2.6 Rational Functions and Asymptotes

Introduction to Rational Functions / Horizontal and Vertical Asymptotes / Applications

Introduction to Rational Functions

A **rational function** can be written in the form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials and $q(x)$ is not the zero polynomial. In this section we assume that $p(x)$ and $q(x)$ have no common factors.

In general, the *domain* of a rational function of x includes all real numbers except x-values that make the denominator zero. Much of our discussion of rational functions will focus on their graphical behavior near these x-values.

Library of Functions

A rational function $f(x)$ is the quotient of two polynomials, $f(x) = p(x)/q(x)$. A rational function is not defined at points x for which the denominator $q(x) = 0$. Near these points, the graph of the rational function may increase or decrease without bound.

EXAMPLE 1 **Finding the Domain of a Rational Function**

Find the domain of

$$f(x) = \frac{1}{x}$$

and discuss the behavior of f near any excluded x-values.

Solution

Because the denominator is zero when $x = 0$, the domain of f is all real numbers except $x = 0$. To determine the behavior of f near this excluded value, evaluate $f(x)$ to the left and right of $x = 0$, as indicated in the following two tables.

x	-1	-0.5	-0.1	-0.01	-0.001	$\to 0$
$f(x)$	-1	-2	-10	-100	-1000	$\to -\infty$

x	$0 \leftarrow$	0.001	0.01	0.1	0.5	1
$f(x)$	$\infty \leftarrow$	1000	100	10	2	1

Note that as x approaches 0 *from the left*, $f(x)$ decreases without bound, whereas as x approaches 0 *from the right*, $f(x)$ increases without bound. The graph of f is shown in Figure 2.34.

Figure 2.34

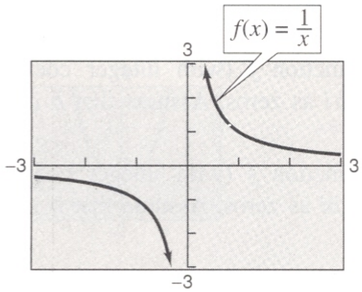
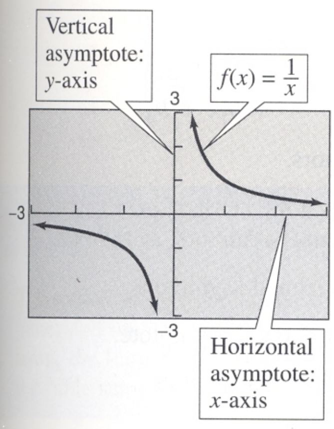


Figure 2.35



Horizontal and Vertical Asymptotes

In Example 1, the behavior of $f(x) = 1/x$ near $x = 0$ is denoted as follows.

$f(x) \to -\infty$ as $x \to 0^-$ — $f(x)$ decreases without bound as x approaches 0 from the left.

$f(x) \to \infty$ as $x \to 0^+$ — $f(x)$ increases without bound as x approaches 0 from the right.

The line $x = 0$ is a **vertical asymptote** of the graph of f, as shown in Figure 2.35. The graph of f also has a **horizontal asymptote**—the line $y = 0$. This means that the values of $f(x) = 1/x$ approach zero as x increases or decreases without bound.

$f(x) \to 0$ as $x \to -\infty$ — $f(x)$ approaches 0 as x decreases without bound.

$f(x) \to 0$ as $x \to \infty$ — $f(x)$ approaches 0 as x increases without bound.

Definition of Vertical and Horizontal Asymptotes

1. The line $x = a$ is a **vertical asymptote** of the graph of f if

$$f(x) \to \infty \quad \text{or} \quad f(x) \to -\infty$$

as $x \to a$, either from the right or from the left.

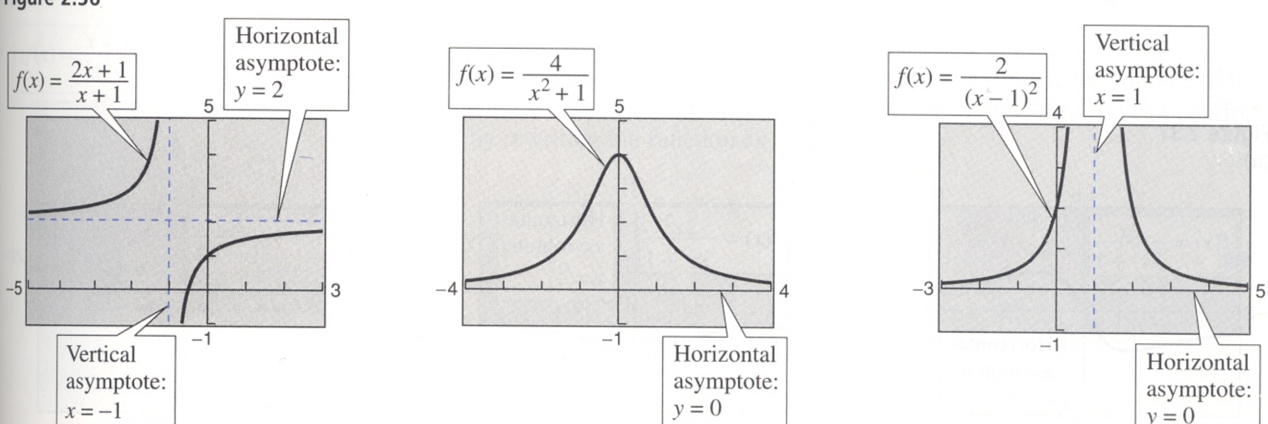
2. The line $y = b$ is a **horizontal asymptote** of the graph of f if

$$f(x) \to b$$

as $x \to \infty$ or $x \to -\infty$.

Eventually (as $x \to \infty$ or $x \to -\infty$), the distance between the horizontal asymptote and the points on the graph must approach zero. Figure 2.36 shows the horizontal and vertical asymptotes of the graphs of three rational functions.

Figure 2.36



Asymptotes of a Rational Function

Let f be the rational function given by

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdot \cdot \cdot + b_1 x + b_0}$$

where $p(x)$ and $q(x)$ have no common factors.

1. The graph of f has vertical asymptotes at the zeros of $q(x)$.
2. The graph of f has at most one horizontal asymptote, as follows.
   - **a.** If $n < m$, the x-axis ($y = 0$) is a horizontal asymptote.
   - **b.** If $n = m$, the line $y = a_n/b_m$ is a horizontal asymptote.
   - **c.** If $n > m$, the graph of f has no horizontal asymptote.

EXPLORATION

Use a graphing utility to compare the graphs of

$$y = \frac{3x^3 - 5x^2 + 4x - 5}{2x^2 - 6x + 7}$$

and

$$y = \frac{3x^3}{2x^2}.$$

First, use a viewing rectangle in which $-5 \le x \le 5$ and $-10 \le y \le 10$, then zoom out. Write a convincing argument that the shape of the graph of a rational function eventually behaves like the graph of $y = a_n x^n / b_m x^m$, where $a_n x^n$ is the leading term of the numerator and $b_m x^m$ is the leading term of the denominator.

You can apply this theorem by comparing the degrees of the numerator and denominator, as illustrated in Figure 2.37.

a. The graph of

$$f(x) = \frac{2x}{3x^2 + 1}$$ See Figure 2.37(a).

has the x-axis as a horizontal asymptote.

b. The graph of

$$f(x) = \frac{2x^2}{3x^2 + 1}$$ See Figure 2.37(b).

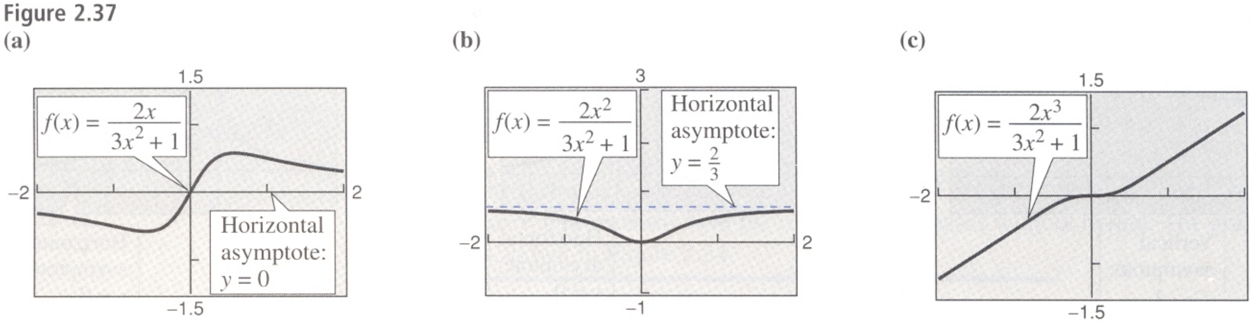
has the line $y = \frac{2}{3}$ as a horizontal asymptote.

c. The graph of

$$f(x) = \frac{2x^3}{3x^2 + 1}$$ See Figure 2.37(c).

has no horizontal asymptote.

Figure 2.37



EXAMPLE 2 A Calculator Experiment

Find the horizontal asymptote of the graph of $f(x) = \dfrac{3x^3 + 7x^2 + 2}{-4x^3 + 5}$.

Solution

Because the degree of the numerator and denominator are the same, the horizontal asymptote is given by the ratio of the leading coefficients.

$$y = \frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{3}{-4} = -\frac{3}{4}$$

The tables at the right can be generated on a *TI-82* or *TI-83* using the Indpnt:Ask feature in the table setup menu.

The following tables show how the values of $f(x)$ become closer and closer to $-\frac{3}{4}$ as x becomes increasingly large or small.

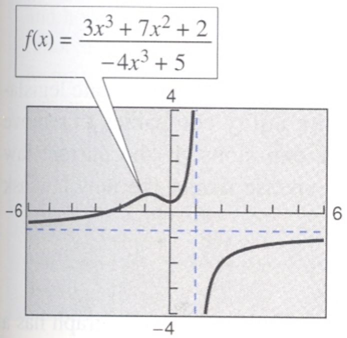
x Decreases Without Bound

X	Y1	
−1	0.66667	
−10	−0.5738	
−100	−0.7325	
−1,000	−0.7482	
−10,000	−0.7498	
X =		

x Increases Without Bound

X	Y1	
1	12	
10	−0.9267	
100	−0.7675	
1,000	−0.7518	
10,000	−0.7502	
X =		

Figure 2.38



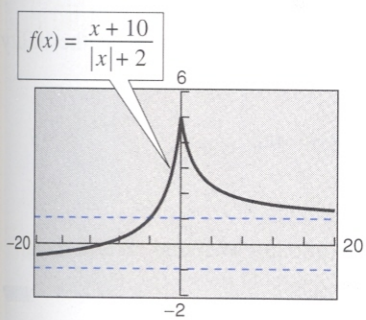
The graph is shown in Figure 2.38. Try using a graphing utility to confirm the graph.

EXAMPLE 3 A Graph with Two Horizontal Asymptotes

A function that is not rational can have two horizontal asymptotes—one to the left and one to the right. For instance, the graph of

$$f(x) = \frac{x + 10}{|x| + 2}$$

Figure 2.39



is shown in Figure 2.39. It has the line $y = -1$ as a horizontal asymptote to the left and the line $y = 1$ as a horizontal asymptote to the right. You can confirm this by rewriting the function as follows.

$$f(x) = \begin{cases} \dfrac{x + 10}{-x + 2}, & x \le 0 \\[2ex] \dfrac{x + 10}{x + 2}, & x > 0 \end{cases}$$

Applications

There are many examples of asymptotic behavior in real life. For instance, Example 4 shows how a vertical asymptote can be used to analyze the cost of removing pollutants from smokestack emissions.

EXAMPLE 4 **Cost-Benefit Model** *Real Life*

A utility company burns coal to generate electricity. The cost of removing a certain *percent* of the pollutants from the smokestack emissions is typically not a linear function. That is, if it costs C dollars to remove 25% of the pollutants, it would cost more than $2C$ dollars to remove 50% of the pollutants. As the percent of removed pollutants approaches 100%, the cost tends to become prohibitive. Suppose that the cost C of removing $p\%$ of the smokestack pollutants is given by

$$C = \frac{80{,}000p}{100 - p}, \qquad 0 \le p < 100.$$

Sketch the graph of this function. Suppose you are a member of a state legislature that is considering a law that would require utility companies to remove 90% of the pollutants from their smokestack emissions. If the current law requires 85% removal, how much additional expense would the new law ask the utility company to spend?

Solution

The graph of this function is shown in Figure 2.40. Note that the graph has a vertical asymptote at $p = 100$. Because the current law requires 85% removal, the current cost to the utility company is

$$C = \frac{80{,}000(85)}{100 - 85} \qquad \text{Evaluate } C \text{ at } p = 85.$$

$$\approx \$453{,}333.$$

If the new law increased the percent removal to 90%, the cost to the utility company would be

$$C = \frac{80{,}000(90)}{100 - 90} \qquad \text{Evaluate } C \text{ at } p = 90.$$

$$= \$720{,}000.$$

Thus, the new law would require the utility company to spend an additional

$$720{,}000 - 453{,}333 = \$266{,}667.$$

Figure 2.40

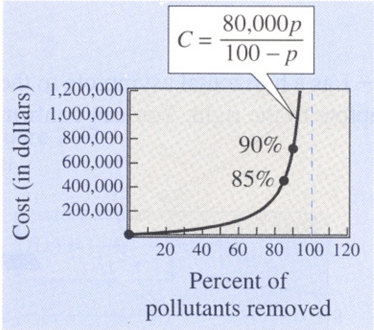
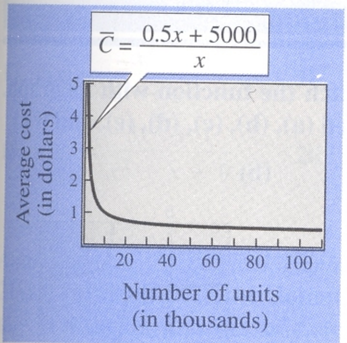


Figure 2.41



EXAMPLE 5 **Average Cost of Producing a Product** *Real Life*

A business has a cost function of $C = 0.5x + 5000$, where C is measured in dollars and x is the number of units produced. The *average cost per unit* is

$$\overline{C} = \frac{C}{x} = \frac{0.5x + 5000}{x}.$$

Find the average cost per unit when $x = 1000$, 5000, 10,000, and 100,000. What is the horizontal asymptote for this function, and what does it represent?

Solution

When $x = 1000$, $\overline{C} = \dfrac{0.5(1000) + 5000}{1000} = \$5.50.$

When $x = 5000$, $\overline{C} = \dfrac{0.5(5000) + 5000}{5000} = \$1.50.$

When $x = 10{,}000$, $\overline{C} = \dfrac{0.5(10{,}000) + 5000}{10{,}000} = \$1.00.$

When $x = 100{,}000$, $\overline{C} = \dfrac{0.5(100{,}000) + 5000}{100{,}000} = \$0.55.$

As shown in Figure 2.41, the horizontal asymptote is given by the line $\overline{C} = 0.50$. This line represents the least possible unit cost for the product.

The table feature of a graphing utility can be used to find vertical and horizontal asymptotes of rational functions. Use the table feature to find any horizontal or vertical asymptotes of

$$f(x) = \frac{2}{x + 1}.$$

Write a statement explaining how you found the asymptote(s) using the table.

Note that this example points out one of the major problems of a small business. That is, it is difficult to have competitively low prices when the production level is low.

Group Activity

Common Factors in the Numerator and Denominator

When sketching the graph of a rational function, be sure that the rational function has no factor that is common to its numerator and denominator. To see why, consider the function given by

$$f(x) = \frac{x(x - 1)}{x}$$

which has a common factor of x in the numerator and denominator. Sketch the graph of this function. Does it have a vertical asymptote at $x = 0$?