Rational Functions

Objectives

Graph rational functions.

Transform rational functions by changing parameters.

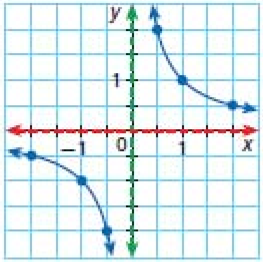
Vocabulary

rational function
discontinuous function
continuous function
hole (in a graph)

Why learn this?

Rational functions can be used to model the cost per person for group events, such as a band trip to a bowl game. (See Exercise 32.)

A **rational function** is a function whose rule can be written as a ratio of two polynomials. The parent rational function is $f(x) = \frac{1}{x}$. Its graph is a *hyperbola*, which has two separate branches. You will learn more about hyperbolas in Chapter 10.



Like logarithmic and exponential functions, rational functions may have asymptotes. The function $f(x) = \frac{1}{x}$ has a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$.

The rational function $f(x) = \frac{1}{x}$ can be transformed by using methods similar to those used to transform other types of functions.

Know it! Note

$$f(x) = \frac{a}{x - h} + k$$

$|a| \longrightarrow$ vertical stretch or compression factor
$a < 0 \longrightarrow$ reflection across the x-axis

$k \longrightarrow$ vertical translation
down for $k < 0$; up for $k > 0$

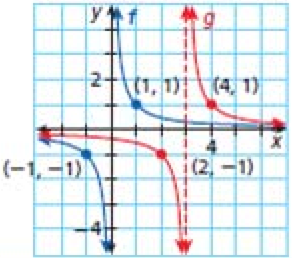
$h \longrightarrow$ horizontal translation
left for $h < 0$; right for $h > 0$

EXAMPLE 1 Transforming Rational Functions

Using the graph of $f(x) = \frac{1}{x}$ as a guide, describe the transformation and graph each function.

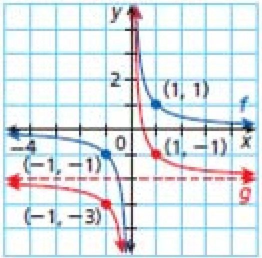
A $g(x) = \dfrac{1}{x - 3}$

Because $h = 3$, translate f 3 units right.



B $g(x) = \dfrac{1}{x} - 2$

Because $k = -2$, translate f 2 units down.



Know it! Note

Rational Functions

For a rational function of the form $f(x) = \frac{a}{x - h} + k$,

- the graph is a hyperbola.
- there is a vertical asymptote at the line $x = h$, and the domain is $\{x \mid x \neq h\}$.
- there is a horizontal asymptote at the line $y = k$, and the range is $\{y \mid y \neq k\}$.

EXAMPLE 2 Determining Properties of Hyperbolas

Identify the asymptotes, domain, and range of the function $g(x) = \frac{1}{x+2} + 4$.

$g(x) = \frac{1}{x - (-2)} + 4$ *$h = -2, k = 4$*

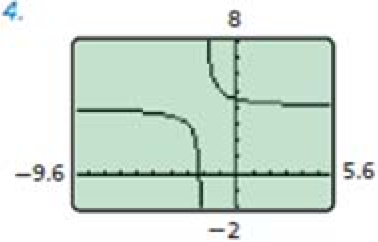
Vertical asymptote: $x = -2$ *The value of h is −2.*

Domain: $\{x \mid x \neq -2\}$

Horizontal asymptote: $y = 4$ *The value of k is 4.*

Range: $\{y \mid y \neq 4\}$

Check Graph the function on a graphing calculator. The graph suggests that the function has asymptotes at $x = -2$ and $y = 4$.



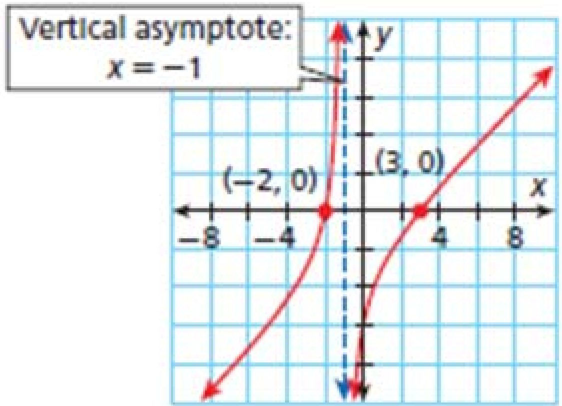
Caution!

Graphing calculators may incorrectly connect two branches of the graph of a rational function with a nearly vertical segment that looks like an asymptote.

A **discontinuous function** is a function whose graph has one or more gaps or breaks. The hyperbola graphed above and many other rational functions are discontinuous functions.

A **continuous function** is a function whose graph has no gaps or breaks. The functions you have studied before this, including linear, quadratic, polynomial, exponential, and logarithmic functions, are continuous functions.

The graphs of some rational functions are not hyperbolas. Consider the rational function $f(x) = \frac{(x-3)(x+2)}{x+1}$ and its graph. The numerator of this function is 0 when $x = 3$ or $x = -2$. Therefore, the function has x-intercepts at -2 and 3. The denominator of this function is 0 when $x = -1$. As a result, the graph of the function has a vertical asymptote at the line $x = -1$.



Know it! Note — Zeros and Vertical Asymptotes — Rational Functions

If $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomial functions in standard form with no common factors other than 1, then the function f has

- zeros at each real value of x for which $p(x) = 0$.
- a vertical asymptote at each real value of x for which $q(x) = 0$.

EXAMPLE 3 Graphing Rational Functions with Vertical Asymptotes

Identify the zeros and vertical asymptotes of $f(x) = \frac{x^2 - 2x - 3}{x - 2}$. Then graph.

Step 1 Find the zeros and vertical asymptotes.

$$f(x) = \frac{(x+1)(x-3)}{x-2}$$ *Factor the numerator.*

Zeros: -1 and 3 — *The numerator is 0 when $x = -1$ or $x = 3$.*

Vertical asymptote: $x = 2$ — *The denominator is 0 when $x = 2$.*

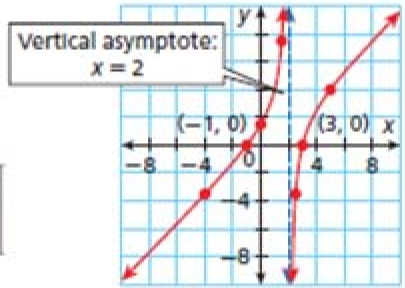
Remember!

For a review of factoring and finding zeros of polynomial functions, see Lessons 6-4 and 6-5.

Step 2 Graph the function.

Plot the zeros and draw the asymptote. Then make a table of values to fill in missing points.

x	−4	−1	0	1.5	2.5	3	5
y	−3.5	0	1.5	7.5	−3.5	0	4



Some rational functions, including those whose graphs are hyperbolas, have a horizontal asymptote. The existence and location of a horizontal asymptote depends on the degrees of the polynomials that make up the rational function.

Note that the graph of a rational function can sometimes cross a horizontal asymptote. However, the graph will approach the asymptote when $|x|$ is large.

Know it! Note

Horizontal Asymptotes — Rational Functions

Let $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomial functions in standard form with no common factors other than 1. The graph of f has at most one horizontal asymptote.

- If degree of $p >$ degree of q, there is no horizontal asymptote.
- If degree of $p <$ degree of q, the horizontal asymptote is the line $y = 0$.
- If degree of $p =$ degree of q, the horizontal asymptote is the line

$$y = \frac{\text{leading coefficient of } p}{\text{leading coefficient of } q}.$$

EXAMPLE 4

Graphing Rational Functions with Vertical and Horizontal Asymptotes

Identify the zeros and asymptotes of each function. Then graph.

A $f(x) = \dfrac{x^2 + x - 6}{x}$

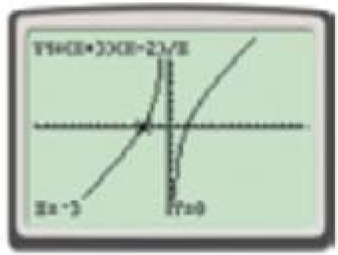
$f(x) = \dfrac{(x+3)(x-2)}{x}$ — *Factor the numerator.*

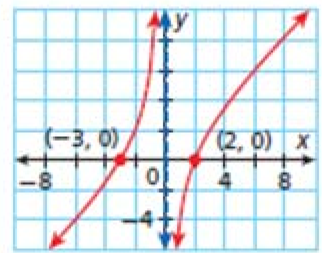
Zeros: -3 and 2 — *The numerator is 0 when $x = -3$ or $x = 2$.*

Vertical asymptote: $x = 0$ — *The denominator is 0 when $x = 0$.*

Horizontal asymptote: none — *Degree of p > degree of q*

Graph with a graphing calculator or by using a table of values.



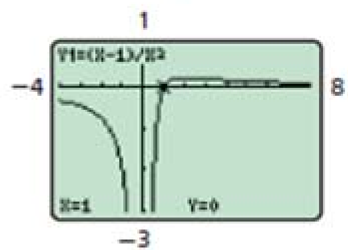


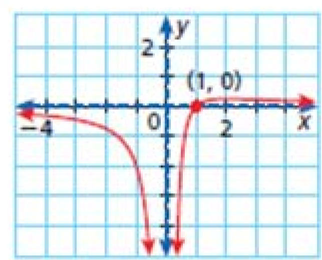
B $f(x) = \dfrac{x - 1}{x^2}$

Zero: 1 — *The numerator is 0 when $x = 1$.*

Vertical asymptote: $x = 0$ — *The denominator is 0 when $x = 0$.*

Horizontal asymptote: $y = 0$ — *Degree of p < degree of q*





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Remember!

Recall from Lesson 6-1 that the leading coefficient of a polynomial is the coefficient of the first term when the polynomial is written in standard form.

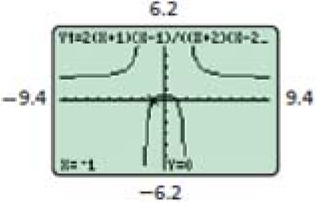
C $f(x) = \dfrac{2x^2 - 2}{x^2 - 4}$

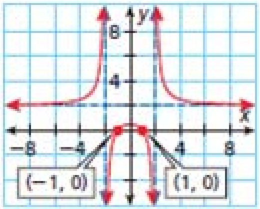
$f(x) = \dfrac{2(x+1)(x-1)}{(x+2)(x-2)}$ *Factor the numerator and denominator.*

Zeros: -1 and 1 *The numerator is 0 when $x = -1$ or $x = 1$.*

Vertical asymptotes: $x = -2$, $x = 2$ *The denominator is 0 when $x = \pm 2$.*

Horizontal asymptote: $y = 2$ *The horizontal asymptote is $y = \frac{\text{leading coefficient of } p}{\text{leading coefficient of } q} = \frac{2}{1} = 2$.*





In some cases, both the numerator and the denominator of a rational function will equal 0 for a particular value of x. As a result, the function will be undefined at this x-value. If this is the case, the graph of the function may have a *hole*. A **hole** is an omitted point in a graph.

Know it! Note

Holes in Graphs | **Rational Functions**

If a rational function has the same factor $x - b$ in both the numerator and the denominator, then there is a hole in the graph at the point where $x = b$, unless the line $x = b$ is a vertical asymptote.

EXAMPLE 5 Graphing Rational Functions with Holes

Identify holes in the graph of $f(x) = \frac{x^2 - 4}{x + 2}$. Then graph.

$$f(x) = \frac{(x-2)(x+2)}{(x+2)}$$

Factor the numerator.

There is a hole in the graph at $x = -2$.

The expression $x + 2$ is a factor of both the numerator and the denominator.

For $x \neq -2$, $f(x) = \frac{(x-2)\cancel{(x+2)}}{\cancel{(x+2)}} = x - 2$

Divide out common factors.

The graph of f is the same as the graph of $y = x - 2$, except for the hole at $x = -2$. On the graph, indicate the hole with an open circle. The domain of f is $\{x \mid x \neq -2\}$.