2.7 Graphs of Rational Functions

The Graph of a Rational Function / Slant Asymptotes / Application

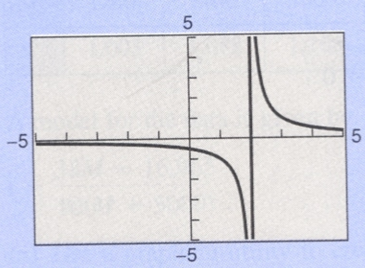
The Graph of a Rational Function

Analyzing Graphs of Rational Functions

Let $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials with no common factors.

1. The y-intercept (if any) is the value of $f(0)$.
2. The x-intercepts (if any) are the zeros of the numerator—that is, the solutions of the equation $p(x) = 0$.
3. The vertical asymptotes (if any) are the zeros of the denominator—that is, the solutions of the equation $q(x) = 0$.
4. The horizontal asymptote (if any) is the value that $f(x)$ approaches as x increases or decreases without bound.
5. Determining the behavior of the graph *between* and *beyond* each x-intercept and vertical asymptote is crucial for describing the complete graph of a rational function.

Note Testing for symmetry can be useful, especially for simple rational functions. For example, the graph of $f(x) = 1/x$ is symmetrical with respect to the origin, and the graph of $g(x) = 1/x^2$ is symmetrical with respect to the y-axis.

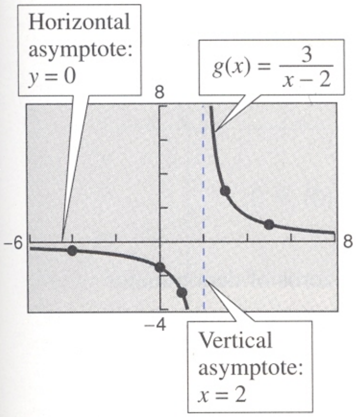


Graphing utilities have difficulty sketching graphs of rational functions that have vertical asymptotes. Often, the utility will connect parts of the graph that are not supposed to be connected. For instance, the screen at the left shows the graph of

$$f(x) = \frac{1}{x-2}.$$

Notice that the graph should consist of two *unconnected* portions—one to the left of $x = 2$ and the other to the right of $x = 2$. To eliminate this problem, you can try changing the *mode* of the graphing utility to *dot mode.* The problem with this is that the graph is then represented as a collection of dots rather than as a smooth curve.

Figure 2.42



EXAMPLE 1 **Analyzing a Rational Function**

Analyze the function $g(x) = \dfrac{3}{x-2}$.

Solution

y-Intercept: $\left(0, -\frac{3}{2}\right)$, because $g(0) = -\frac{3}{2}$.

x-Intercept: None, because $3 \neq 0$.

Vertical Asymptote: $x = 2$, zero of denominator

Horizontal Asymptote: $y = 0$, degree of $p(x) <$ degree of $q(x)$

Additional Points:

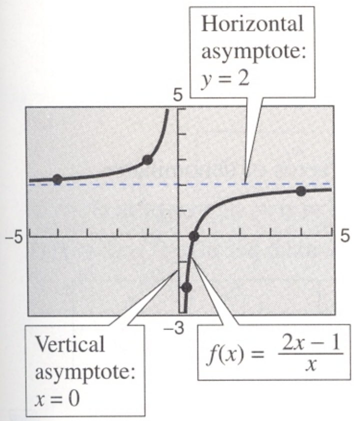
x	-4	1	3	5
$g(x)$	-0.5	-3	3	1

By plotting the intercepts, asymptotes, and a few additional points, you can obtain the graph shown in Figure 2.42. Confirm this with a graphing utility.

Note The graph of g in Example 1 is a vertical stretch and a right shift of the graph of $f(x) = 1/x$ because

$$g(x) = \frac{3}{x-2} = 3\left(\frac{1}{x-2}\right) = 3f(x-2).$$

Figure 2.43



EXAMPLE 2 **Analyzing a Rational Function**

Analyze the function $f(x) = \dfrac{2x-1}{x}$.

Solution

y-Intercept: None, because $x = 0$ is not in the domain.

x-Intercept: $\left(\frac{1}{2}, 0\right)$, because $2x - 1 = 0$.

Vertical Asymptote: $x = 0$, zero of denominator

Horizontal Asymptote: $y = 2$, degree of $p(x) =$ degree of $q(x)$

Additional Points:

x	-4	-1	$\frac{1}{4}$	4
$f(x)$	2.25	3	-2	1.75

By plotting the intercepts, asymptotes, and a few additional points, you can obtain the graph shown in Figure 2.43. Confirm this with a graphing utility.

EXAMPLE 3 Analyzing a Rational Function

Analyze the function $f(x) = \dfrac{x}{x^2 - x - 2}$.

Solution

By factoring the denominator, you have

$$f(x) = \frac{x}{x^2 - x - 2} = \frac{x}{(x+1)(x-2)}.$$

y-Intercept: (0, 0), because $f(0) = 0$.

x-Intercept: (0, 0)

Vertical Asymptotes: $x = -1$, $x = 2$, zeros of denominator

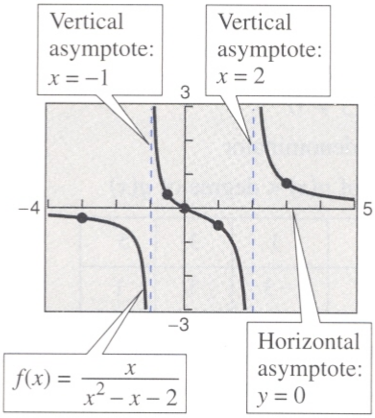
Horizontal Asymptote: $y = 0$, degree of $p(x) <$ degree of $q(x)$

Additional Points:

x	-3	-0.5	1	3
$f(x)$	-0.3	0.4	-0.5	0.75

The graph is shown in Figure 2.44.

Figure 2.44



EXAMPLE 4 Analyzing a Rational Function

Analyze the function $f(x) = \dfrac{2(x^2 - 9)}{x^2 - 4}$.

Solution

By factoring the numerator and denominator, you have

$$f(x) = \frac{2(x^2 - 9)}{x^2 - 4} = \frac{2(x-3)(x+3)}{(x-2)(x+2)}.$$

y-Intercept: $\left(0, \frac{9}{2}\right)$, because $f(0) = \frac{9}{2}$.

x-Intercepts: $(-3, 0)$, $(3, 0)$

Vertical Asymptotes: $x = -2$, $x = 2$, zeros of denominator

Horizontal Asymptote: $y = 2$, degree of $p(x) =$ degree of $q(x)$

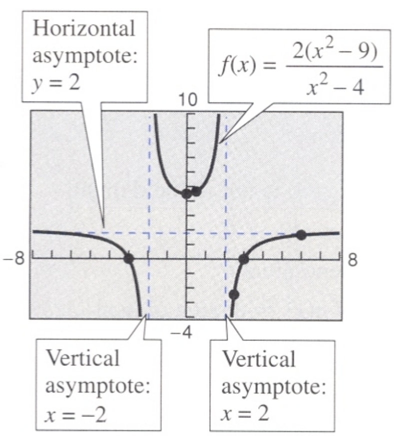
Symmetry: With respect to y-axis, because $f(-x) = f(x)$.

Additional Points:

x	0.5	2.5	6
$f(x)$	4.67	-2.44	1.69

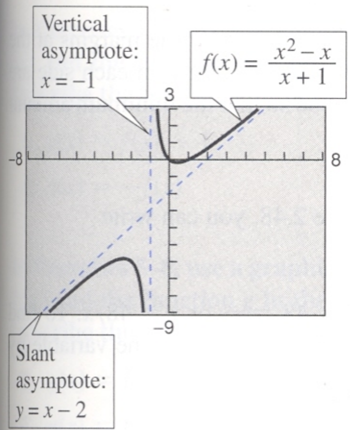
The graph is shown in Figure 2.45.

Figure 2.45



Slant Asymptotes

Figure 2.46



If the degree of the numerator of a rational function is exactly *one more* than the degree of its denominator, the graph of the function has a **slant asymptote.** For example, the graph of

$$f(x) = \frac{x^2 - x}{x + 1}$$

has a slant asymptote, as shown in Figure 2.46. To find the equation of a slant asymptote, use long division. For instance, by dividing $x + 1$ into $x^2 - x$, you have

$$f(x) = \frac{x^2 - x}{x + 1} = \underbrace{x - 2}_{} + \frac{2}{x + 1}. \qquad \frac{2}{x+1} \rightarrow 0 \text{ as } x \rightarrow \infty$$

Slant asymptote ($y = x - 2$)

In Figure 2.46, notice that the graph of f approaches the line $y = x - 2$ as x moves to the right or left.

EXAMPLE 5 A Rational Function with a Slant Asymptote

Graph the function $f(x) = \dfrac{x^2 - x - 2}{x - 1}$.

Solution

First write $f(x)$ in two different ways. Factoring the numerator

$$f(x) = \frac{x^2 - x - 2}{x - 1} = \frac{(x - 2)(x + 1)}{x - 1}$$

allows you to recognize the x-intercepts, and long division

$$f(x) = \frac{x^2 - x - 2}{x - 1} = x - \frac{2}{x - 1} \qquad \frac{2}{x-1} \rightarrow 0 \text{ as } x \rightarrow \infty$$

allows you to recognize that the line $y = x$ is a slant asymptote of the graph.

y-Intercept: $(0, 2)$, because $f(0) = 2$.

x-Intercepts: $(-1, 0)$, $(2, 0)$

Vertical Asymptote: $x = 1$

Slant Asymptote: $y = x$

Additional Points:

x	-2	0.5	1.5	3
$f(x)$	-1.33	4.5	-2.5	2

The graph is shown in Figure 2.47.

Figure 2.47

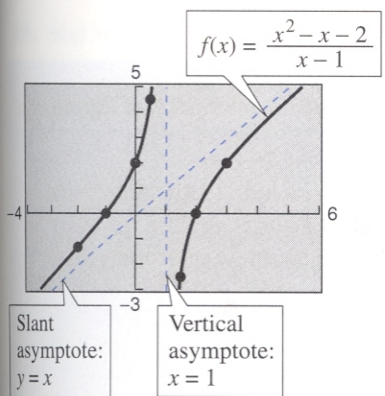


Figure 2.48

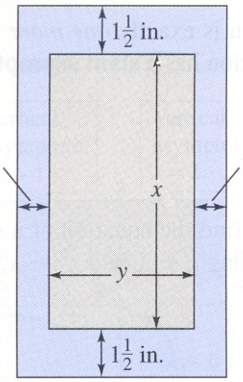
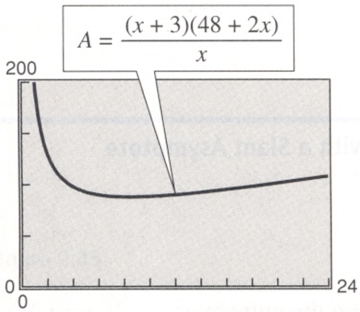


Figure 2.49



Application

Real Life

EXAMPLE 3 **Finding a Minimum Area**

A rectangular page is to contain 48 square inches of print. The margins at the top and bottom of the page are each $1\frac{1}{2}$ inches. The margins on each side are 1 inch. What should the dimensions of the page be so that the minimum amount of paper is used?

Solution

Let A be the area to be minimized. From Figure 2.48, you can write

$$A = (x + 3)(y + 2).$$

The printed area inside the margins is given by $48 = xy$ or $y = 48/x$. To find the minimum area, rewrite the equation for A in terms of just one variable by substituting $48/x$ for y.

$$A = (x + 3)\left(\frac{48}{x} + 2\right) = \frac{(x + 3)(48 + 2x)}{x}, \quad x > 0$$

The graph of this rational function is shown in Figure 2.49. Because x represents the height of the printed area, you need consider only the portion of the graph for which x is positive. Using the zoom and trace features of a graphing utility, you can approximate the minimum value of A to occur when $x \approx 8.5$ inches. The corresponding value of y is $48/8.5 \approx 5.6$ inches. Thus, the dimensions should be

$$x + 3 \approx 11.5 \text{ inches} \quad \text{by} \quad y + 2 \approx 7.6 \text{ inches}.$$

If you go on to take a course in calculus, you will learn a technique for finding the exact value of x that produces a minimum area. In this case, that value is $x = 6\sqrt{2} \approx 8.485$.

Group Activity *Asymptotes of Graphs of Rational Functions*

Discuss whether or not it is possible for the graph of a rational function to cross its horizontal asymptote or its slant asymptote. Use the graphs of the following functions to investigate these questions. What can you conclude?

$$f(x) = \frac{x}{x^2 + 1} \quad \text{and} \quad g(x) = \frac{x^3}{x^2 + 1}$$