Definition of Continuity

1. $f(c)$ must be defined

2. $\lim_{x \to c} f(x)$ Must exist

Additional Info

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$$

3. $\lim_{x \to c} f(x) = f(c)$

Problem-Solving Strategy: Determining Continuity at a Point

1. Check to see if $f(a)$ is defined. If $f(a)$ is undefined, we need go no further. The function is not continuous at a. If $f(a)$ is defined, continue to step 2.
2. Compute $\lim_{x \to a} f(x)$. In some cases, we may need to do this by first computing $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$. If $\lim_{x \to a} f(x)$ does not exist (that is, it is not a real number), then the function is not continuous at a and the problem is solved. If $\lim_{x \to a} f(x)$ exists, then continue to step 3.
3. Compare $f(a)$ and $\lim_{x \to a} f(x)$. If $\lim_{x \to a} f(x) \neq f(a)$, then the function is not continuous at a. If $\lim_{x \to a} f(x) = f(a)$, then the function is continuous at a.

Types Of Discontinuities

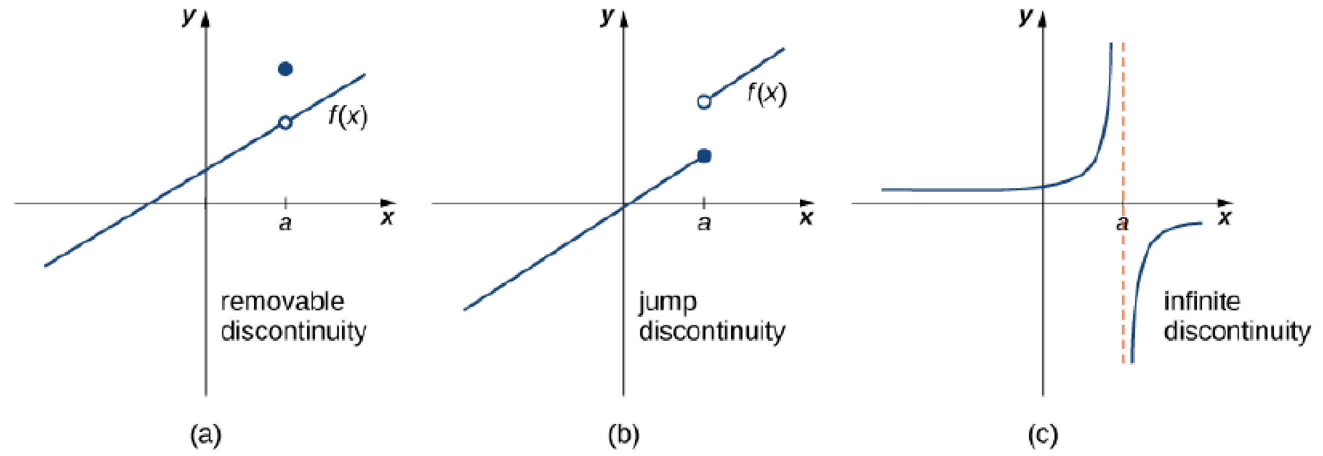


Figure 2.37 Discontinuities are classified as (a) removable, (b) jump, or (c) infinite.

Definition

If $f(x)$ is discontinuous at a, then

1. f has a **removable discontinuity** at a if $\lim_{x \to a} f(x)$ exists. (Note: When we state that $\lim_{x \to a} f(x)$ exists, we mean that $\lim_{x \to a} f(x) = L$, where L is a real number.)
2. f has a **jump discontinuity** at a if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$. (Note: When we state that $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, we mean that both are real-valued and that

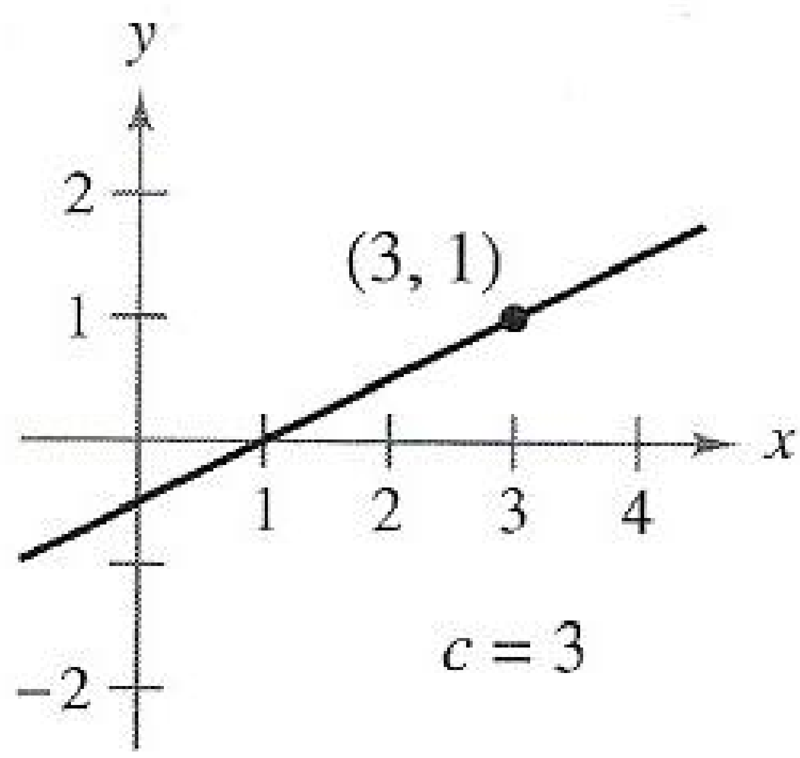
neither take on the values $\pm\infty$.)

3. f has an **infinite discontinuity** at a if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.

In Exercises 1–6, use the graph to determine the limit, and discuss the continuity of the function.

(a) $\lim_{x \to c^+} f(x)$ (b) $\lim_{x \to c^-} f(x)$ (c) $\lim_{x \to c} f(x)$

1.



Solution

a) $\lim_{x \to 3^+} f(x) = 1$

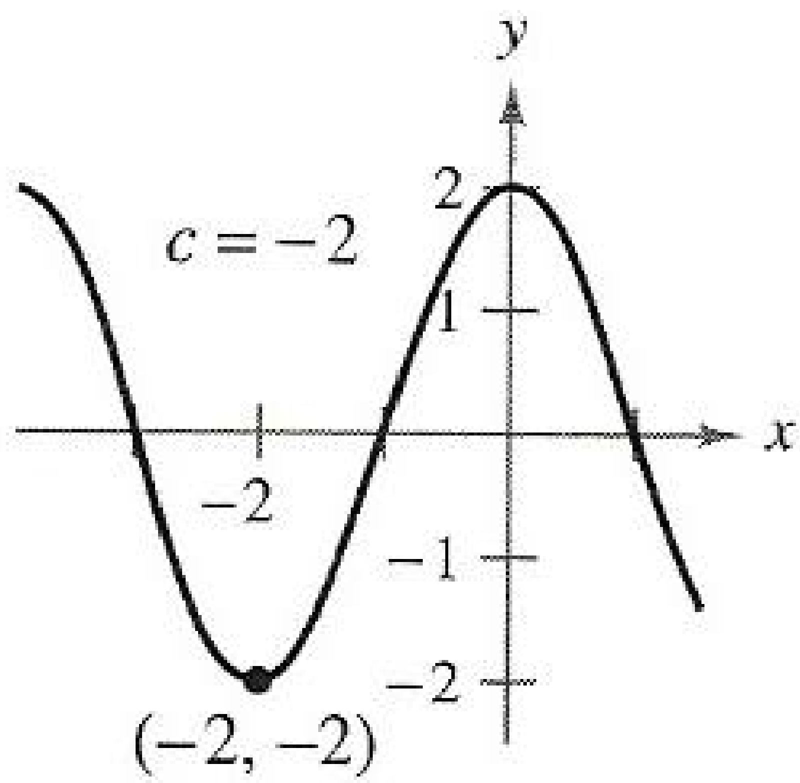
b) $\lim_{x \to 3^-} f(x) = 1$

c) $\lim_{x \to 3} f(x) = 1$

$$\lim_{x \to 3} f(x) = f(3)$$

The function is continuous at $x=3$ because all three criteria of the Definition of Continuity are met.

2.



Solution

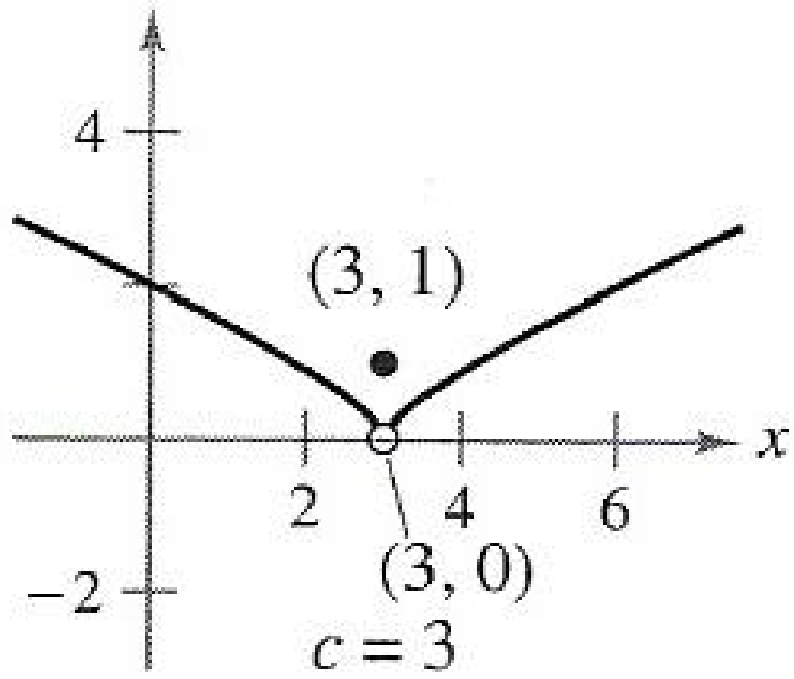
a) $\lim_{x \to -2^+} f(x) = -2$

b) $\lim_{x \to -2^-} f(x) = -2$

c.) $\lim_{x \to -2} f(x) = -2 = f(-2)$

The function is continuous at $x=-2$ because all three criteria of the Definition of Continuity are met.

3.



$f(3)=1$ Defined at $x=3$

(a) $\lim_{x \to 3^+} f(x) = 0$

(b) $\lim_{x \to 3^-} f(x) = 0$

(c) $\lim_{x \to 3} f(x) = 0$ (The limit exists)

Because $\lim_{x \to 3^-} f(x) = 0 = \lim_{x \to 3^+} f(x)$

$\lim_{x \to 3} f(x) \neq f(3)$

$\therefore$ $f(x)$ is NOT continuous at $x=3$

because $\lim_{x \to 3} f(x) \neq f(3)$

$f(3) = 1$

$\lim_{x \to 3} f(x) = 0$

This a removable discontinuity.

$\boxed{\text{Definition of Continuity}}$

Definition of Continuity

1. $f(c)$ must be defined

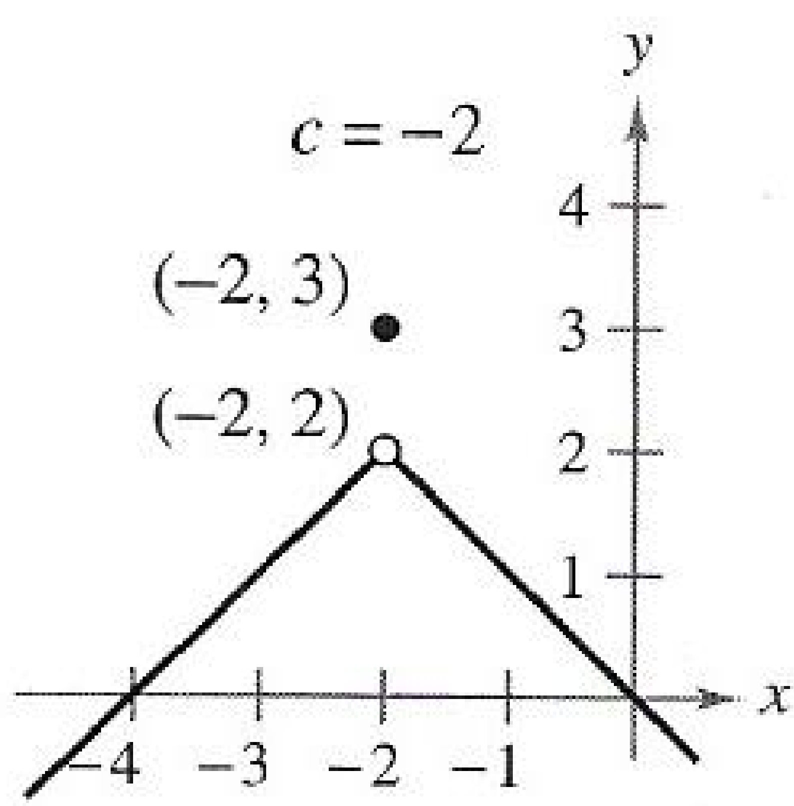
2. $\lim_{x \to c} f(x)$ Must exist

Additional Info

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$$

3. $\lim_{x \to c} f(x) = f(c)$

4.



Solution

we are applying the definition of Continuity

1. $f(-2)=3$ The function is defined at $x=-2$

2. $\lim_{x\to -2^+} f(x)=2$ & $\lim_{x\to -2^-} f(x)=2$

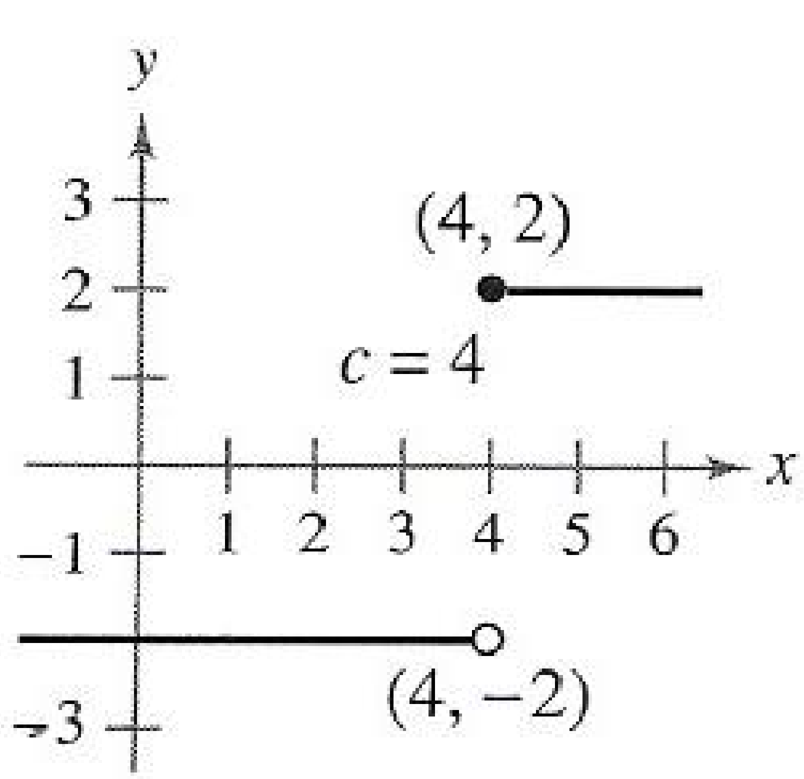
$\therefore \lim_{x\to -2} f(x)=2$ [The limit exists]

3. $\lim_{x\to 2} f(x) \neq f(2)$

$\therefore$ $f(x)$ is discontinuous at $x=-2$ by criteria 3.

This a removable discontinuity.

5.



Solution

We will be using the definition of continuity.

1. $f(4)=2$; $f(x)$ is defined at $x=4$

2. $\lim_{x\to 4^+} f(x) = 2$ & $\lim_{x\to 4^-} f(x) = -2$

Since $\lim_{x\to 4^+} f(x) \neq \lim_{x\to 4^-} f(x)$

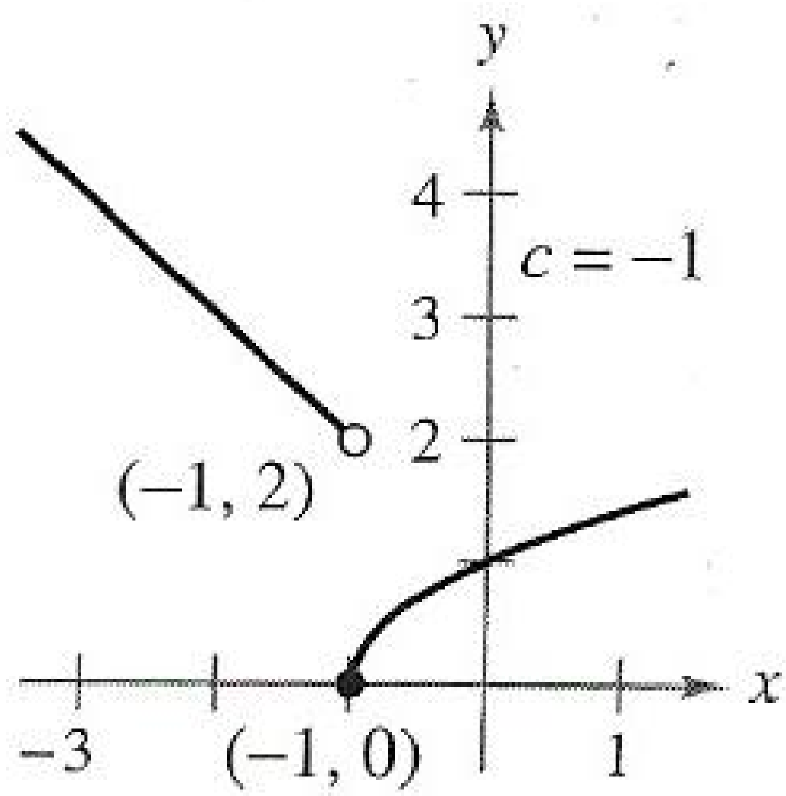
$\lim_{x\to 4} f(x)$ does not exist

$\therefore$ $f(x)$ is discontinuous at $x=4$

This is a jump discontinuity.

Criteria 2 is not fulfilled.

6.



Solution

1. $f(-1) = 0$; $f(x)$ is defined at $x = -1$

2. $\lim_{x \to -1^+} f(x) = 0$ & $\lim_{x \to -1^-} f(x) = 2$

Since

$$\lim_{x \to -1^+} f(x) \neq \lim_{x \to -1^-} f(x),$$

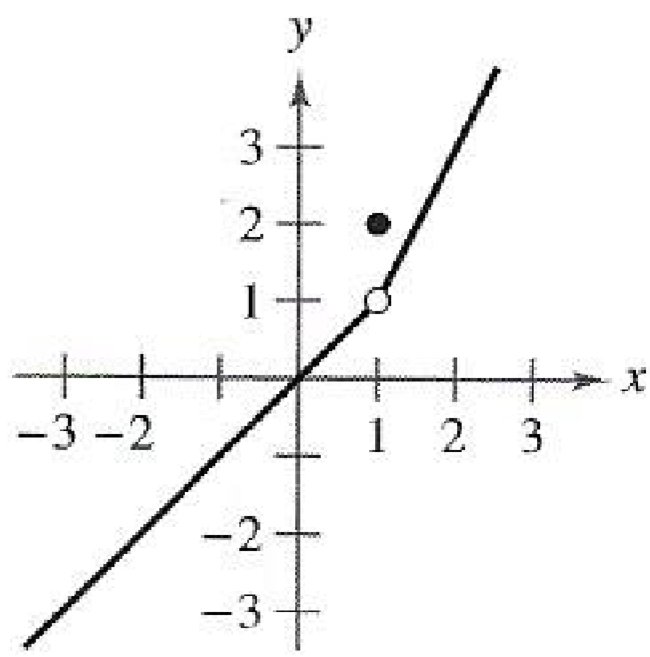
$$\lim_{x \to -1} f(x) \text{ does not exist}$$

Since criteria 2 is not fulfilled, $f(x)$ is discontinuous at $x = -1$.

This is a jump discontinuity.

7.

$$f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$$



Solution

1. $f(1) = 2$; $f(x)$ is defined at $x = 1$.

2. $\lim_{x \to 1^+} f(x) = 1$ & $\lim_{x \to 1^-} f(x) = 1$

Since

$$\lim_{x \to 1^+} f(x) = 1 = \lim_{x \to 1^-} f(x)$$

$\lim_{x \to 1} f(x) = 1$; {The limit exists}

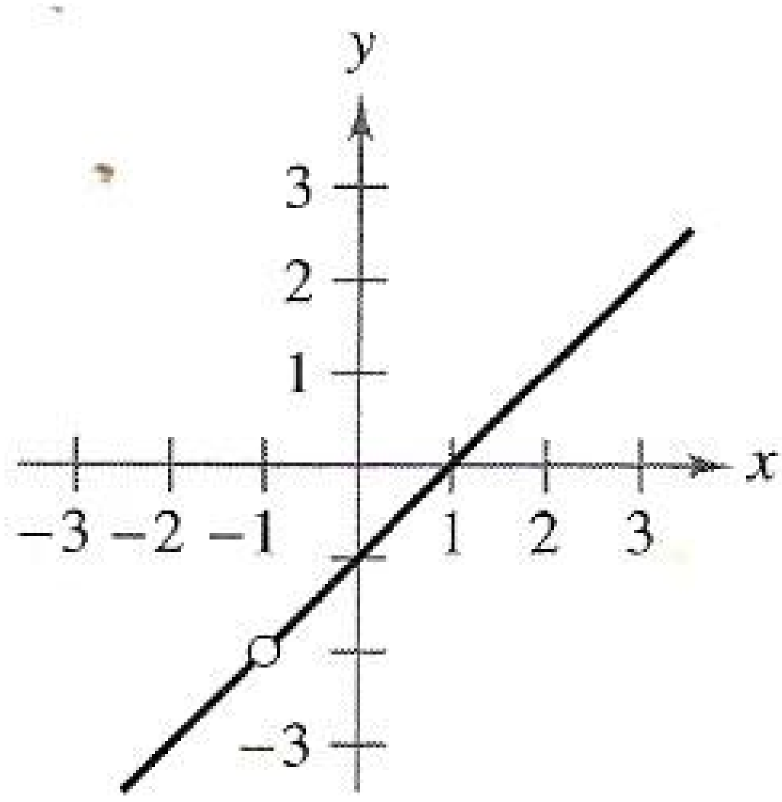
3. $\lim_{x \to 1} f(x) \neq f(1)$

$\therefore$ Since Criteria 3 is NOT fulfilled,

$f(x)$ is discontinuous at $x=1$

8.

$$f(x) = \frac{x^2 - 1}{x + 1}$$



$f(x) = \frac{x^2-1}{x+1}$ For the "ORIGINAL" function,

when $x=-1$, $f(x)$ is undefined or

indeterminate.

Factor $f(x)$ and simplify.

$$f(x) = \frac{x^2-1}{x+1} = \frac{(x+1)(x-1)}{(x+1)} = x-1$$

$x=-1$ is a removable discontinuity

We will now use the definition of continuity to verify that $f(x)$ is discontinuous at $x=1$.

1. $f(x)$ is NOT defined at $x=-1$

2. $\lim\limits_{x\to -1^+} f(x) = -3$ & $\lim\limits_{x\to -1^-} f(x) = -3$

Since $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^-} f(x)$

$\lim\limits_{x\to -1} f(x) = -3$ {The limit exists}

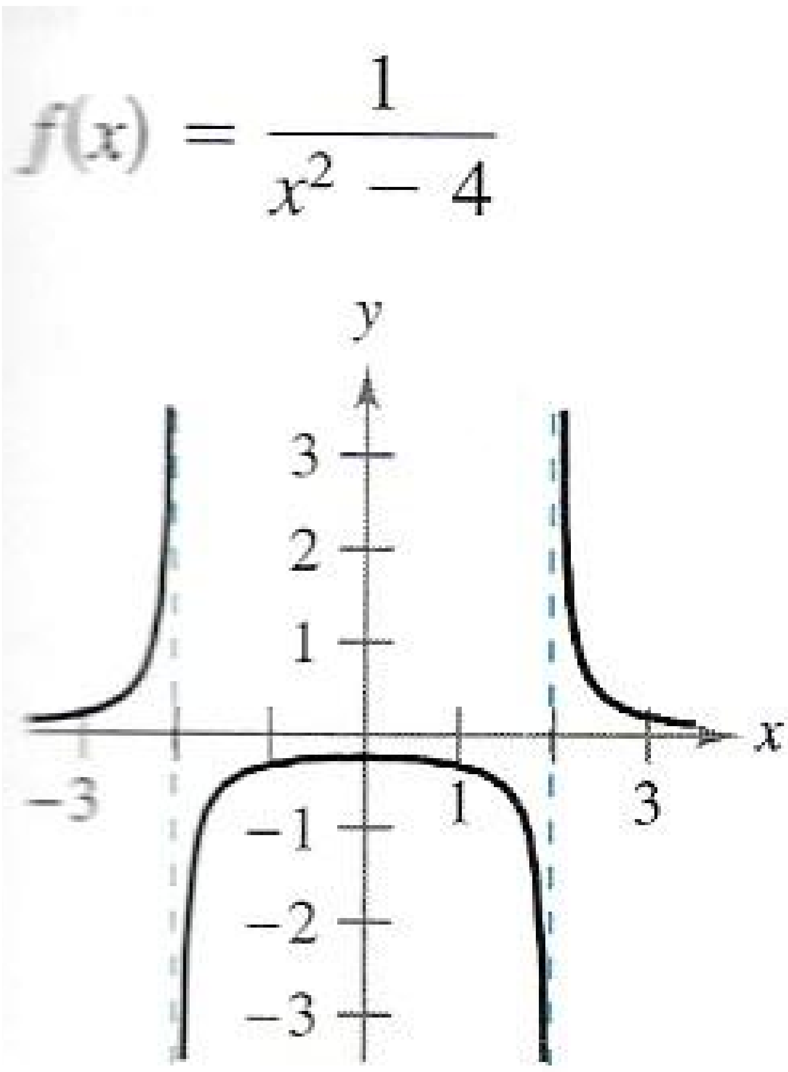
3. Since $f(x)$ is NOT defined at $x=-1$,

criteria 3 failed.

$$\lim_{x\to -1} f(x) \neq f(-1)$$

We have shown by the definition of continuity that $f(x)$ is discontinuous at $x=-1$.

9.



Solution

$$f(x) = \frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)}$$

$x=-2$ & $x=2$ are vertical asymptotes.

$x=-2$ & $x=2$ are infinite discontinuities

$$\lim_{x \to -2^-} f(x) \Rightarrow \infty \quad \& \quad \lim_{x \to -2^+} f(x) \Rightarrow -\infty$$

$$\lim_{x \to 2^-} f(x) \Rightarrow -\infty \quad \& \quad \lim_{x \to 2^+} f(x) \Rightarrow \infty$$

Prelude to 2.5

THEOREM 6: Suppose that a function f is defined on an open interval containing c, except perhaps at c itself. Then $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \text{if and only if} \quad \lim_{x\to c^-} f(x) = L$$

$$\text{and} \quad \lim_{x\to c^+} f(x) = L$$

By the above theorem, a function $f(x)$ is continuous at an INTERIOR POINT of c of an interval in its domain if and only if it is both right-continuous and left-continuous at c

We say that a function is CONTINUOUS OVER A CLOSED INTERVAL $[a,b]$ if it is right-continuous at a, left-continuous at b, and continuous at all interior points of the interval.

This definition applies to the infinite closed intervals $[a,\infty)$ and $(-\infty,b]$ as well, but only one endpoint is involved.

If a function is not continuous at point c of its domain, we say that f is DISCONTINUOUS AT c, and that f has a discontinuity at c.

OBSERVATION: A function $f(x)$ can be continuous, right-continuous, or left-continuous only at a point c for which $f(c)$ is defined.

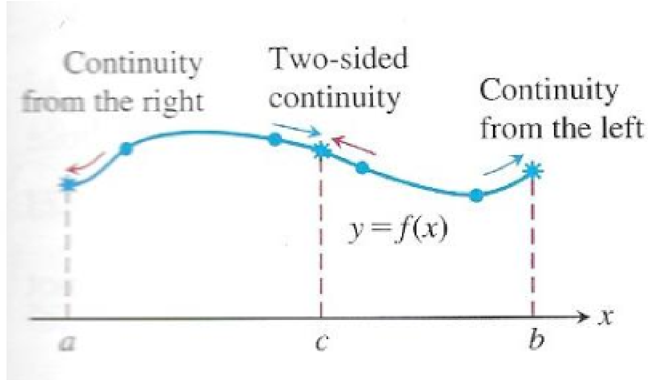
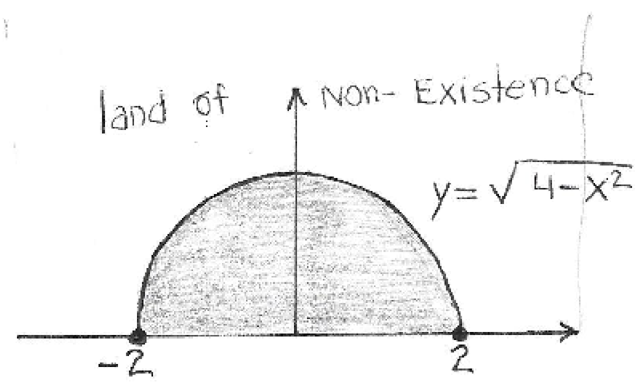


FIGURE 2.36 Continuity at points a, b, and c.



The domain of $y=\sqrt{4-x^2}$ only exists between $-2 \leq x \leq 2$

$\lim_{x\to -2^-} \sqrt{4-x^2}$ Does not exist or is NOT defined because $\sqrt{4-x^2}$ does Not exist when $x<-2$

$\lim_{x\to 2^+} \sqrt{4-x^2}$ Does NOT exist when $x>2$

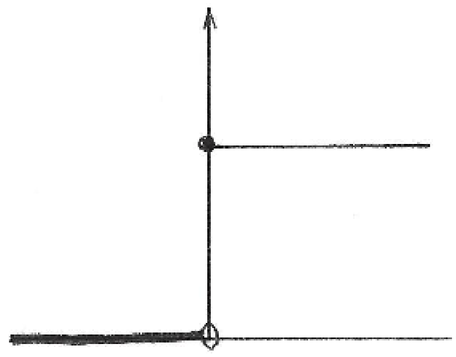
The function $f(x)=\sqrt{4-x^2}$ is continuous over its domain $[-2,2]$

$f(x)=\sqrt{4-x^2}$ is right-continuous at $x=-2$

$$\lim_{x\to -2^+} \sqrt{4-x^2} = 0$$

$f(x)=\sqrt{4-x^2}$ is left-continuous at $x=2$

$$\lim_{x\to 2^-} \sqrt{4-x^2} = 0$$



The unit step function $U(x)$, graphed to the left is right-continuous at $x=0$, but is neither left-continuous nor continuous there.

It has JUMP discontinuity at $x=0$.

$$U(x)=\begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x<0 \end{cases}$$

EXAMPLE 4 The function $y = \lfloor x \rfloor$ introduced in Section 1.1 is graphed in Figure 2.39. It is discontinuous at every integer n, because the left-hand and right-hand limits are not equal as $x \to n$:

$$\lim_{x\to n^-} \lfloor x \rfloor = n - 1 \quad \text{and} \quad \lim_{x\to n^+} \lfloor x \rfloor = n.$$

Since $\lfloor n \rfloor = n$, the greatest integer function is right-continuous at every integer n (but not left-continuous).

The greatest integer function is continuous at every real number other than the integers. For example,

$$\lim_{x\to 1.5} \lfloor x \rfloor = 1 = \lfloor 1.5 \rfloor.$$

In general, if $n - 1 < c < n$, n an integer, then

$$\lim_{x\to c} \lfloor x \rfloor = n - 1 = \lfloor c \rfloor.$$

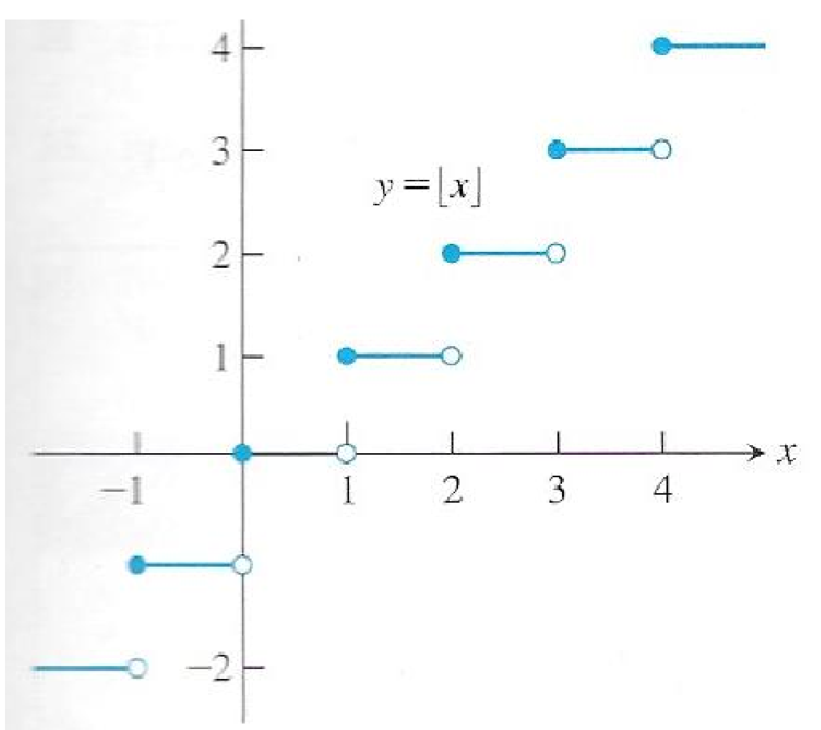
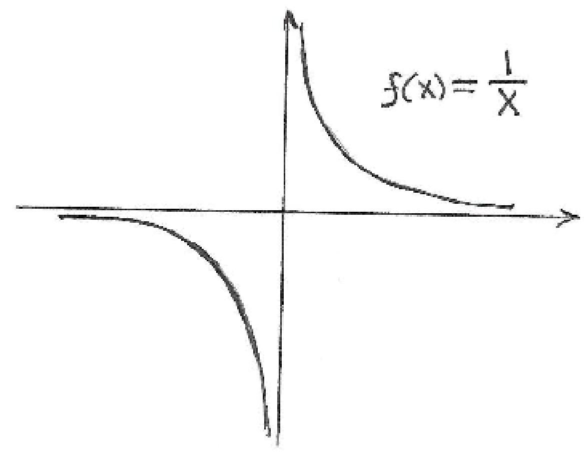


FIGURE 2.39 The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point (Example 4).



$f(x)=\frac{1}{x}$ is discontinuous at $x=0$, but it is continuous over the union of open intervals:

$$(-\infty,0)\cup(0,\infty)$$

$x=0$ is an infinite discontinuity.

Every polynomial $P(x)=a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is continuous because

$$\lim_{x\to c} P(x) = P(c)$$

by Theorem 2, Section 2.2

The function $f(x)=|x|$ is continuous

step#1: $f(0)=0$

$f(x)$ is defined at $x=0$

step#2:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -x = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0$$

$$\lim_{x\to 0^-} f(x) = 0 = \lim_{x\to 0^+} f(x)$$

step#2:

$$\therefore \lim_{x\to 0} f(x) = 0$$

step#3:

$$\lim_{x\to 0} f(x) = 0 = f(0)$$

$f(x) = |x|$ is continuous in $(-\infty, \infty)$.

Inverse Functions & Continuity

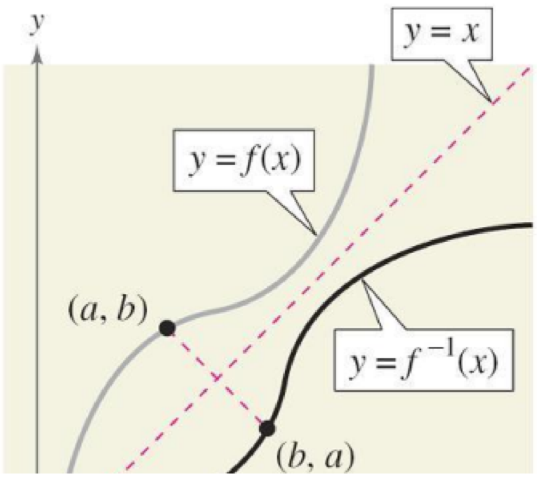
when a continuous function defined on an interval has an inverse, the inverse function is itself a continuous function over its own domain.

This result is suggested by the observation that the graph of $f^{-1}(x)$, being the reflection of the graph of f across the line $y = x$, cannot have any breaks in it when the graph of f has no breaks.

The Graph of an Inverse Function

If the point (a, b) lies on the graph of f, then the point (b, a) must lie on the graph of f^{-1}, and vice versa.

This means that the graph of f^{-1} is a *reflection* of the graph of f in the line $y = x$,



Example 3 – *Finding Inverse Functions Graphically*

Sketch the graphs of the inverse functions $f(x) = 2x - 3$ and $f^{-1}(x) = \frac{1}{2}(x + 3)$ on the same rectangular coordinate system and show that the graphs are reflections of each other in the line $y = x$.

The graphs of f and f^{-1} are shown in Figure 1.95.

It appears that the graphs are reflections of each other in the line $y = x$.

You can further verify this reflective property by testing a few points on each graph.

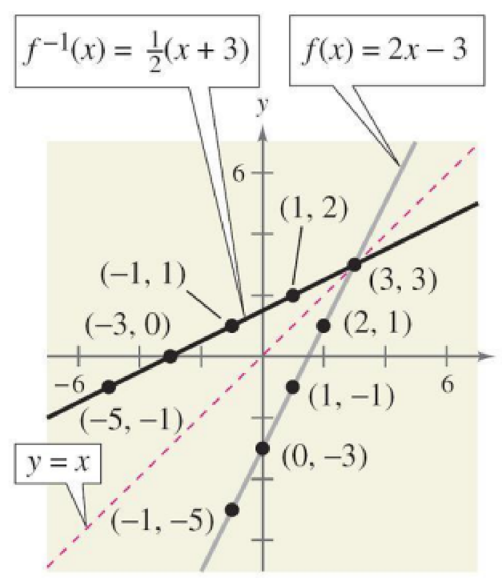


Figure 1.95

Continuity of Compositions of Functions

Functions obtained by composing continuous functions are continuous. If $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is also continuous at $x = c$ (Figure 2.42). In this case, the limit of $g \circ f$ as $x \to c$ is $g(f(c))$.

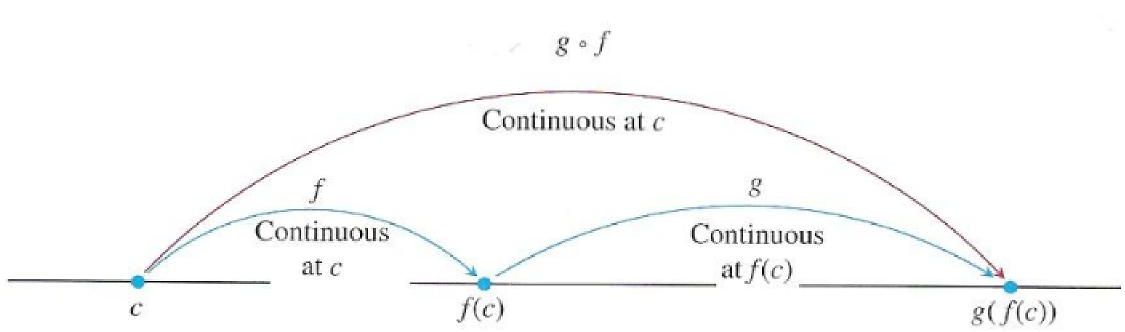


FIGURE 2.42 Compositions of continuous functions are continuous.

> **THEOREM 9—Compositions of Continuous Functions**
> If f is continuous at c and g is continuous at $f(c)$, then the composition $g \circ f$ is continuous at c.

Intuitively, Theorem 9 is reasonable because if x is close to c, then $f(x)$ is close to $f(c)$, and since g is continuous at $f(c)$, it follows that $g(f(x))$ is close to $g(f(c))$.

The continuity of compositions holds for any finite number of compositions of functions. The only requirement is that each function be continuous where it is applied. An outline of a proof of Theorem 9 is given in Exercise 6 in Appendix 4.

show that the following functions are continuous on their natural domains.

(a) $y=\sqrt{x^2-2x-5}$

Let $f(x)=x^2-2x-5$ & $g(x)=\sqrt{x}$

Let us look at the discriminant of $f(x)$

$a=1$, $b=-2$, $c=-5$

$D=b^2-4ac$

$D=(-2)^2-4(1)(-5)$

$=4+20$

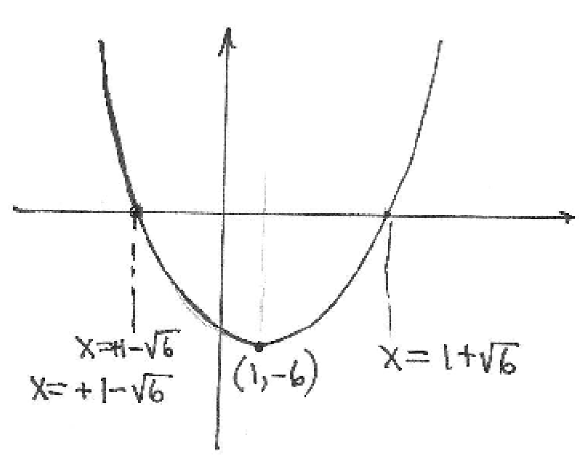
$D=24$

using completing the square, we have

$y=(x-1)^2+(-6)$

$V(1,-6)$

The zeros are $x=1\pm\sqrt{6}$



$f(x)>0$ when

$x<1-\sqrt{6}$ & $x>1+\sqrt{6}$

$g(x)=\sqrt{x}$ is continuous on its natural domain

$(-\infty, 1-\sqrt{6})\cup(1+\sqrt{6}, \infty)$

(b) $y = \dfrac{x^{2/3}}{1+x^4}$

$1+x^4 > 0$ This will always be greater than zero.

$$\lim_{x \to \pm\infty} \frac{x^{2/3}}{1+x^4} = \lim_{x \to \pm\infty} \frac{x^{2/3}}{1+x^4}$$

$$= \lim_{x \to \pm\infty} \frac{\frac{x^{2/3}}{x^4}}{\frac{1}{x^4} + 1}$$

$$= \lim_{x \to \pm\infty} \frac{\frac{1}{x^{10/3}}}{\frac{1}{x^4} + 1} \Rightarrow \frac{0}{1} = 0$$

The quotient is continuous.

THEOREM 10 — Limits of Continuous Functions

If $\lim_{x \to c} f(x) = b$ and g is continuous at the point b, then

$$\lim_{x \to c} g(f(x)) = g(b)$$

EXAMPLE 9 As an application of Theorem 10, we have the following calculations.

(a)
$$\begin{aligned}\lim_{x \to \pi/2} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right) &= \cos\left(\lim_{x \to \pi/2} 2x + \lim_{x \to \pi/2} \sin\left(\frac{3\pi}{2} + x\right)\right) \\ &= \cos\left(\pi + \sin 2\pi\right) = \cos \pi = -1.\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x \to 1} \sin^{-1}\left(\frac{1-x}{1-x^2}\right) &= \sin^{-1}\left(\lim_{x \to 1} \frac{1-x}{1-x^2}\right) && \text{Arcsine is continuous.} \\ &= \sin^{-1}\left(\lim_{x \to 1} \frac{1}{1+x}\right) && \text{Cancel common factor } (1-x). \\ &= \sin^{-1}\frac{1}{2} = \frac{\pi}{6}\end{aligned}$$

(c)
$$\begin{aligned}\lim_{x \to 0} \sqrt{x+1}\, e^{\tan x} &= \lim_{x \to 0} \sqrt{x+1} \cdot \exp\left(\lim_{x \to 0} \tan x\right) && \text{exp is continuous.} \\ &= 1 \cdot e^0 = 1\end{aligned}$$

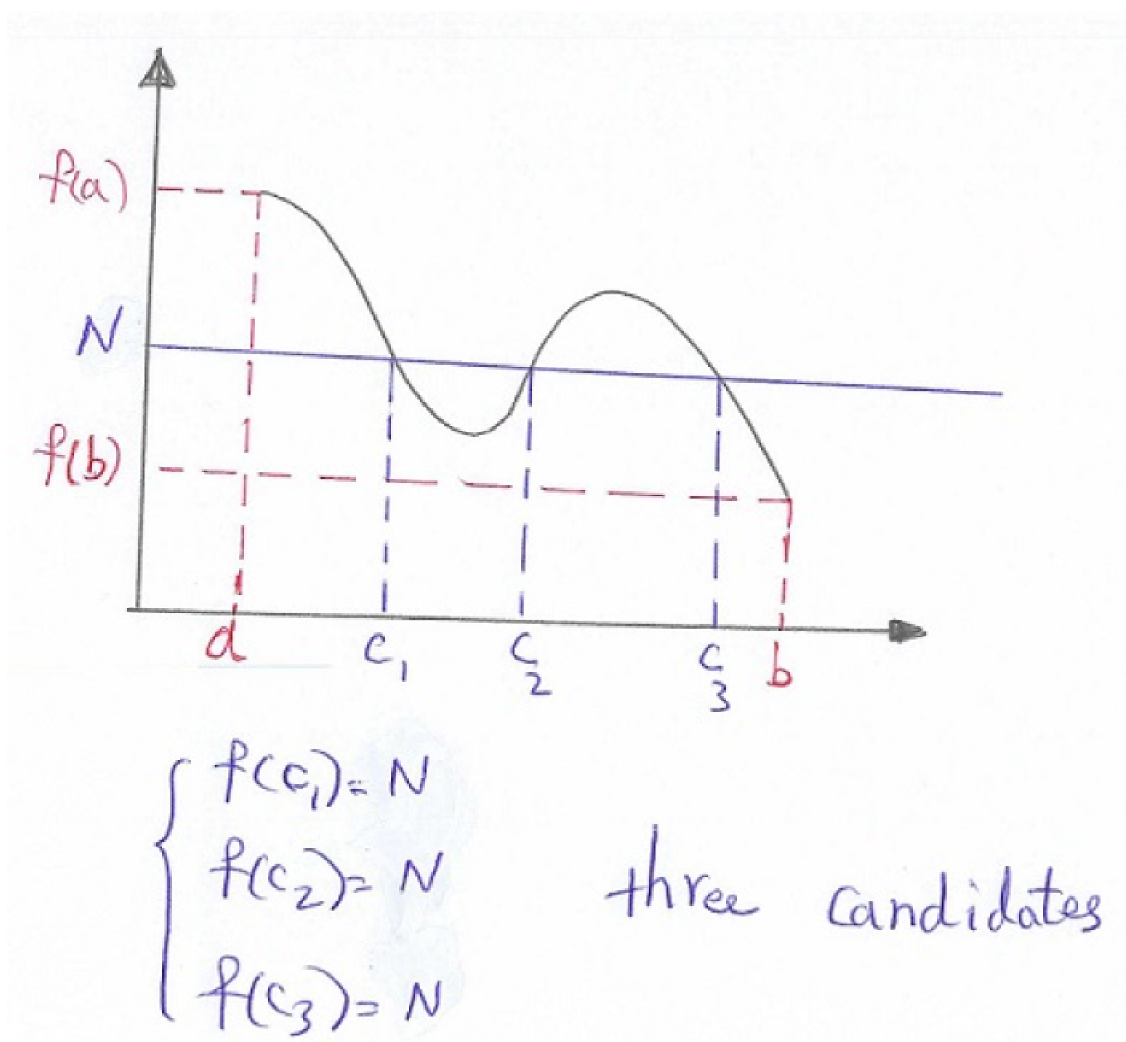
■

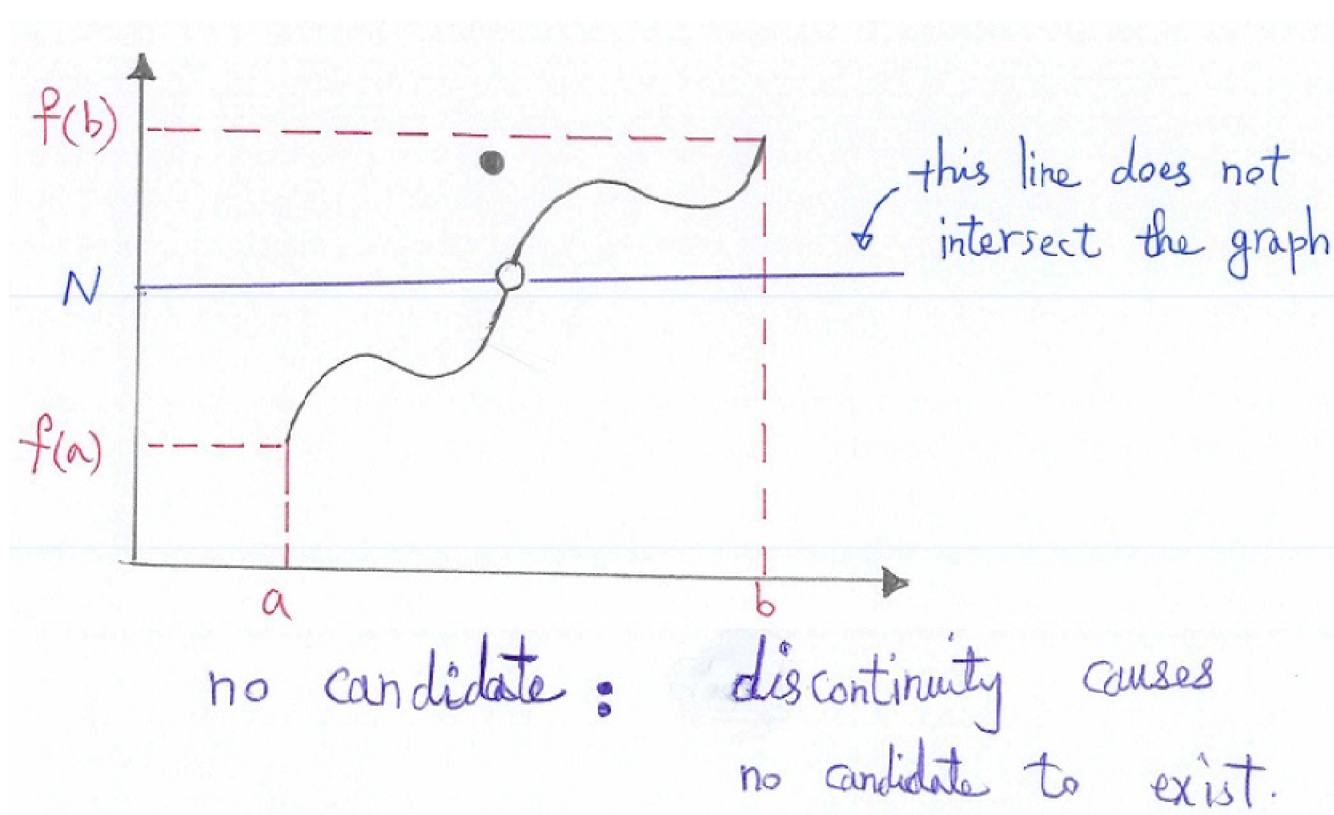
Intermediate Value Theorem

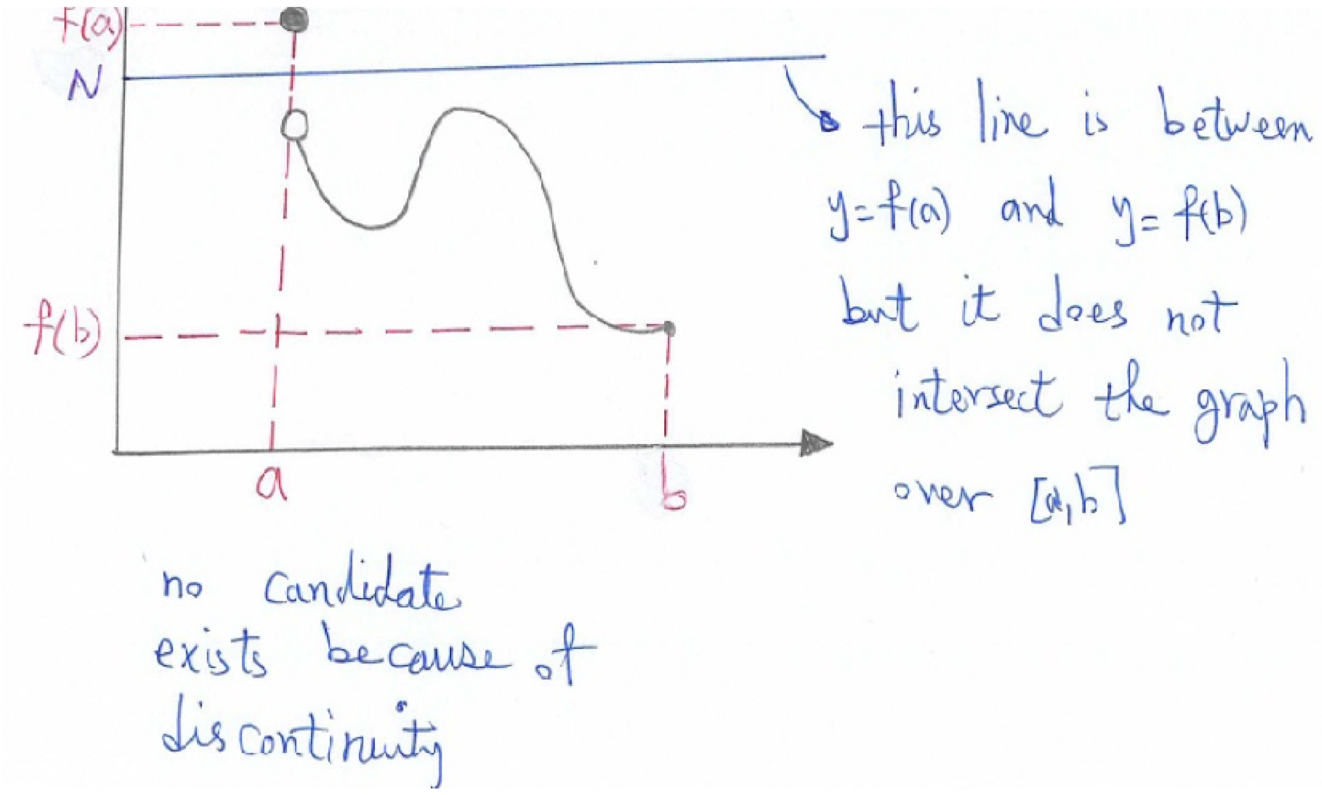
(from section 2.5)

Theorem: Suppose that f is continuous on the interval $[a\,,\,b]$ (it is continuous on the path from a to b). If $f(a) \neq f(b)$ and if N is a number between $f(a)$ and $f(b)$ ($f(a) < N < f(b)$ or $f(b) < N < f(a)$), then there is number c in the open interval $a < c < b$ such that $f(c) = N$.

Note. This theorem says that any horizontal line between the two horizontal lines $y = f(a)$ and $y = f(b)$ intersects the graph of f somewhere between a and b. See figures 8 and 9 on page 126. Also see the following figures:







Example (from the textbook). Use the Intermediate Value Theorem to show that there is root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

in the interval $[1\ ,\ 2]$.

Solution:

Consider the function $f(x) = 4x^3 - 6x^2 + 3x - 2$ over the closed interval $[1\ ,\ 2]$.

The function f is a polynomial, therefore it is continuous over $[1\ ,\ 2]$

We have

$$\begin{cases} f(1) &= 4 - 6 + 3 - 2 = -1 \\ f(2) &= 32 - 24 + 6 - 2 = 12 \end{cases}$$

Since:

$$f(1) < 0 < f(2)$$

by the Intermediate VT there exists a value c in the interval $(1, 2)$ such that $f(c) = 0$, i.e. there is a solution for the equation $f(x) = 0$ in the interval $(1, 2)$.

Intermediate Value Theorem

Use the Intermediate Value Theorem to prove that the equation

$$\sqrt{2x+5} = 4 - x^2$$

has a solution.

Solution

Rewrite the equation as

Rewrite the equation as

$$\sqrt{2x+5}+x^2-4=0$$

Let $f(x)=\sqrt{2x+5}+x^2-4=0$

Now $g(x)=\sqrt{2x+5}$ is continuous on the interval $[-5/2, \infty)$ since it is formed as the composition of two functions,

the square root function $g(x)=\sqrt{x}$

with nonnegative function $y=2x+5$

or

$u(x)=2x+5$

$$g(u(x))=\sqrt{u(x)}$$

$$=\sqrt{2x+5}$$

The $f(x)$ is the sum of the function g and the quadratic function $y=x^2-4$.

$y=x^2-4$ is continuous for all values of x.

$f(x)=\sqrt{2x+5}+x^2-4$ is continuous on the interval $[-5/2, \infty)$.

By trial and error, we find the function values

$f(0)=\sqrt{5}-4\approx -1.76$

&

$f(2)=\sqrt{9}=3$

Observe that $f(x)$ is continuous on the finite closed interval $[0,2]\subset[-5/2, \infty)$.

Since the value $y_0=0$ is between the numbers

numbers

$f(0) = -1.76$

&

$f(2) = 3$

by the Intermediate Value Theorem, there is a number $c \in [0,2]$ such that $f(c) = 0$

The number c solves the original equation.

15

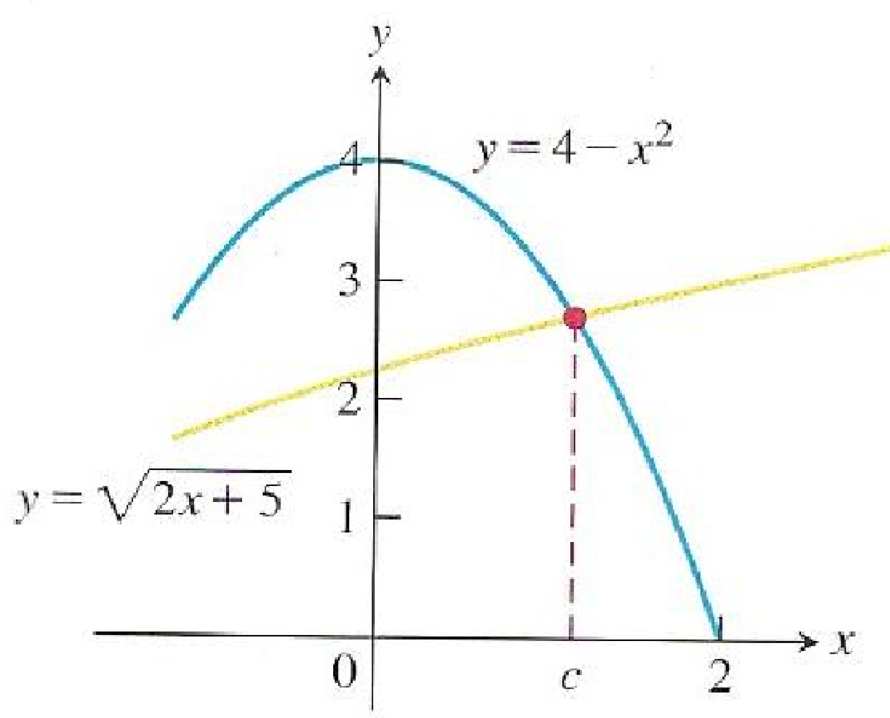


FIGURE 2.46 The curves $y = \sqrt{2x+5}$ and $y = 4 - x^2$ have the same value at $x = c$ where $\sqrt{2x+5} + x^2 - 4 = 0$ (Example 11).

Continuous Extension to a Point

$$f(x) = \frac{\sin x}{x}$$

is continuous at every point except $x = 0$.

$x = 0$ is NOT in its domain.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Since this limit is finite, we can extend the function's domain to include the point $x = 0$ in such a way that the extended function is continuous at x=0.

we define the new function

$$F(x)=\begin{cases}\dfrac{\sin x}{x}, & x\neq 0\\ 1, & x=0\end{cases}$$

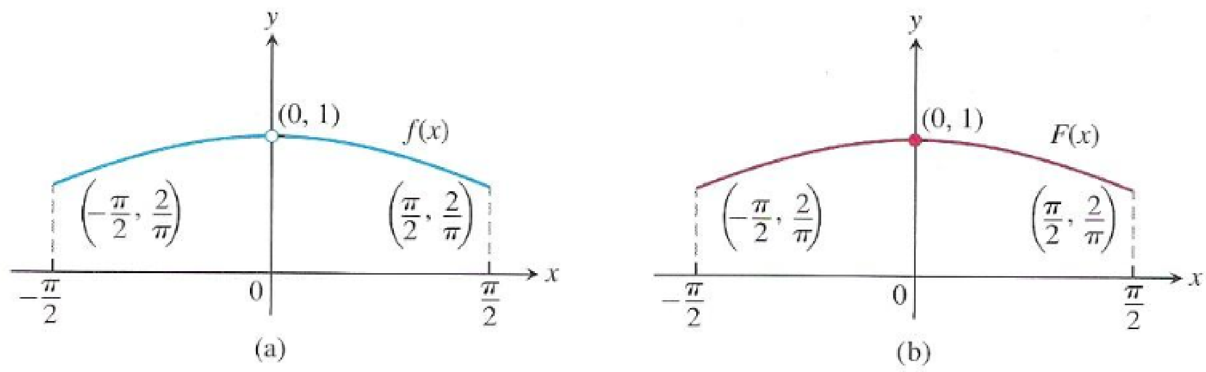


FIGURE 2.47 (a) The graph of $f(x) = (\sin x)/x$ for $-\pi/2 \le x \le \pi/2$ does not include the point (0, 1) because the function is not defined at $x = 0$. (b) We can extend the domain to include $x = 0$ by defining the new function $F(x)$ with $F(0) = 1$ and $F(x) = f(x)$ everywhere else. Note that $F(0) = \lim_{x\to 0} f(x)$ and $F(x)$ is a continuous function at $x = 0$.

The new function F(x) is continuous at x=0 because

$$\lim_{x\to 0} \frac{\sin x}{x} = F(0)$$

so it meets the requirements for continuity.

More generally, a function (such as a rational function) may have a limit at a point where it is NOT defined.

If $f(c)$ is not defined, but $\lim_{x\to c} f(x)= L$ exists, we can define a new function F(x) by the rule

$$F(x)=\begin{cases} f(x), & \text{if } x \text{ is in the domain of } f(x)\\ L, & \text{if } x=c\end{cases}$$

The function F is continuous at x=c. It is called the CONTINUOUS EXTENSION of f(x) to

$x = c$.

For rational functions f, continuous extensions are often found by canceling common factors in the numerator and denominator.

EXAMPLE 12 Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}, \quad x \neq 2$$

has a continuous extension to $x = 2$, and find that extension.

Solution Although $f(2)$ is not defined, if $x \neq 2$ we have

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x - 2)(x + 3)}{(x - 2)(x + 2)} = \frac{x + 3}{x + 2}.$$

The new function

$$F(x) = \frac{x + 3}{x + 2}$$

is equal to $f(x)$ for $x \neq 2$, but is continuous at $x = 2$, having there the value of $5/4$. Thus F is the continuous extension of f to $x = 2$, and

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \to 2} f(x) = \frac{5}{4}.$$

The graph of f is shown in Figure 2.48. The continuous extension F has the same graph except with no hole at $(2, 5/4)$. Effectively, F is the function f extended across the missing domain point at $x = 2$ so as to give a continuous function over the larger domain. ■

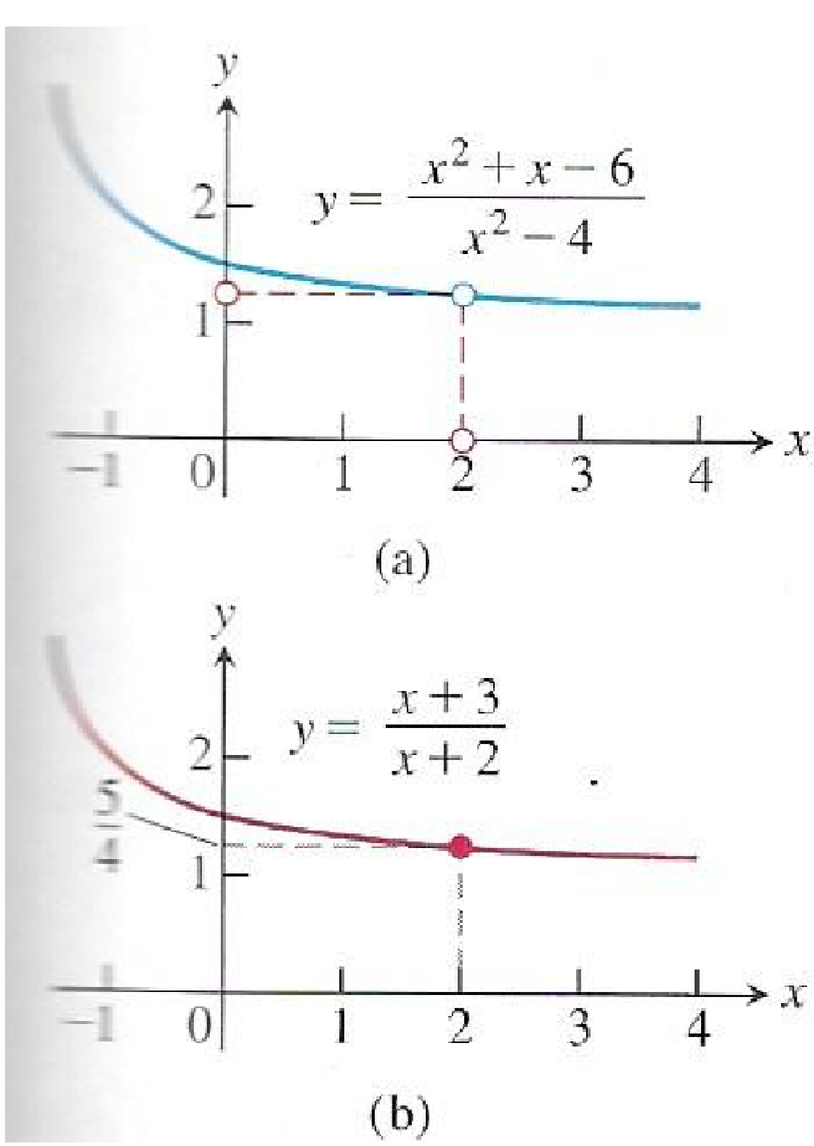


FIGURE 2.48 (a) The graph of $f(x)$ and (b) the graph of its continuous extension $F(x)$ (Example 12).

Example (Midterm 1 - Fall 2016). Show that the equation $2 - e^x = 4x$ has a solution in the interval $[0, 1]$. Justify your answer.

Solution: We take everything to one side and set the equation in the equivalent form $4x - 2 + e^x = 0$. Consider the function $f(x) = 4x - 2 + e^x$. Equivalently, we must show that $f(x) = 0$ for some x in the interval $(0\,,\ 1)$

Because f is a difference of a polynomial and an exponential function, it is continuous everywhere, in particular, on the closed interval $[0\ ,\ 1]$

$$\begin{cases} f(0) &= 0 - 2 + e^0 = -2 + 1 = -1 \\ f(1) &= 4 - 2 + e^1 = 2 + e \approx 2 + 2.718 = 4.718 \end{cases}$$

Since:

$$f(0) < 0 < f(1)$$

by the Intermediate VT there exists a value c in the interval $(0\,,\,1)$ such that $f(c) = 0$, i.e. there is a solution for the equation $f(x) = 0$ in the interval $(0\,,\,1)$.

Example (exrcise 53 of section 2.5). Use the Intermediate Value Theorem to show that there is root of the equation

$$4x^4 + x - 3 = 0$$

in the interval $[1\ ,\ 2]$.

Solution:

Consider the function $f(x) = 4x^3 - 6x^2 + 3x - 2$ over the closed interval $[1\,,\,2]$.

The function f is a polynomial, therefore it is continuous over $[1\ ,\ 2]$

We have

$$\begin{cases} f(1) &= 4 - 6 + 3 - 2 = -1 \\ f(2) &= 32 - 24 + 6 - 2 = 12 \end{cases}$$

Since:

$$f(1) < 0 < f(2)$$

by the Intermediate VT there exists a value c in the interval $(1\,,\,2)$ such that $f(c) = 0$, i.e. there is a solution for the equation $f(x) = 0$ in the interval $(1\,,\,2)$.