SECTION 2.8: CONTINUITY

LEARNING OBJECTIVES

• Understand and know the definitions of continuity at a point (in a one-sided and two-sided sense), on an open interval, on a closed interval, and variations thereof.
• Be able to identify discontinuities and classify them as removable, jump, or infinite.
• Know properties of continuity, and use them to analyze the continuity of rational, algebraic, and trigonometric functions and compositions thereof.
• Understand the Intermediate Value Theorem (IVT) and apply it to solutions of equations and real zeros of functions.

PART A: CONTINUITY AT A POINT

Informally, a function f with domain $\mathbb{R}$ is everywhere continuous (on $\mathbb{R}$) $\Leftrightarrow$ we can take a pencil and **trace** the graph of f between any two distinct points on the graph **without** having to lift up our pencil.

We will make this idea more precise by first defining continuity **at a point** a $(a \in \mathbb{R})$ and then continuity **on intervals**.

Continuity at a Point a

f is continuous at $x = a$ $\Leftrightarrow$

1) $f(a)$ is **defined (real)**; that is, $a \in \text{Dom}(f)$,

2) $\lim_{x \to a} f(x)$ **exists (is real)**, and

3) $\lim_{x \to a} f(x) = f(a)$.

f is discontinuous at $x = a$ $\Leftrightarrow$ f is not continuous at $x = a$.

Comments

1) ensures that there is literally a **point** at $x = a$.
2) **constrains** the behavior of f **immediately around** $x = a$.
3) then ensures **"safe passage"** through the point $(a, f(a))$ on the graph of $y = f(x)$. Some sources just state 3) in the definition, since the form of 3) implies 1) and 2).

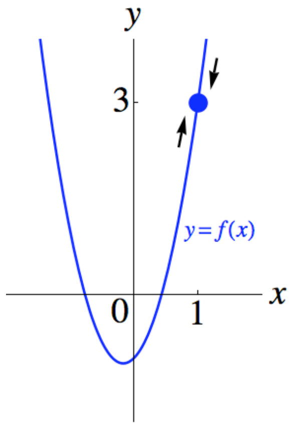
Example 1 (Continuity at a Point; Revisiting Section 2.1, Example 1)

Let $f(x) = 3x^2 + x - 1$. Show that f is continuous at $x = 1$.

§ Solution

1) $f(1) = 3$, a real number $(1 \in \text{Dom}(f))$
2) $\lim_{x \to 1} f(x) = 3$, a real number, and
3) $\lim_{x \to 1} f(x) = f(1)$.

Therefore, f is continuous at $x = 1$. The graph of $y = f(x)$ is below.



Note: The **Basic Limit Theorem for Rational Functions** in Section 2.1 basically states that a **rational** function is **continuous** at any number in its **domain**. §

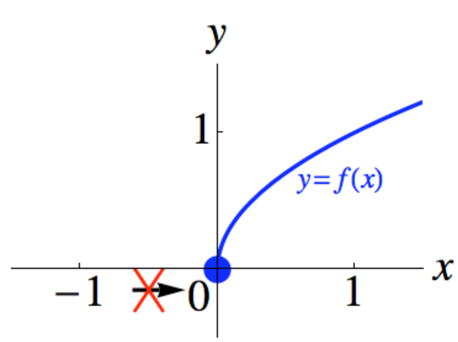
Example 2 (Discontinuities at a Point; Revisiting Section 2.2, Example 2)

Let $f(x) = \sqrt{x}$. Explain why f is discontinuous at $x = -1$ and $x = 0$.

§ Solution

- $f(-1)$ is **not real** $(-1 \notin \text{Dom}(f))$, so f is **discontinuous** at $x = -1$.
- $f(0) = 0$, but $\lim_{x \to 0} \sqrt{x}$ **does not exist (DNE)**, so f is **discontinuous** at $x = 0$.

The graph of $y = f(x)$ is below.



Some sources do not even bother calling -1 and 0 "discontinuities" of f, since f is **not even defined** on a **punctured neighborhood** of $x = -1$ or of $x = 0$. §

PART B: CLASSIFYING DISCONTINUITIES

We now consider cases where a function f is **discontinuous** at $x = a$, even though f is defined on a punctured neighborhood of $x = a$.

We will **classify** such discontinuities as removable, jump, or infinite.
(See Footnotes 1 and 2 for another type of discontinuity.)

> Removable Discontinuities
>
> A function f has a removable discontinuity at $x = a$ $\Leftrightarrow$
>
> 1) $\lim_{x \to a} f(x)$ **exists** (call this limit L), but
>
> 2) f is still **discontinuous** at $x = a$.
>
> • Then, the graph of $y = f(x)$ has a **hole** at the point (a, L).

Example 3 (Removable Discontinuity at a Point; Revisiting Section 2.1, Ex. 7)

Let $g(x) = x + 3,\ (x \neq 3)$. **Classify** the discontinuity at $x = 3$.

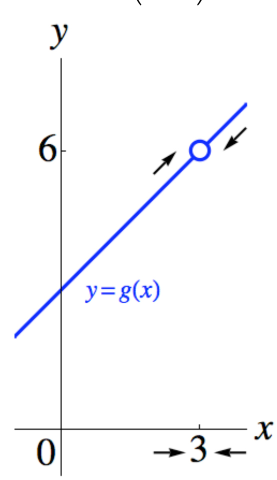
§ Solution

g has a **removable discontinuity** at $x = 3$, because:

1) $\lim_{x \to 3} g(x) = 6$, but

2) g is still discontinuous at $x = 3$; here, $g(3)$ is undefined.

The graph of $y = g(x)$ below has a **hole** at the point $(3, 6)$.



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Example 4 (Removable Discontinuity at a Point; Revisiting Section 2.1, Ex. 8)

Let $h(x) = \begin{cases} x+3, & x \neq 3 \\ 7, & x = 3 \end{cases}$. **Classify** the discontinuity at $x = 3$.

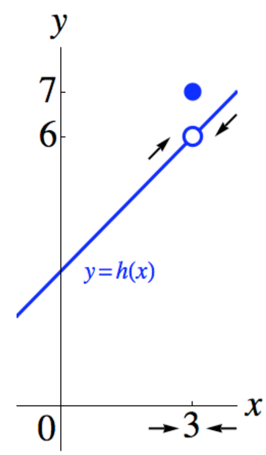
§ Solution

h has a **removable discontinuity** at $x = 3$, because:

1) $\lim_{x \to 3} h(x) = 6$, but

2) h is still discontinuous at $x = 3$;
   here, $\lim_{x \to 3} h(x) \neq h(3)$,
   because $6 \neq 7$.

The graph of $y = h(x)$ also has a **hole** at the point $(3, 6)$.



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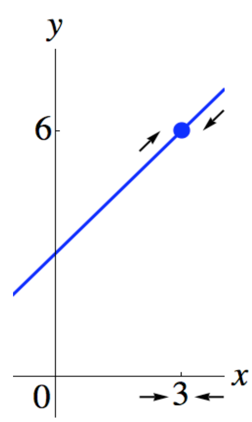
Why are These Discontinuities Called "Removable"?

The term "removable" is a bit of a misnomer here, since we have no business changing the function at hand.

The idea is that a removable discontinuity at *a* can be removed by (re)defining the function at *a*; the new function will then be continuous at *a*.

For example, if we were to **define** $g(3) = 6$ in Example 3 and **redefine** $h(3) = 6$ in Example 4, then we would remove the discontinuity at $x = 3$ in both situations. We would obtain the graph below.



Jump Discontinuities

A function f has a jump discontinuity at $x = a$ $\Leftrightarrow$

1) $\lim\limits_{x \to a^-} f(x)$ **exists**, and (call this limit L_1)

2) $\lim\limits_{x \to a^+} f(x)$ **exists**, but (call this limit L_2)

3) $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$. $(L_1 \neq L_2)$

• Therefore, $\lim\limits_{x \to a} f(x)$ **does not exist (DNE)**.

• It is **irrelevant** how $f(a)$ is defined, or if it is defined at all.

Example 5 (Jump Discontinuity at a Point; Revisiting Section 2.1, Example 14)

Let $f(x) = \dfrac{|x|}{x} = \begin{cases} \dfrac{x}{x} = 1, & \text{if } x > 0 \\ \dfrac{-x}{x} = -1, & \text{if } x < 0 \end{cases}$. **Classify** the discontinuity at $x = 0$.

§ Solution

f has a **jump discontinuity** at $x = 0$, because:

1) $\lim\limits_{x \to 0^-} f(x) = -1$, and

2) $\lim\limits_{x \to 0^+} f(x) = 1$, but

3) $\lim\limits_{x \to 0} f(x)$ does not exist (DNE), because $-1 \neq 1$.

We **cannot remove** this discontinuity by defining $f(0)$.

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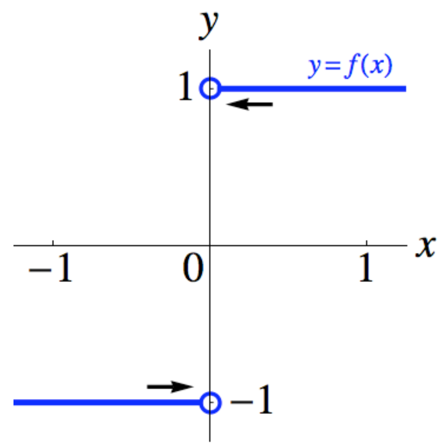
The graph of $y = f(x)$ is below.



Infinite Discontinuities

A function f has an infinite discontinuity at $x = a$ $\Leftrightarrow$

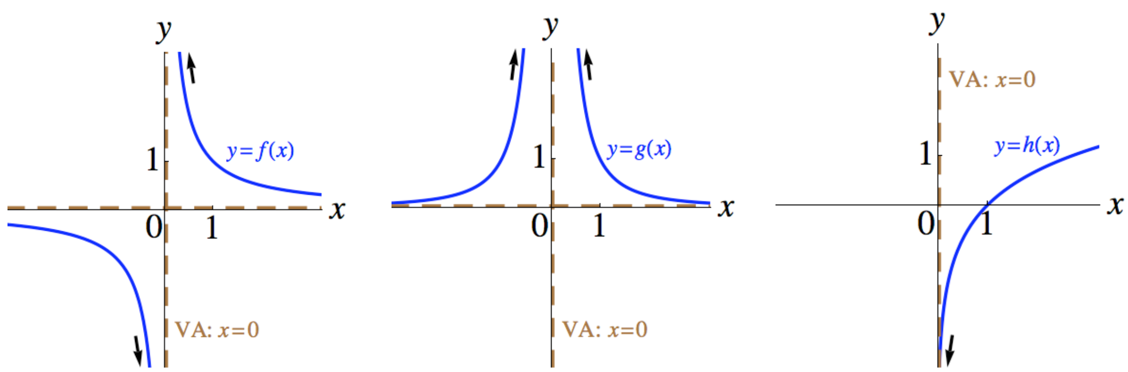
$\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ is ∞ or $-\infty$.

- That is, the graph of $y = f(x)$ has a **VA** at $x = a$.
- It is **irrelevant** how $f(a)$ is defined, or if it is defined at all.

Example 6 (Infinite Discontinuities at a Point; Revisiting Section 2.4, Exs. 1 and 2)

The functions described below have **infinite discontinuities** at $x = 0$.
We will study $\ln x$ in Chapter 7 (see also the Precalculus notes, Section 3.2).

$f(x) = \dfrac{1}{x}$ $\quad g(x) = \dfrac{1}{x^2}$ $\quad h(x) = \ln x$



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PART C: CONTINUITY ON AN OPEN INTERVAL

We can extend the concept of continuity in various ways.
(For the remainder for this section, assume $a < b$.)

<u>Continuity on an Open Interval</u>

A function f is continuous on the **open interval** (a, b) $\Leftrightarrow$
f is continuous at **every number (point)** in (a, b).

• This extends to **unbounded** open intervals of the form (a, ∞), $(-\infty, b)$, and $(-\infty, \infty)$.

In Example 6, all three functions are continuous on the interval $(0, \infty)$.

The first two functions are also continuous on the interval $(-\infty, 0)$.

We will say that the "<u>continuity intervals</u>" of the first two functions are: $(-\infty, 0)$, $(0, \infty)$. However, this terminology is **not standard**.

• In Footnote 1, f has the singleton (one-element) set $\{0\}$ as a "degenerate continuity interval." See also Footnotes 2 and 3.

• Avoid using the union $(\cup)$ symbol here. In Section 2.1, Example 10, f was continuous on $(-\infty, 0]$ and $(0,1)$, but not on $(-\infty, 1)$.

PART D: CONTINUITY ON OTHER INTERVALS; ONE-SIDED CONTINUITY

<u>Continuity on a Closed Interval</u>

A function f is continuous on the **closed interval** $[a, b]$ $\Leftrightarrow$

1) f is defined on $[a, b]$,
2) f is continuous on (a, b),
3) $\lim_{x \to a^+} f(x) = f(a)$, and
4) $\lim_{x \to b^-} f(x) = f(b)$.

3) and 4) **weaken** the continuity requirements at the **endpoints**, *a* and *b*.
Imagine taking limits as we **"push outwards"** towards the endpoints.

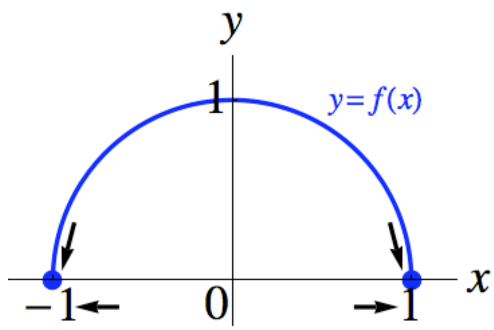
3) implies that f is <u>continuous from the right</u> at *a*.
4) implies that f is <u>continuous from the left</u> at *b*.

Example 7 (Continuity on a Closed Interval)

Let $f(x) = \sqrt{1 - x^2}$. Show that f is continuous on the closed interval $[-1, 1]$.

§ Solution

The graph of $y = f(x)$ is below.



$$y = \sqrt{1 - x^2} \quad \Leftrightarrow$$
$$y^2 = 1 - x^2 \quad (y \ge 0) \quad \Leftrightarrow$$
$$x^2 + y^2 = 1 \quad (y \ge 0)$$

The graph is the upper half of the unit circle centered at the origin, including the points $(-1, 0)$ and $(1, 0)$.

f is continuous on $[-1, 1]$, because:

1) f is defined on $[-1, 1]$,

2) f is continuous on $(-1, 1)$,

3) $\lim_{x \to -1^+} f(x) = f(-1)$, so f is continuous from the **right** at -1, and

4) $\lim_{x \to 1^-} f(x) = f(1)$, so f is continuous from the **left** at 1.

Note: $f(-1) = 0$, and $f(1) = 0$, but they need not be equal.

f has $[-1, 1]$ as its sole "**continuity interval.**" When giving "continuity intervals," we include brackets where appropriate, even though f is **not continuous** (in a two-sided sense) at -1 and at 1 (**WARNING 1**).

• Some sources would call $(-1, 1)$ the continuity set of f; it is the set of all real numbers at which f is continuous. (See Footnotes 2 and 3.) §

Challenge to the Reader: Draw a graph where f is defined on $[a, b]$, and f is continuous on (a, b), but f is **not** continuous on the closed interval $[a, b]$.

Continuity on Half-Open, Half-Closed Intervals

f is continuous on an interval of the form $[a, b)$ or $[a, \infty)$ $\Leftrightarrow$
f is continuous on (a, b) or (a, ∞), respectively, and it is continuous **from the right** at a.

f is continuous on an interval of the form $(a, b]$ or $(-\infty, b]$ $\Leftrightarrow$
f is continuous on (a, b) or $(-\infty, b)$, respectively, and it is continuous **from the left** at b.

Example 8 (Continuity from the Right; Revisiting Example 2)

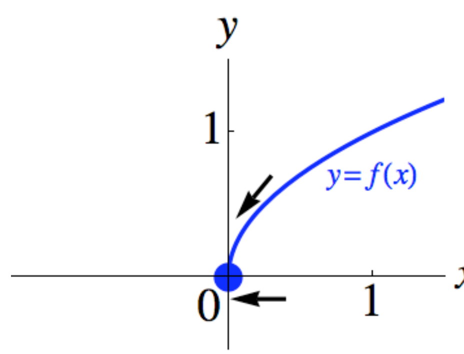
Let $f(x) = \sqrt{x}$.

f is continuous on $(0, \infty)$.

$\lim_{x \to 0^+} \sqrt{x} = 0 = f(0)$, so f is continuous **from the right** at 0.

The sole **"continuity interval"** of f is $[0, \infty)$.

The graph of $y = f(x)$ is below.



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Example 9 (Continuity from the Left)

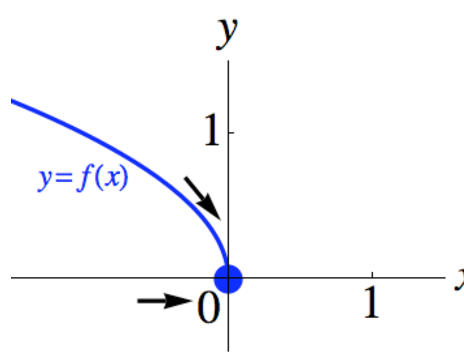
Let $f(x) = \sqrt{-x}$.

f is continuous on $(-\infty, 0)$.

$\lim_{x \to 0^-} \sqrt{-x} = 0 = f(0)$, so f is continuous **from the left** at 0.

The sole **"continuity interval"** of f is $(-\infty, 0]$.

The graph of $y = f(x)$ is below.



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PART E: CONTINUITY THEOREMS

Properties of Continuity / Algebra of Continuity Theorems

If f and g are functions that are continuous at $x = a$, then so are the functions:

- $f + g$, $f - g$, and fg.
- $\frac{f}{g}$, if $g(a) \neq 0$.
- f^n, if n is a positive integer exponent $(n \in \mathbb{Z}^+)$.
- $\sqrt[n]{f}$, if:
  - (n is an **odd** positive integer), or
  - (n is an **even** positive integer, **and** $f(a) > 0$).

In Section 2.2, we showed how similar properties of **limits** justified the Basic Limit Theorem for Rational Functions. Similarly, the properties above, together with the fact that **constant** functions and the **identity** function (represented by $f(x) = x$) are **everywhere continuous** on $\mathbb{R}$, justify the following:

Continuity of Rational Functions

A rational function is continuous on its domain.

- That is, the "continuity interval(s)" of a rational function f **are** its domain interval(s).

In particular, **polynomial** functions are **everywhere continuous** (on $\mathbb{R}$).

Although this is typically true for **algebraic** functions in general, there are counterexamples (see Footnote 4).

Example 10 (Continuity of a Rational Function; Revisiting Example 6)

If $f(x) = \frac{1}{x}$, then $\text{Dom}(f) = (-\infty, 0) \cup (0, \infty)$.

f is **rational**, so the "**continuity intervals**" of f are: $(-\infty, 0)$, $(0, \infty)$. §

When analyzing the continuity of functions that are **not rational**, we may need to check for **one-sided** continuity at **endpoints** of domain intervals.

Example 11 (Continuity of an Algebraic Function; Revisiting Chapter 1, Ex. 6)

Let $h(x) = \dfrac{\sqrt{x+3}}{x-10}$. What are the **"continuity intervals"** of *h*?

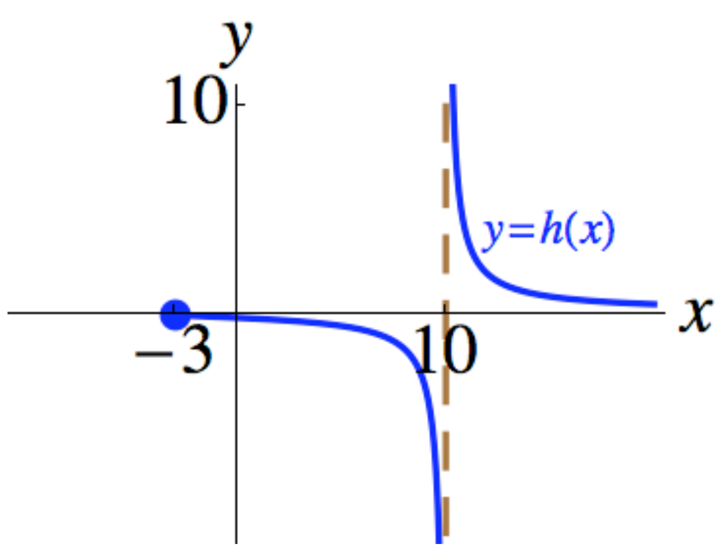
§ Solution

In Chapter 1, we found that $\text{Dom}(h) = [-3, 10) \cup (10, \infty)$.
We will show that the **"continuity intervals"** are, in fact, the **domain intervals**, $[-3, 10)$ and $(10, \infty)$.

By the **Algebra of Continuity Theorems**, we find that *h* is continuous on $(-3, 10)$ and $(10, \infty)$.

Now, $\lim\limits_{x \to -3^+} h(x) = 0 = h(-3)$, because $\left(\text{Limit Form } \dfrac{\sqrt{0^+}}{-13} \right) \Rightarrow 0$.
Therefore, *h* is continuous **from the right** at $x = -3$, and its **"continuity intervals"** are: $[-3, 10)$ and $(10, \infty)$.

The graph of $y = h(x)$ is below.



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Continuity of Composite Functions

If g is continuous at a, and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

(See Footnote 5.)

Continuity of Basic Trigonometric Functions

The six basic **trigonometric** functions (sine, cosine, tangent, cosecant, secant, and cotangent) are **continuous** on their **domain intervals**.

Example 12 (Continuity of a Composite Function)

Let $h(x) = \sec\left(\frac{1}{x}\right)$. Where is h continuous?

§ Solution

Observe that $h(x) = (f \circ g)(x) = f(g(x))$, where:

the **"inside"** function is given by $g(x) = \frac{1}{x}$, and

the **"outside"** function f is given by $f(\theta) = \sec\theta$, where $\theta = \frac{1}{x}$.

g is continuous at all real numbers **except** 0 $(x \neq 0)$.
f is continuous on its domain intervals.

$$\sec\theta \text{ is real} \quad \Leftrightarrow \quad \cos\theta \neq 0, \text{ and } x \neq 0$$

$$\Leftrightarrow \quad \theta \neq \frac{\pi}{2} + \pi n \ \ (n \in \mathbb{Z}), \text{ and } x \neq 0$$

$$\Leftrightarrow \quad \frac{1}{x} \neq \frac{\pi}{2} + \pi n \ \ (n \in \mathbb{Z}), \text{ and } x \neq 0$$

We can replace both sides of the inequation with their **reciprocals**, because we exclude the case $x = 0$, and both sides are never 0.

$$\Leftrightarrow \quad x \neq \frac{1}{\frac{\pi}{2} + \pi n} \ \ (n \in \mathbb{Z}), \text{ and } x \neq 0$$

$$\Leftrightarrow \quad x \neq \frac{1}{\left(\frac{\pi}{2} + \pi n\right)} \cdot \frac{2}{2} \quad (n \in \mathbb{Z}), \text{ and } x \neq 0$$

$$\Leftrightarrow \quad x \neq \frac{2}{\pi + 2\pi n} \quad (n \in \mathbb{Z}), \text{ and } x \neq 0$$

h is continuous on:

$$\left\{ x \in \mathbb{R} \,\middle|\, x \neq \frac{2}{\pi + 2\pi n} \;(n \in \mathbb{Z}), \text{ and } x \neq 0 \right\}, \text{ or}$$

$$\left\{ x \in \mathbb{R} \,\middle|\, x \neq \frac{2}{\pi(2n+1)} \;(n \in \mathbb{Z}), \text{ and } x \neq 0 \right\}.$$

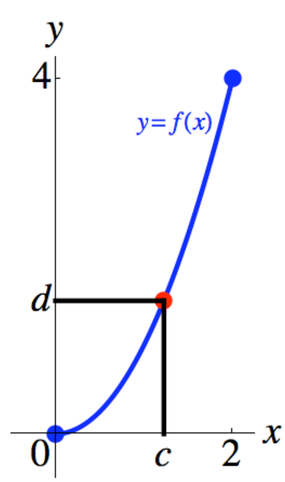
§

PART F: THE INTERMEDIATE VALUE THEOREM (IVT)

Continuity of a function **constrains** its behavior in important (and useful) ways. Continuity is central to some key theorems in calculus. We will see the Extreme Value Theorem (EVT) in Chapter 4 and Mean Value Theorems (MVTs) in Chapters 4 and 5. We now discuss the Intermediate Value Theorem (IVT), which directly relates to the **meaning** of continuity. We will motivate it before stating it.

Example 13 (Motivating the IVT)

Let $f(x) = x^2$ on the x-interval $[0, 2]$. The graph of $y = f(x)$ is below.



f is **continuous** on $[0, 2]$,

$f(0) = 0$, and

$f(2) = 4$.

The **IVT** guarantees that **every** real number (d) **between** 0 and 4 is a value of (is taken on by) f at **some** x-value (c) in $[0, 2]$. §

<u>The Intermediate Value Theorem (IVT): Informal Statement</u>

If a function f is **continuous** on the **closed** interval $\left[a, b\right]$,

then f takes on **every** real number **between** $f\left(a\right)$ and $f\left(b\right)$ on $\left[a, b\right]$.

<u>The Intermediate Value Theorem (IVT): Precise Statement</u>

Let $\min\left(f\left(a\right), f\left(b\right)\right)$ be the **lesser** of $f\left(a\right)$ and $f\left(b\right)$;
if they are equal, then we take their common value.

Let $\max\left(f\left(a\right), f\left(b\right)\right)$ be the **greater** of $f\left(a\right)$ and $f\left(b\right)$;
if they are equal, then we take their common value.

A function f is **continuous** on $\left[a, b\right] \Rightarrow$
$\forall d \in \left[\min\left(f\left(a\right), f\left(b\right)\right),\ \max\left(f\left(a\right), f\left(b\right)\right)\right], \exists c \in \left[a, b\right] \ni f\left(c\right) = d$.

<u>Example 14 (Applying the IVT to Solutions of Equations)</u>

Prove that $x^2 = 3$ has a **solution** in $\left[0, 2\right]$.

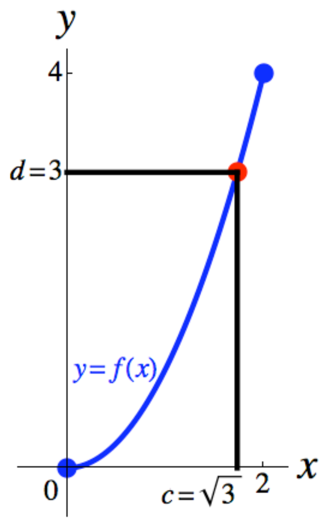
<u>§ Solution</u>

Let $f\left(x\right) = x^2$. (We also let the desired **function value**, $d = 3$.)

f is **continuous** on $\left[0, 2\right]$,
$f\left(0\right) = 0$,
$f\left(2\right) = 4$, **and**
$3 \in \left[0, 4\right]$.

Therefore, by the **IVT**, $\exists c \in \left[0, 2\right] \ni$ (such that) $f\left(c\right) = 3$.
That is, $x^2 = 3$ has a **solution** (c) in $\left[0, 2\right]$.
Q.E.D. §

In Example 14, $c=\sqrt{3}$ was our solution to $x^2 = 3$ in $[0,2]$; $d = 3$ here.



To **verify the conclusion of the IVT** in general, we can give a **formula** for c given **any** real number d in $[0,4]$, where $c \in [0,2]$ and $f(c) = d$.

Example 15 (Verifying the Conclusion of the IVT; Revisiting Examples 13 and 14)

Verify the conclusion of the IVT for $f(x) = x^2$ on the x-interval $[0,2]$.

§ Solution

f is **continuous** on $[0,2]$, so the IVT applies. $f(0) = 0$, and $f(2) = 4$.
Let $d \in [0,4]$, and let $c = \sqrt{d}$.

- The following justifies our **formula** for c:

$$\begin{aligned} f(c) = d \text{ and } c \in [0,2] &\iff \\ c^2 = d \text{ and } c \in [0,2] &\iff \\ c = \sqrt{d}, \text{ a real number in } [0,2] & \end{aligned}$$

WARNING 2: We do not write $c = \pm\sqrt{d}$, because either $d = 0$, or a value for c would fall outside of $[0,2]$.

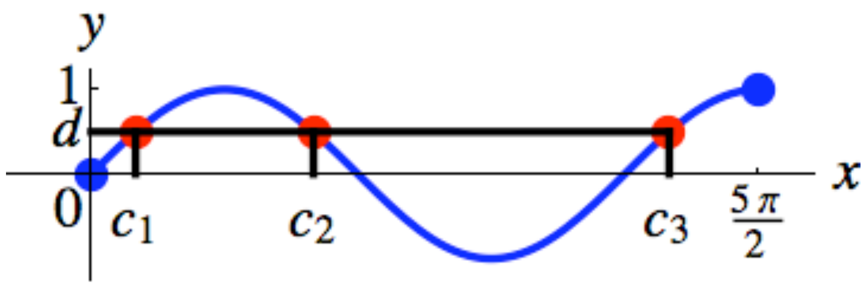
Observe: $0 \le d \le 4 \Rightarrow 0 \le \sqrt{d} \le 2$.

Then, $c \in [0,2]$, and $f(c) = c^2 = \left(\sqrt{d}\right)^2 = d$.

Therefore, $\forall d \in [0,4], \exists c \in [0,2] \ni f(c) = d$. §

Example 16 (c Might Not Be Unique)

Let $f(x) = \sin x$ on the x-interval $\left[0, \frac{5\pi}{2}\right]$. The graph of $y = f(x)$ is below.



$f(0) = 0$, and $f\left(\frac{5\pi}{2}\right) = 1$. Because f is **continuous** on $\left[0, \frac{5\pi}{2}\right]$, the IVT guarantees that **every** real number d **between** 0 and 1 is taken on by f at **some** x-value c in $\left[0, \frac{5\pi}{2}\right]$.

WARNING 3: Given an appropriate value for d, there **might be more than one** appropriate choice for c. The IVT does not forbid that.

WARNING 4: Also, there are real numbers **outside of** $[0, 1]$ that are taken on by f on the x-interval $\left[0, \frac{5\pi}{2}\right]$. The IVT does not forbid that, either. §

PART G: THE BISECTION METHOD FOR APPROXIMATING A ZERO OF A FUNCTION

Our ability to **solve equations** is equivalent to our ability to **find zeros** of functions. For example, $f(x) = g(x) \Leftrightarrow f(x) - g(x) = 0$; we can solve the first equation by finding the zeros of $h(x)$, where $h(x) = f(x) - g(x)$.

We may have to use computer algorithms to **approximate zeros** of functions if we can't find them exactly.

- While we do have (nastier) analogs of the Quadratic Formula for 3^{rd}- and 4^{th}-degree polynomial functions, it has actually been proven that there is **no similar formula** for higher-degree polynomial functions.

The Bisection Method, which is the basis for some of these algorithms, uses the **IVT** to produce a **sequence of smaller and smaller intervals** that are guaranteed to contain a **zero** of a given function.

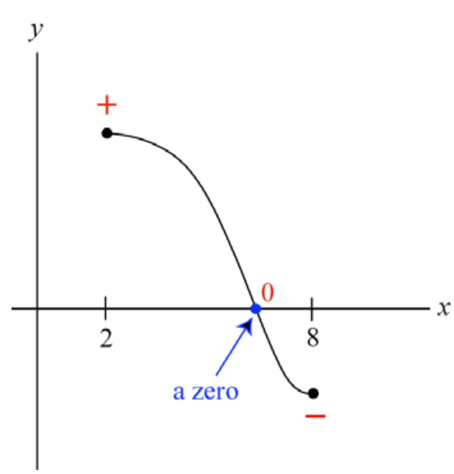
The Bisection Method for Approximating a Zero of a Continuous Function f

Let's say we want to **approximate a zero** of a function f.

Find x-values a_1 and b_1 $(a_1 < b_1)$ such that $f(a_1)$ and $f(b_1)$ have **opposite signs** and f is **continuous** on $[a_1, b_1]$. (The method fails if such x-values cannot be found.)

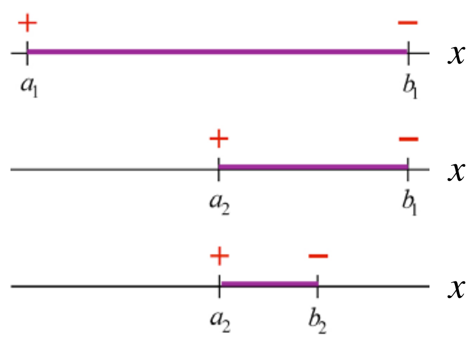
According to the **IVT**, there must be a **zero** of f in $[a_1, b_1]$, which we call our **"search interval."**

For example, consider the graph of $y = f(x)$ below. Our search interval is apparently $[2, 8]$.



If $f(a_1)$ or $f(b_1)$ were 0, then we would have found a **zero** of f, and we could either stop or try to approximate another zero.

If neither is 0, then we take the **midpoint** of the search interval and determine the sign of $f(x)$ there (in red below). We can then **shrink the search interval** (in purple below) and **repeat** the process. We call the Bisection Method an iterative method because of this repetition.



We stop when we **find a zero**, or until the search interval is **small enough** so that we are satisfied with taking its **midpoint** as our approximation.

A key drawback to numerical methods such as the Bisection Method is that, unless we manage to find n distinct real zeros of an n^{th}-degree polynomial $f(x)$, we may need other techniques to be sure that we have found **all** of the real zeros, if we are looking for all of them. §

Example 17 (Applying the Bisection Method; Revisiting Example 14)

We can approximate $\sqrt{3}$ by approximating the positive real **solution** of $x^2 = 3$, or the positive real **zero** of $h(x)$, where $h(x) = x^2 - 3$.

Search interval $[a, b]$	Sign of $h(a)$	Sign of $h(b)$	Midpoint	Sign of h there
$[0, 2]$	−	+	1	−
$[1, 2]$	−	+	1.5	−
$[1.5, 2]$	−	+	1.75	+
$[1.5, 1.75]$	−	+	1.625	−

etc. §

In Section 4.8, we will use Newton's Method for approximating zeros of a function, which tends to be more efficient. However, Newton's Method requires **differentiability** of a function, an idea we will develop in Chapter 3.

FOOTNOTES

1. **A function with domain $\mathbb{R}$ that is only continuous at 0.** (Revisiting Footnote 1 in Section 2.1.) Let $f(x)=\begin{cases} 0, & \text{if } x \text{ is a rational number } (x\in\mathbb{Q}) \\ x, & \text{if } x \text{ is an irrational number } (x\notin\mathbb{Q};\text{ really, } x\in\mathbb{R}\setminus\mathbb{Q}) \end{cases}$.
f is continuous at $x=0$, because $f(0)=0$, and we can use the Squeeze (Sandwich) Theorem to prove that $\lim_{x\to 0} f(x)=0$, also. The discontinuities at the nonzero real numbers are not categorized as removable, jump, or infinite.

2. **Continuity sets and a nowhere continuous function.** See *Cardinality of the Set of Real Functions With a Given Continuity Set* by Jiaming Chen and Sam Smith. The 19th-century German mathematician Dirichlet came up with a nowhere continuous function, D:
$$D(x)=\begin{cases} 0, & \text{if } x \text{ is a rational number } (x\in\mathbb{Q}) \\ 1, & \text{if } x \text{ is an irrational number } (x\notin\mathbb{Q};\text{ really, } x\in\mathbb{R}\setminus\mathbb{Q}) \end{cases}$$

3. **Continuity on a set.** This is tricky to define! See "Continuity on a Set" by R. Bruce Crofoot, *The College Mathematics Journal*, Vol. 26, No. 1 (Jan. 1995) by the Mathematical Association of America (MAA). Also see Louis A. Talman, *The Teacher's Guide to Calculus* (web). Talman suggests:
Let S be a subset of $\text{Dom}(f)$; that is, $S\subseteq\text{Dom}(f)$. f is continuous on S $\Leftrightarrow$
$\forall a\in S$, $\forall\varepsilon>0$, $\exists\delta>0$ $\ni$ $\left[\left(x\in S \text{ and } |x-a|<\delta\right) \Rightarrow \left|f(x)-f(a)\right|<\varepsilon\right]$.
• The definition essentially states that, for every number *a* in the set of interest, its function value is arbitrarily close to the function values of nearby *x*-values in the set. Note that we use $f(a)$ instead of *L*, which we used to represent $\lim_{x\to a} f(x)$, because we need $\lim_{x\to a} f(x)=f(a)$ (or possibly some one-sided variation) in order to have continuity on *S*.
• This definition covers / subsumes our definitions of continuity on open intervals; closed intervals; half-open, half-closed intervals; and unions (collections) thereof.
• One possible criticism against this definition is that it implies that the functions described in Footnote 4 are, in fact, continuous on the singleton set {0}. This conflicts with our definition of continuity at a point in Part A because of the issue of nonexistent limits. Perhaps we should require that f be defined on some interval of the form $[a, c)$ with $c>a$ or the form $(c, a]$ with $c<a$.
• Crofoot argues for the following definition: f is continuous on *S* if the restriction of f to *S* is continuous at each number in *S*. He acknowledges the use of one-sided continuity when dealing with closed intervals.

4. **An algebraic function that is not continuous on its domain.** Let $f(x)=\sqrt{x}+\sqrt{-x}$.
$\text{Dom}(f)=\{0\}$, a singleton (a set consisting of a single element), but f is not continuous at 0 (by Part A), because $\lim_{x\to 0} f(x)$ does not exist (DNE). The same is true for $f(x)=\sqrt{-x^2}$.

5. Continuity and the limit properties in Section 2.2, Part A. Let $a, K \in \mathbb{R}$.

If $\lim_{x \to a} g(x) = K$, and f is continuous at *K*, then:

$\lim_{x \to a} (f \circ g)(x) = \lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(K)$. Basically, continuity allows f to commute with a limit operator: $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$. Think: "The limit of a (blank) is the (blank) of the limit." This relates to Property 5) on the limit of a power, Property 6) on the limit of a constant multiple, and Property 7) on the limit of a root in Section 2.2.
For example, f could represent the squaring function.

6. A function that is continuous at every irrational point and discontinuous at every rational point. See Gelbaum and Olmsted, *Counterexamples in Analysis* (Dover), p.27. Also see Tom Vogel, http://www.math.tamu.edu/~tvogel/gallery/node6.html (web). If *x* is rational, where $x = \frac{a}{b}$ $(a, b \in \mathbb{Z})$, $b > 0$, and the fraction is simplified, then let $f(x) = \frac{1}{b}$. If *x* is irrational, let $f(x) = 0$. Vogel calls this the "ruler function," appealing to the image of markings on a ruler. However, there does not exist a function that is continuous at every rational point and discontinuous at every irrational point.

7. An everywhere continuous function that is nowhere monotonic (either increasing or decreasing). See Gelbaum and Olmsted, *Counterexamples in Analysis* (Dover), p.29. There is no open interval on which the function described there is either increasing or decreasing.