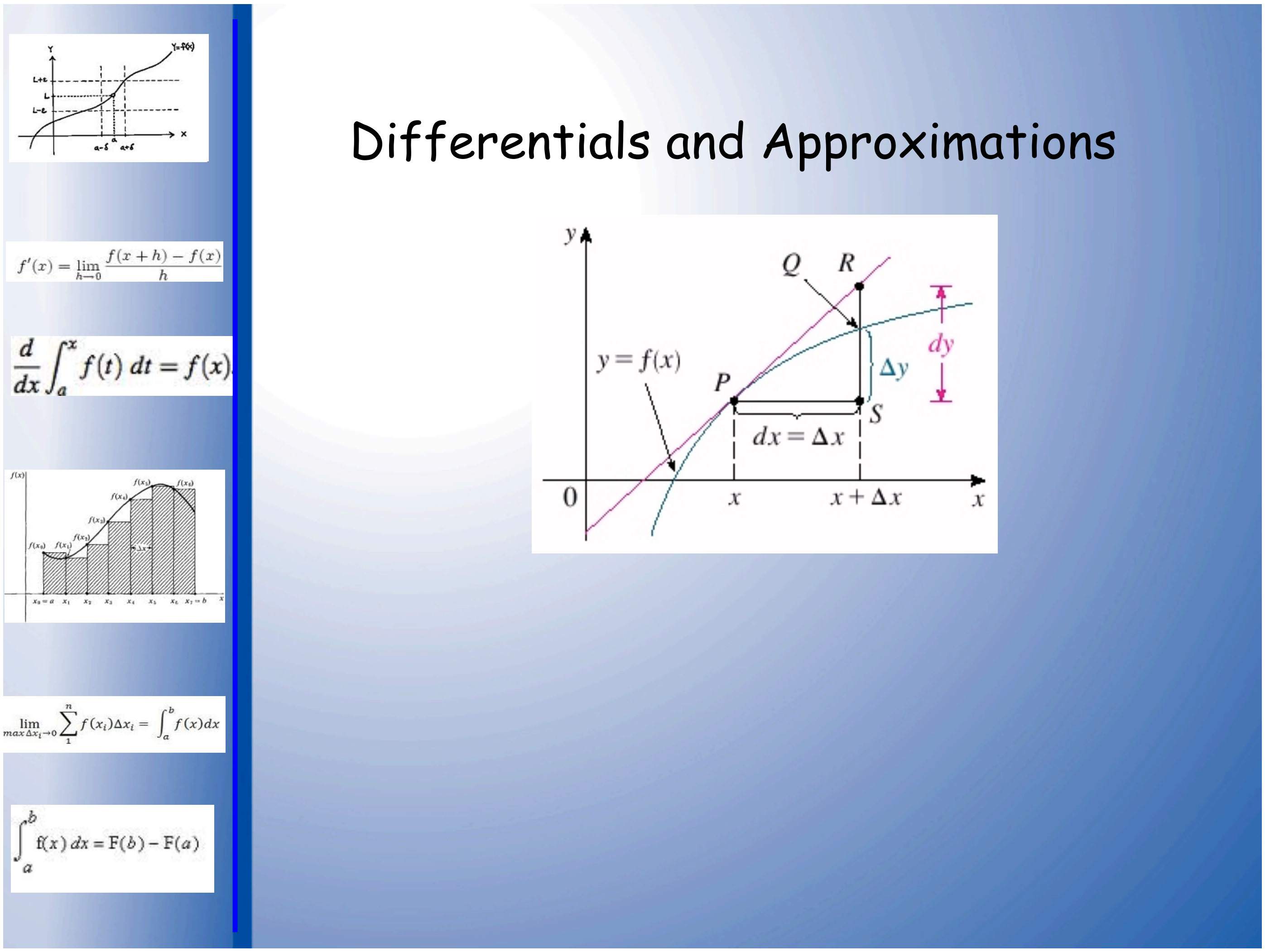

Differentials and Approximations
$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
$\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$
$\lim_{max \Delta x_i \to 0} \sum_1^n f(x_i) \Delta x_i = \int_a^b f(x) dx$
$\int_a^b f(x)\,dx = F(b) - F(a)$
y
Q
R
dy
Δy
$y = f(x)$
P
S
$dx = \Delta x$
0
x
$x + \Delta x$
x

15.5B Differentials

Differentials and Approximations

We have seen the notation dy/dx and we've never separated the symbols.
Now, we'll give meaning to dy and dx as separate entities.

We know $\boxed{\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = f'(x_0)}$ gives the derivative (slope) of the function $f(x)$ at $x=x_0$.

If Δx is really small, then $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \approx f'(x_0)$ approximately true

and $f(x_0+\Delta x)-f(x) \approx f'(x_0)\Delta x$

Differentials

Let $y=f(x)$ be a differentiable function of x. Δx is an arbitrary increment of x.

$dx = \Delta x$ (dx is called a differential of x.) (a small increment of x)

Δy is actual change in y as x goes from x to $x+\Delta x$.

i.e. $\Delta y = f(x+\Delta x)-f(x)$

$\boxed{dy = f'(x)dx}$ (dy is called the differential of y.)

$$\frac{dy}{dx} = f'(x)$$

$$dy = f'(x)\,dx$$

$$\boxed{\Delta y \approx dy}$$

15.5B Differentials

EX 1 Find dy.

a) $y = 4x^3-2x+5$

$$\frac{dy}{dx} = 12x^2-2 \Rightarrow \boxed{dy=(12x^2-2)dx}$$

b) $y = 2\sqrt{x^4+6x} = 2(x^4+6x)^{1/2}$

$$\frac{dy}{dx} = \cancel{2}\left(\tfrac{1}{\cancel{2}}\right)(x^4+6x)^{-1/2}(4x^3+6)$$

$$\boxed{dy = \frac{(4x^3+6)}{\sqrt{x^4+6x}}\,dx}$$

c) $y = \cos(x^3-5x+11)$

$$\frac{dy}{dx} = -\sin(x^3-5x+11)(3x^2-5) \Rightarrow \boxed{dy = -\sin(x^3-5x+11)(3x^2-5)\,dx}$$

d) $y = (x^{10} + \sqrt{\sin(2x)}\,)^2$

$$\frac{dy}{dx} = 2(x^{10}+\sqrt{\sin(2x)})^1\left(10x^9+\frac{\cos(2x)\cancel{(2)}}{\cancel{2}\sqrt{\sin(2x)}}\right)$$

$$\boxed{dy = 2(x^{10}+\sqrt{\sin(2x)})\left(10x^9+\frac{\cos(2x)}{\sqrt{\sin(2x)}}\right)dx}$$

15.5B Differentials

Differentials can be used for approximations.

If $\quad f(x+\Delta x)-f(x) \approx f'(x)\,\Delta x,$

then $\quad f(x+\Delta x) \approx f(x) + f'(x)\,\Delta x.$

EX 2 Find a good approximation for $\sqrt{9.2}$ *without using a calculator.*

let $f(x)=\sqrt{x}$, $x=9$ $\quad \Delta x = dx = 0.2$

$f(x+\Delta x) \approx f(x)+f'(x)\Delta x \quad x+\Delta x = 9+0.2=9.2$

$f(9.2) \approx f(9)+f'(9)(0.2) \quad f'(x)=\frac{1}{2\sqrt{x}}$

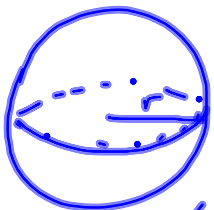
$\approx \sqrt{9}+\frac{1}{2\sqrt{9}}(0.2)$

$= 3+\frac{1}{6}(0.2)$

$= 3+\frac{0.1}{3}$

$= 3+0.0\overline{3} = 3.0\overline{3} \Rightarrow \boxed{\sqrt{9.2} \approx 3.0\overline{3}}$

EX 3 Use differentials to approximate the increase in the surface area of a soap bubble when its radius increases from 4 inches to 4.1 inches.



$SA = f(r) = 4\pi r^2$, $r = 4\text{in}$, $\Delta r = dr = 0.1\text{in}$

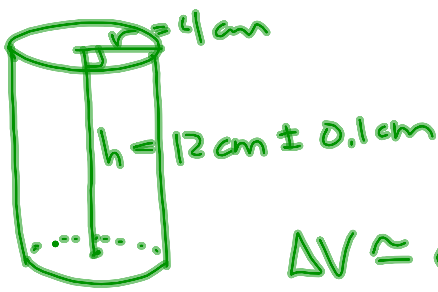
want to approximate Δf.

(Δf = actual change in SA, $f(4.1) - f(4)$)

$\Delta f \simeq df = f'(r)\,dr$, $f'(r) = 8\pi r$

$\Delta f \simeq 8\pi(4)(0.1) = 3.2\pi \simeq \boxed{10.05 \text{ in}^2}$

EX 4 The height of a cylinder is measured as 12 cm with a possible error of ± 0.1 cm. Evaluate the volume of the cylinder with radius 4 cm and give an estimate for the possible error in this value.



$V = \pi r^2 h = 16\pi h$, $dh = \pm 0.1\text{cm}$

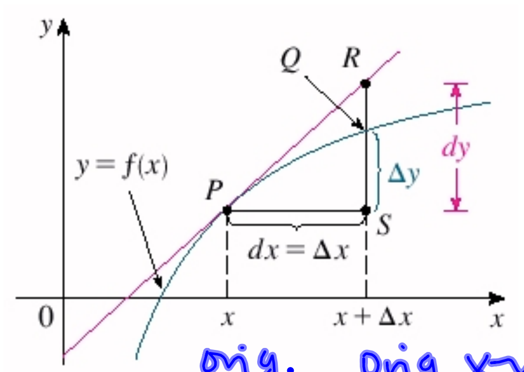
$V(h) = 16\pi h$ $V(12) = 16\pi(12)$

$\Delta V \simeq dV = V'(h)\,dh = 16\pi(\pm 0.1)$

$dV = \pm 1.6\pi \simeq 5.03 \text{ cm}^3$

$V = 192\pi \simeq 603.2 \text{cm}^3$

$\Rightarrow V \simeq 603.2 \pm 5.03 \text{ cm}^3$



if Δx very small,

$dy \approx \Delta y$.

orig.

orig. x-value
+ a little bit of change