CHAPTER 2:

Limits and Continuity

- The conventional approach to calculus is founded on limits.
- In this chapter, we will develop the concept of a limit by example.
- Properties of limits will be established along the way.
- We will use limits to analyze asymptotic behaviors of functions and their graphs.
- Limits will be formally defined near the end of the chapter.
- Continuity of a function (at a point and on an interval) will be defined using limits.

SECTION 2.1: AN INTRODUCTION TO LIMITS

LEARNING OBJECTIVES

- Understand the concept of (and notation for) a limit of a rational function at a point in its domain, and understand that "limits are local."
- Evaluate such limits.
- Distinguish between one-sided (left-hand and right-hand) limits and two-sided limits – and what it means for such limits to exist.
- Use numerical / tabular methods to guess at limit values.
- Distinguish between limit values and function values at a point.
- Understand the use of neighborhoods and punctured neighborhoods in the evaluation of one-sided and two-sided limits.
- Evaluate some limits involving piecewise-defined functions.

PART A: THE LIMIT OF A FUNCTION AT A POINT

Our study of calculus begins with an understanding of the expression $\lim_{x \to a} f(x)$, where a is a real number (in short, $a \in \mathbb{R}$) and f is a function. This is read as:

"the limit of $f(x)$ as x approaches a."

- **WARNING 1:** $\to$ means "approaches." Avoid using this symbol outside the context of limits.

- $\lim_{x \to a}$ is called a limit operator. Here, it is applied to the function f.

$\lim_{x \to a} f(x)$ is the real number that $f(x)$ approaches as x approaches a, **if such a number exists**. If $f(x)$ does, indeed, approach a real number, we denote that number by L (for limit value). We say the limit **exists**, and we write:

$$\lim_{x \to a} f(x) = L, \quad \text{or} \quad f(x) \to L \text{ as } x \to a.$$

These statements will be **rigorously defined** in Section 2.7.

When we **evaluate** $\lim_{x \to a} f(x)$, we do one of the following:

- We find the limit value L (in simplified form).

  We write: $\lim_{x \to a} f(x) = L$.

- We say the limit is ∞ (infinity) or $-\infty$ (negative infinity).

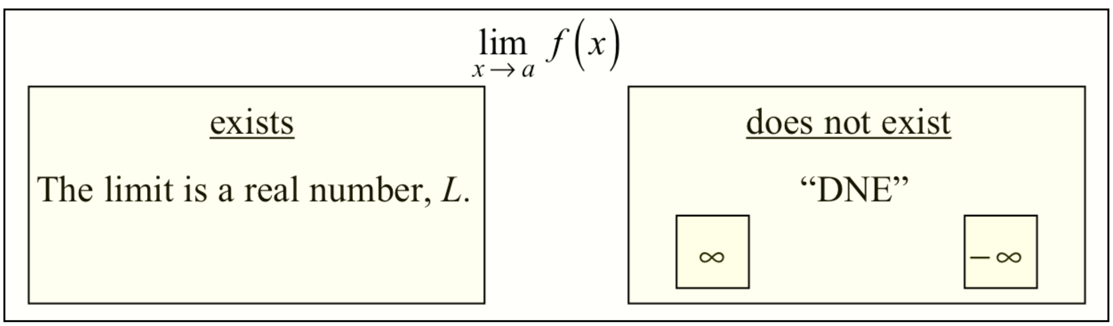
  We write: $\lim_{x \to a} f(x) = \infty$, or $\lim_{x \to a} f(x) = -\infty$.

- We say the limit **does not exist ("DNE")** in some other way.

  We write: $\lim_{x \to a} f(x)$ DNE.

  (The "DNE" notation is used by Swokowski but few other authors.)

If we say the limit is ∞ or $-\infty$, the limit is still **nonexistent**. Think of ∞ and $-\infty$ as "special cases of DNE" that we do write when appropriate; they indicate **why** the limit does not exist.



$\lim_{x \to a} f(x)$ is called a limit at a point, because $x = a$ corresponds to a **point** on the real number line. Sometimes, this is related to a point on the graph of f.

Example 1 (Evaluating the Limit of a Polynomial Function at a Point)

Let $f(x) = 3x^2 + x - 1$. Evaluate $\lim_{x \to 1} f(x)$.

§ Solution

f is a **polynomial** function with implied domain $\text{Dom}(f) = \mathbb{R}$.

We **substitute** ("plug in") $x = 1$ and evaluate $f(1)$.

WARNING 2: Sometimes, the **limit value** $\lim_{x \to a} f(x)$ does not equal the **function value** $f(a)$. (See Part C.)

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \left(3x^2 + x - 1\right)$$

WARNING 3: Use **grouping symbols** when taking the limit of an expression consisting of **more than one term**.

$$= 3(1)^2 + (1) - 1$$

WARNING 4: Do not omit the limit operator $\lim_{x \to 1}$ until this substitution phase.

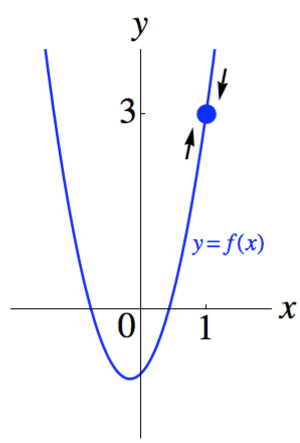
WARNING 5: When performing **substitutions**, be prepared to use **grouping symbols**. Omit them only if you are sure they are unnecessary.

$$= 3$$

We can write: $\lim_{x \to 1} f(x) = 3$, or $f(x) \to 3$ as $x \to 1$.

• Be prepared to work with function and variable names other than f and x. For example, if $g(t) = 3t^2 + t - 1$, then $\lim_{t \to 1} g(t) = 3$, also.

The graph of $y = f(x)$ is below.



Imagine that the arrows in the figure represent two lovers running towards each other along the parabola. What is the y-coordinate of the point they are approaching as they approach $x = 1$?
It is 3, the limit value.

TIP 1: Remember that ***y*-coordinates** of points along the graph correspond to **function values**. §

Example 2 (Evaluating the Limit of a Rational Function at a Point)

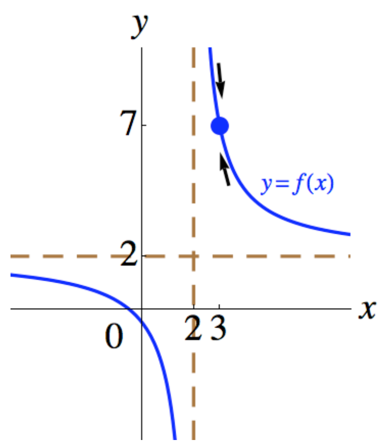
Let $f(x)=\dfrac{2x+1}{x-2}$. Evaluate $\lim\limits_{x\to 3} f(x)$.

§ Solution

f is a **rational** function with implied domain $\mathrm{Dom}(f)=\{x\in\mathbb{R}\,|\,x\neq 2\}$.
We observe that 3 is in the **domain** of f (in short, $3\in\mathrm{Dom}(f)$), so we **substitute** ("plug in") $x=3$ and evaluate $f(3)$.

$$\begin{aligned}\lim_{x\to 3} f(x) &= \lim_{x\to 3}\frac{2x+1}{x-2}\\ &= \frac{2(3)+1}{(3)-2}\\ &= 7\end{aligned}$$

The graph of $y=f(x)$ is below.



Note: As is often the case, you might not know how to draw the graph until later.

• **Asymptotes.** The dashed lines are asymptotes, which are lines that a graph approaches

- in a "long-run" sense
(see the horizontal asymptote, or "HA," at $y=2$), or

- in an "explosive" sense
(see the vertical asymptote, or "VA," at $x=2$).

"HA"s and "VA"s will be defined using limits in Sections 2.3 and 2.4, respectively.

• **"Limits are Local."** What if the lover on the left is running along the left branch of the graph? In fact, we ignore the left branch, because of the following key principle of limits.

"Limits [at a Point] are Local"

When analyzing $\lim_{x \to a} f(x)$, we only consider the behavior of f in the **"immediate vicinity"** of $x = a$.

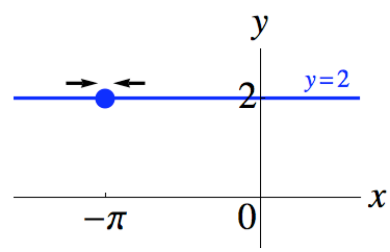
In fact, we may exclude consideration of $x = a$ itself, as we will see in Part C.

In the graph, we only care what happens **"immediately around"** $x = 3$. Section 2.7 will feature a rigorous approach. §

Example 3 (Evaluating the Limit of a Constant Function at a Point)

$$\lim_{x \to -\pi} 2 = 2.$$

(Observe that substituting $x = -\pi$ technically works here, since there is no "x" in "2," anyway.)



• **A constant approaches itself.** We can write $2 \to 2$ ("2 approaches 2") as $x \to -\pi$. When we think of a sequence of numbers approaching 2, we may think of distinct numbers such as 2.1, 2.01, 2.001, …. However, the **constant sequence** 2, 2, 2, … is also said to approach 2. §

All **constant** functions are also **polynomial** functions, and all **polynomial** functions are also **rational** functions. The following theorem applies to all three Examples thus far.

Basic Limit Theorem for Rational Functions

If f is a rational function, and $a \in \text{Dom}(f)$, then $\lim_{x \to a} f(x) = f(a)$.

• To evaluate the limit, substitute ("plug in") $x = a$, and evaluate $f(a)$.

We will justify this theorem in Section 2.2.

PART B: ONE- AND TWO-SIDED LIMITS; EXISTENCE OF LIMITS

$\lim_{x \to a}$ is a **two-sided** limit operator in $\lim_{x \to a} f(x)$, because we must consider the behavior of f as x approaches a from **both** the left **and** the right.

$\lim_{x \to a^-}$ is a **one-sided** left-hand limit operator. $\lim_{x \to a^-} f(x)$ is read as:

"the limit of $f(x)$ as x approaches a **from the left**."

$\lim_{x \to a^+}$ is a **one-sided** right-hand limit operator. $\lim_{x \to a^+} f(x)$ is read as:

"the limit of $f(x)$ as x approaches a **from the right**."

Example 4 (Using a Numerical / Tabular Approach to Guess a Left-Hand Limit Value)

Guess the value of $\lim_{x \to 3^-} (x+3)$ using a **table** of function values.

§ Solution

Let $f(x) = x+3$. $\lim_{x \to 3^-} f(x)$ is the real number, if any, that $f(x)$ approaches as x approaches 3 from **lesser (or lower) numbers**. That is, we approach $x = 3$ from the **left** along the real number line.

We select an **increasing** sequence of real numbers (x values) approaching 3 such that all the numbers are **close to (but less than) 3**. We evaluate the function at those numbers, and we **guess** the limit value, if any, the function values are approaching. For example:

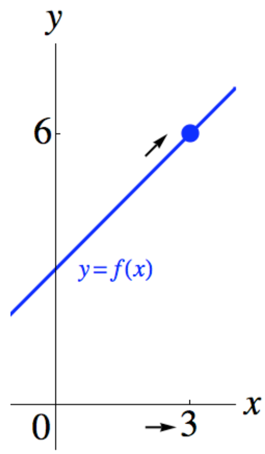
x	2.9	2.99	2.999	$\to 3^-$
$f(x) = x+3$	5.9	5.99	5.999	$\to 6$ (?)

We guess: $\lim_{x \to 3^-} (x+3) = 6$.

WARNING 6: Do not confuse superscripts with signs of numbers. Be careful about associating the "$-$" superscript with negative numbers. Here, we consider **positive** numbers that are close to 3.

- If we were taking a limit as ***x* approached 0**, then we would associate the "$-$" superscript with **negative** numbers and the "+" superscript with **positive** numbers.

The graph of $y = f(x)$ is below. We only consider the behavior of f **"immediately" to the left of** $x = 3$.



WARNING 7: The numerical / tabular approach is **unreliable**, and it is typically **unacceptable** as a method for evaluating limits on exams. (See Part D, Example 11 to witness a failure of this method.) However, it may help us guess at limit values, and it strengthens our understanding of limits. §

Example 5 (Using a Numerical / Tabular Approach to Guess a Right-Hand Limit Value)

Guess the value of $\lim_{x \to 3^+} (x+3)$ using a **table** of function values.

§ Solution

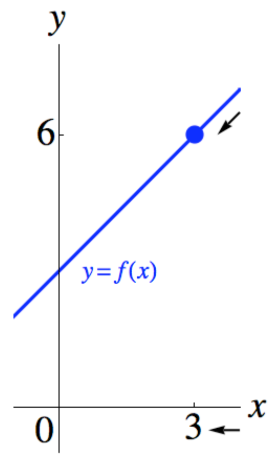
Let $f(x) = x + 3$. $\lim_{x \to 3^+} f(x)$ is the real number, if any, that $f(x)$ approaches as x approaches 3 from **greater (or higher) numbers**. That is, we approach $x = 3$ from the **right** along the real number line.

We select a **decreasing** sequence of real numbers (x values) approaching 3 such that all the numbers are **close to (but greater than) 3**. We evaluate the function at those numbers, and we **guess** the limit value, if any, the function values are approaching. For example:

x	$3^+ \leftarrow$	3.001	3.01	3.1
$f(x) = x+3$	6 (?) $\leftarrow$	6.001	6.01	6.1

We guess: $\lim_{x \to 3^+} (x+3) = 6$.

The graph of $y = f(x)$ is below. We only consider the behavior of f **"immediately" to the right of** $x = 3$.



§

<u>Existence of a Two-Sided Limit at a Point</u>

$$\lim_{x \to a} f(x) = L \Leftrightarrow \left[\lim_{x \to a^-} f(x) = L, \text{ and } \lim_{x \to a^+} f(x) = L \right], \quad (a, L \in \mathbb{R}).$$

• A two-sided limit **exists** $\Leftrightarrow$ the corresponding left-hand and right-hand limits **exist**, and they are **equal**.

• If either one-sided limit **does not exist (DNE)**, or if the two one-sided limits are **unequal**, then the two-sided limit **does not exist (DNE)**.

Our guesses, $\lim_{x \to 3^-} (x+3) = 6$ and $\lim_{x \to 3^+} (x+3) = 6$, imply $\lim_{x \to 3} (x+3) = 6$.

In fact, all three limits can be evaluated by **substituting** $x = 3$ into $(x+3)$:

$$\lim_{x \to 3^-} (x+3) = 3+3 = 6; \ \lim_{x \to 3^+} (x+3) = 3+3 = 6; \ \lim_{x \to 3} (x+3) = 3+3 = 6.$$

This procedure is generalized in the following theorem.

<u>Extended Limit Theorem for Rational Functions</u>

If f is a rational function, and $a \in \text{Dom}(f)$,

then $\lim_{x \to a^-} f(x) = f(a)$, $\lim_{x \to a^+} f(x) = f(a)$, and $\lim_{x \to a} f(x) = f(a)$.

• To evaluate each limit, substitute ("plug in") $x = a$, and evaluate $f(a)$.

WARNING 8: Substitution might not work if f is not a rational function.

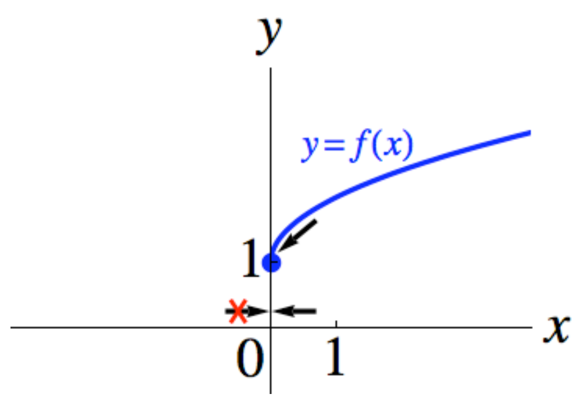
Example 6 (Pitfalls of Substituting into a Function that is Not Rational)

Let $f(x) = \sqrt{x} + 1$. Evaluate $\lim_{x \to 0^+} f(x)$, $\lim_{x \to 0^-} f(x)$, and $\lim_{x \to 0} f(x)$.

§ Solution

Observe that $\text{Dom}(f) = \{x \in \mathbb{R} \mid x \geq 0\} = [0, \infty)$, because $\sqrt{x}$ is **real** when $x \geq 0$, but it is **not real** when $x < 0$.

This is important, because x is only allowed to approach 0 (or whatever a is) **through** $\text{Dom}(f)$. Here, x is allowed to approach 0 from the right but **not** from the left.



Right-Hand Limit: $\lim_{x \to 0^+} f(x) = 1$.

Substituting $x = 0$ works: $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\sqrt{x} + 1\right) = \sqrt{0} + 1 = 1$.

Left-Hand Limit: $\lim_{x \to 0^-} f(x)$ does not exist (DNE).

Substituting $x = 0$ **does not work** here.

Two-Sided Limit: $\lim_{x \to 0} f(x)$ does not exist (DNE).

This is because the corresponding left-hand limit does not exist (DNE).

Observe that f is **not** a rational function, so the aforementioned theorem does **not** apply, even though $0 \in \text{Dom}(f)$. f is, however, an **algebraic** function, and we will discuss algebraic functions in Section 2.2. §

PART C: IGNORING THE FUNCTION AT *a*

Example 7 (Ignoring the Function at 'a' When Evaluating a Limit; Modifying Examples 4 and 5)

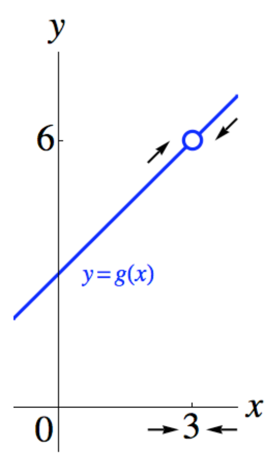
Let $g(x) = x + 3,\ (x \neq 3)$.

(We are deleting 3 from the domain of the function in Examples 4 and 5; this changes the function.)

Evaluate $\lim_{x \to 3^-} g(x)$, $\lim_{x \to 3^+} g(x)$, and $\lim_{x \to 3} g(x)$.

§ Solution

Since $3 \notin \text{Dom}(g)$, we must delete the point $(3, 6)$ from the graph of $y = x + 3$ to obtain the graph of *g* below.



We say that *g* has a removable discontinuity at $x = 3$ (see Section 2.8), and the graph of *g* has a hole at the point $(3, 6)$.

Observe that, as *x* approaches 3 from the left **and** from the right, $g(x)$ **approaches** 6, even though $g(x)$ never equals 6.

$g(3)$ is undefined, yet the following statements are true:

$$\lim_{x \to 3^-} g(x) = 6,$$
$$\lim_{x \to 3^+} g(x) = 6, \text{ and}$$
$$\lim_{x \to 3} g(x) = 6.$$

There literally **does not have to be a point** at $x = 3$ (in general, $x = a$) for these limits to exist! Observe that substituting $x = 3$ into $(x + 3)$ works. §

Example 8 (Ignoring the Function at 'a' When Evaluating a Limit; Modifying Example 7)

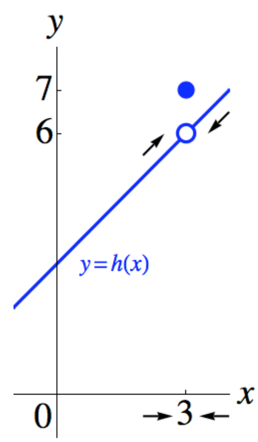
Let the function h be defined **piecewise** as follows: $h(x)=\begin{cases} x+3, & x\neq 3 \\ 7, & x=3 \end{cases}$.

(A piecewise-defined function applies different evaluation rules to different subsets of (groups of numbers in) its domain. This type of function can lead to interesting limit problems.)

Evaluate $\lim_{x\to 3} h(x)$.

§ Solution

h is identical to the function g from Example 7, except that $3\in \mathrm{Dom}(h)$, and $h(3)=7$. As a result, we must add the point $(3,7)$ to the graph of g to obtain the graph of h below.



As with g, h also has a **removable discontinuity** at $x=3$, and its graph also has a **hole** at the point $(3,6)$.

Observe that, as x approaches 3 from the left **and** from the right, $h(x)$ also **approaches** 6.

$\lim_{x\to 3} h(x)=6$ once again, even though $h(3)=7$.

WARNING 2 repeat (applied to f): Sometimes, the **limit value** $\lim_{x\to a} f(x)$ does not equal the **function value** $f(a)$. §

As in Example 7, observe that substituting $x=3$ into $(x+3)$ works. §

The existence (or value) of $\lim_{x \to a} f(x)$ **need not** depend on the existence (or value) of $f(a)$.

• Sometimes, it **does help** to know what $f(a)$ is when evaluating $\lim_{x \to a} f(x)$.
In Section 2.8, we will say that f is <u>continuous</u> at $a \Leftrightarrow \lim_{x \to a} f(x) = f(a)$, provided that $\lim_{x \to a} f(x)$ and $f(a)$ exist. We appreciate **continuity**, because we can then simply **substitute** $x = a$ to evaluate a limit, which was what we did when we applied the **Basic Limit Theorem for Rational Functions** in Part A.

• In Examples 7 and 8, we dealt with functions that were **not** continuous at $x = 3$, yet **substituting** $x = 3$ into $(x+3)$ allowed us to evaluate the one- and two-sided limits at $a = 3$. We will develop theorems that cover these Examples. We first need the following definitions.

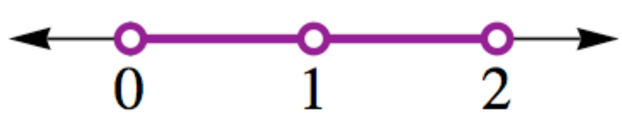
A <u>neighborhood of *a*</u> is an **open interval** along the real number line that is **symmetric** about *a*.

For example, the interval $(0, 2)$ is a **neighborhood** of 1. Since 1 is the **midpoint** of $(0, 2)$, the neighborhood is **symmetric** about 1.

A <u>punctured (or deleted) neighborhood of *a*</u> is constructed by taking a neighborhood of *a* and **deleting** *a* itself.

For example, the set $(0, 2) \setminus \{1\}$, which can be written as $(0, 1) \cup (1, 2)$, is a **punctured neighborhood** of 1. It is a set of numbers that are **"immediately around"** 1 on the real number line.

• The notation $(0, 2) \setminus \{1\}$ indicates that we can construct it by taking the **neighborhood** $(0, 2)$ and **deleting** 1.



“Puncture Theorem” for Limits of Locally Rational Functions

Let r be a rational function, and let $a \in \text{Dom}(r)$.
Let $f(x) = r(x)$ on a punctured neighborhood of $x = a$.
Then, $\lim_{x \to a} f(x) = \lim_{x \to a} r(x) = r(a)$.

• To evaluate the limits, substitute (“plug in”) $x = a$ into $r(x)$, and evaluate $r(a)$.

• That is, if a function rule is given by a **rational** expression $r(x)$ **locally (immediately) around** $x = a$, where $a \in \text{Dom}(r)$, then **evaluate** the rational expression **at** a to obtain the **limit** of the function at a.

Refer to Examples 7 and 8. Let $r(x) = x + 3$. Observe that r is a rational function, and $3 \in \text{Dom}(r)$. Both the g and h functions were defined by $x + 3$ **locally (immediately) around** $x = 3$. More precisely, they were defined by $x + 3$ on some **punctured neighborhood** of $x = 3$, say $(2.9, 3.1) \setminus \{3\}$. Therefore,

$$\lim_{x \to 3} g(x) = \lim_{x \to 3} r(x) = r(3) = 3 + 3 = 6, \text{ and}$$

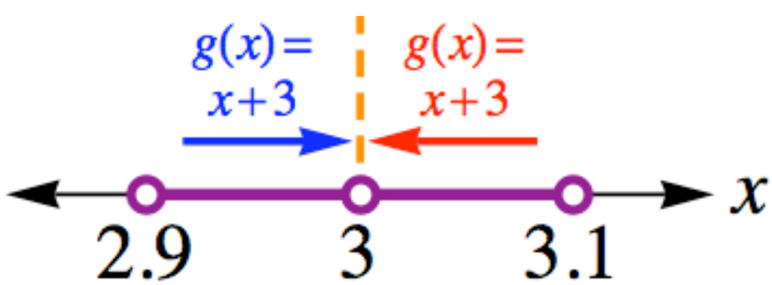
$$\lim_{x \to 3} h(x) = \lim_{x \to 3} r(x) = r(3) = 3 + 3 = 6.$$

It is easier to write:

$$\lim_{x \to 3} g(x) = \lim_{x \to 3} (x + 3) = 3 + 3 = 6, \text{ and}$$

$$\lim_{x \to 3} h(x) = \lim_{x \to 3} (x + 3) = 3 + 3 = 6.$$

The figure below refers to g, but it also applies to h.
The dashed line segment at $x = 3$ reiterates the **puncture** there.



Why does the theorem only require that a function be **locally** rational about *a*? Consider the following Example.

Example 9 (Limits are Local)

Let $f(t)=\begin{cases} t+2, & t<0 \\ \sqrt{t}, & t\geq 0 \end{cases}$. Evaluate $\lim_{t\to -1} f(t)$.

§ Solution

Observe that $f(t)=t+2$ is the **only** rule that is relevant as *t* approaches -1 **locally** from the left **and** from the right. We only consider values of *t* that are **"immediately around"** $a=-1$. **"Limits are Local!"**

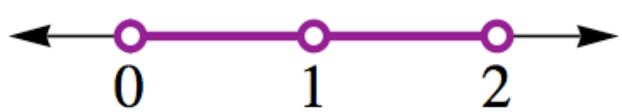
It is **irrelevant** that the rule $f(t)=\sqrt{t}$ is different, or that it is not rational. §

The following definitions will prove helpful in our study of **one-sided limits**.

A left-neighborhood of *a* is an **open interval** of the form (c,a), where $c<a$.
A right-neighborhood of *a* is an **open interval** of the form (a,c), where $c>a$.
A **punctured neighborhood** of *a* consists of **both** a left-neighborhood of *a* **and** a right-neighborhood of *a*.

For example, the interval $(0,1)$ is a **left-neighborhood** of 1. It is a set of numbers that are **"immediately to the left"** of 1 on the real number line.

The interval $(1,2)$ is a **right-neighborhood** of 1. It is a set of numbers that are **"immediately to the right"** of 1 on the real number line.



We now modify the "Puncture Theorem" for **one-sided limits**.

• Basically, when evaluating a **left-hand limit** such as $\lim_{x\to a^-} f(x)$, we use the function rule that governs the *x*-values **"immediately to the left"** of *a* on the real number line.

• Likewise, when evaluating a **right-hand limit** such as $\lim_{x\to a^+} f(x)$, we use the rule that governs the *x*-values **"immediately to the right"** of *a*.

Variation of the "Puncture Theorem" for Left-Hand Limits

Let r be a rational function, and let $a \in \text{Dom}(r)$.
Let $f(x) = r(x)$ on a left-neighborhood of $x = a$.
Then, $\lim_{x \to a^-} f(x) = \lim_{x \to a^-} r(x) = r(a)$.

Variation of the "Puncture Theorem" for Right-Hand Limits

Let r be a rational function, and let $a \in \text{Dom}(r)$.
Let $f(x) = r(x)$ on a right-neighborhood of $x = a$.
Then, $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} r(x) = r(a)$.

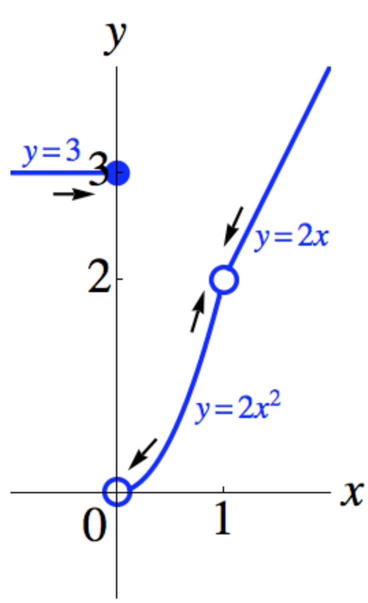
Example 10 (Evaluating One-Sided and Two-Sided Limits of a Piecewise-Defined Function)

Let $f(x) = \begin{cases} 3, & \text{if } x \le 0 \\ 2x^2, & \text{if } 0 < x < 1 \\ 2x, & \text{if } x > 1 \end{cases}$.

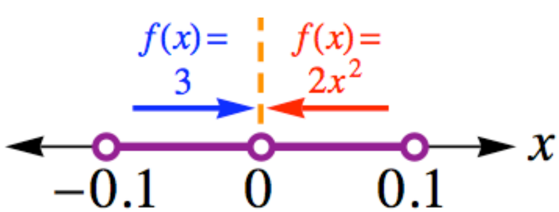
Evaluate the one-sided and two-sided limits of f at 1 and at 0.

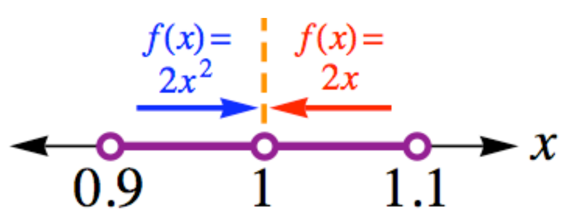
§ Solution

The graph of $y = f(x)$ is below. It helps, but it is **not** required to evaluate limits. Instead, we can evaluate limits of **relevant** function rules.



$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x^2$ $= 2(1)^2$ $= 2$	The left-hand limit as $x \to 1^-$: We use the rule $f(x) = 2x^2$, because it applies to a **left-neighborhood** of 1, say $(0.9, 1)$.
$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x$ $= 2(1)$ $= 2$	The right-hand limit as $x \to 1^+$: We use the rule $f(x) = 2x$, because it applies to a **right-neighborhood** of 1, say $(1, 1.1)$.
$\lim_{x\to 1} f(x) = 2$	The two-sided limit as $x \to 1$: The left-hand and right-hand limits at 1 **exist**, and they are **equal**, so the two-sided limit exists and equals their common value.
$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3$ $= 3$	The left-hand limit as $x \to 0^-$: We use the rule $f(x) = 3$, because it applies to a **left-neighborhood** of 0, say $(-0.1, 0)$.
$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2x^2$ $= 2(0)^2$ $= 0$	The right-hand limit as $x \to 0^+$: We use the rule $f(x) = 2x^2$, because it applies to a **right-neighborhood** of 0, say $(0, 0.1)$.
$\lim_{x\to 0} f(x)$ does not exist (DNE)	The two-sided limit as $x \to 0$: The left-hand and right-hand limits at 0 **exist**, but they are **unequal**, so the two-sided limit does not exist (DNE).





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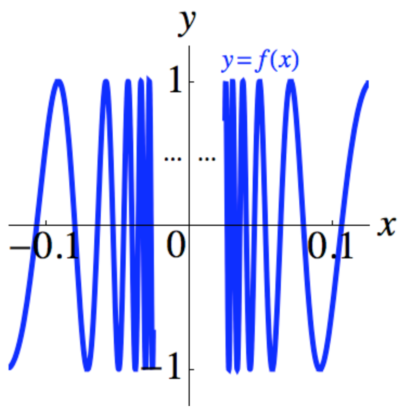
PART D: NONEXISTENT LIMITS

Example 11 (Nonexistent Limits)

Let $f(x) = \sin\left(\frac{1}{x}\right)$. Evaluate $\lim_{x \to 0^+} f(x)$, $\lim_{x \to 0^-} f(x)$, and $\lim_{x \to 0} f(x)$.

§ Solution

The graph of $y = f(x)$ is below. Ask your instructor if s/he might have you even attempt to draw this. In a sense, the classic sine wave is being turned "inside out" relative to the *y*-axis.



As *x* approaches 0 from the right (or from the left), the function values **oscillate** between −1 and 1. They do **not** approach a **single real number**. Therefore,

$\lim_{x \to 0^+} f(x)$ does not exist (DNE),

$\lim_{x \to 0^-} f(x)$ does not exist (DNE), and

$\lim_{x \to 0} f(x)$ does not exist (DNE).

<u>Note 1</u>: The *y*-axis is **not a vertical asymptote (VA)** here, because the graph and the function values are **not "exploding" without bound** around the *y*-axis.

<u>Note 2</u>: Here is an example of how the **numerical / tabular approach** introduced in Part B **might lead us astray**:

x	$0^+ \leftarrow$	$\frac{1}{3\pi}$	$\frac{1}{2\pi}$	$\frac{1}{\pi}$
$f(x) = \sin\left(\frac{1}{x}\right)$	0 (?) $\leftarrow$ NO!	0	0	0

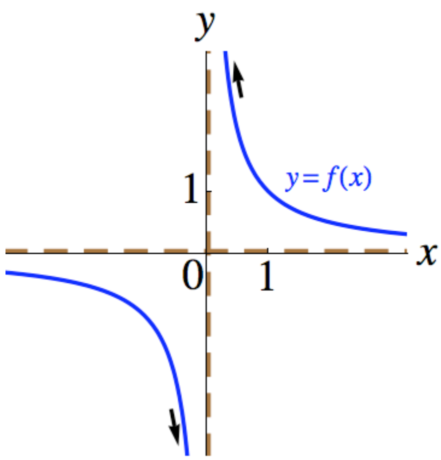
§

Example 12 (Infinite and/or Nonexistent Limits)

Let $f(x) = \frac{1}{x}$. Evaluate $\lim_{x \to 0^+} f(x)$, $\lim_{x \to 0^-} f(x)$, and $\lim_{x \to 0} f(x)$.

§ Solution

The graph of $y = f(x)$ is below. We will discuss this graph in later sections.



As x approaches 0 from the **right**, the function values **increase without bound**.

Therefore, $\lim_{x \to 0^+} f(x) = \infty$.

As x approaches 0 from the **left**, the function values **decrease without bound**.

Therefore, $\lim_{x \to 0^-} f(x) = -\infty$.

∞ and $-\infty$ are **mismatched**.

Therefore, $\lim_{x \to 0} f(x)$ does not exist (DNE).

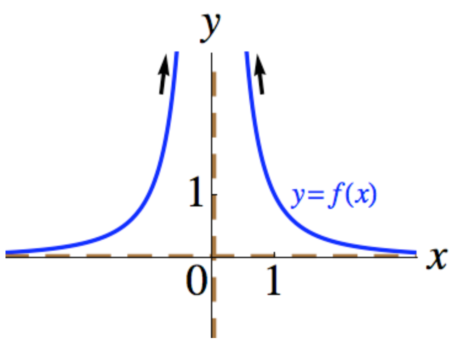
In fact, all three limits **do not exist**. For example, $\lim_{x \to 0^+} f(x)$, **does not exist**, because the function values **do not approach a single real number** as x approaches 0 from the right. The expressions ∞ and $-\infty$ indicate **why** the one-sided limits do not exist, and we write ∞ and $-\infty$ where appropriate. §

Example 13 (Infinite and Nonexistent Limits)

Let $f(x) = \frac{1}{x^2}$. Evaluate $\lim_{x \to 0^+} f(x)$, $\lim_{x \to 0^-} f(x)$, and $\lim_{x \to 0} f(x)$.

§ Solution

The graph of $y = f(x)$ is below. Observe that f is an even function.



$\lim_{x \to 0^+} f(x) = \infty$,

$\lim_{x \to 0^-} f(x) = \infty$, and

$\lim_{x \to 0} f(x) = \infty$. §

Example 14 (A Nonexistent Limit)

Let $f(x) = \dfrac{|x|}{x}$. Evaluate $\lim_{x \to 0^+} f(x)$, $\lim_{x \to 0^-} f(x)$, and $\lim_{x \to 0} f(x)$.

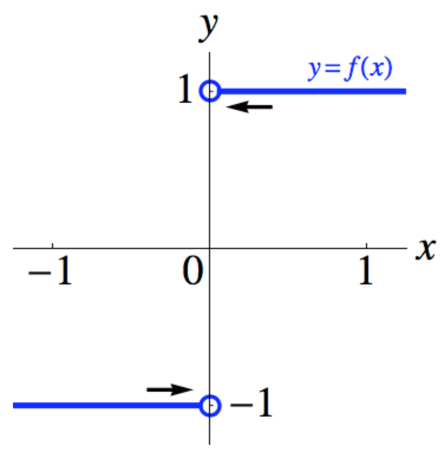
§ Solution

Note: f is **not** a rational function, but it is an **algebraic function**, since $f(x) = \dfrac{|x|}{x} = \dfrac{\sqrt{x^2}}{x}$.

Remember that: $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$.

Then, $f(x) = \dfrac{|x|}{x} = \begin{cases} \dfrac{x}{x} = 1, & \text{if } x > 0 \\ \dfrac{-x}{x} = -1, & \text{if } x < 0 \end{cases}$, and $f(0)$ is undefined.

The graph of $y = f(x)$ is below.



$\lim_{x \to 0^+} f(x) = 1$,

$\lim_{x \to 0^-} f(x) = -1$, and

$\lim_{x \to 0} f(x)$ does not exist (DNE),

due to the fact that the right-hand and left-hand limits are **unequal**. §

FOOTNOTES

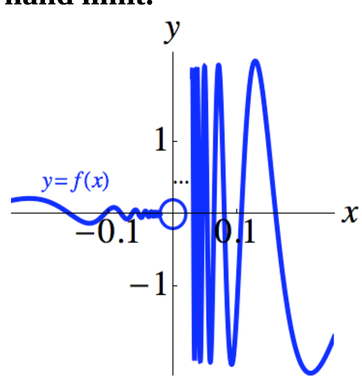
1. **Limits do not require continuity.** In Section 2.8, we will discuss continuity, a property of functions that helps our lovers run along the graph of a function without having to jump or hop. In Exercises 1-3, we could imagine the lovers running towards each other (one from the left, one from the right) while staying on the graph of f and without having to jump or hop, provided they were placed on appropriate parts of the graph. Sometimes, the "run" requires jumping or hopping. Let $f(x) = \begin{cases} 0, & \text{if } x \text{ is a rational number } (x \in \mathbb{Q}) \\ x, & \text{if } x \text{ is an irrational number } (x \notin \mathbb{Q}; \text{ really, } x \in \mathbb{R} \setminus \mathbb{Q}) \end{cases}$.
   It turns out that $\lim_{x \to 0} f(x) = 0$.

2. **Misconceptions about limits.**
   See "Why Is the Limit Concept So Difficult for Students?" by Sally Jacobs in the Fall 2002 edition (vol.24, No.1) of *The AMATYC Review*, pp.25-34.
   • Students can be misled by the use of the word "limit" in real-world contexts. For example, a speed limit is a bound that is not supposed to be exceeded; there is no such restriction on limits in calculus.
   • Limit values can sometimes be attained. For example, if a function f is continuous at $x = a$ (see Examples 1-3), then the function value takes on the limit value at $x = a$.
   • Limit values do not have to be attained. See Examples 7 and 8.

   Observations:
   • The dynamic view of limits, which involves ideas of motion and "approaching" (for example, our lovers), may be more accessible to students than the static view preferred by many textbook authors. The static view is exemplified by the formal definitions of limits we will see in Section 2.7. The dynamic view greatly assists students in transitioning to the static view and the formal definitions.
   • Leading mathematicians in 18$^{\text{th}}$- and 19$^{\text{th}}$-century Europe heatedly debated ideas of limits.

3. **Multivariable calculus.** When we go to higher dimensions, there may be more than two possible approaches (not just left-hand and right-hand) when analyzing limits at a point! Neighborhoods can take the form of disks or balls.

4. **An example where a left-hand limit exists but not the right-hand limit.**
   Let $f(x) = \dfrac{x + |x|(1+x)}{x} \sin\left(\dfrac{1}{x}\right) = \begin{cases} -x\sin\left(\dfrac{1}{x}\right), & \text{if } x < 0 \\ (2+x)\sin\left(\dfrac{1}{x}\right), & \text{if } x > 0 \end{cases}$.



   Then, $\lim_{x \to 0^-} f(x) = 0$, which can be proven by the Squeeze (Sandwich) Theorem in Section 2.6. However, $\lim_{x \to 0^+} f(x)$ does not exist (DNE).
   See William F. Trench, *Introduction to Real Analysis* (free online at: http://ramanujan.math.trinity.edu/wtrench/texts/TRENCH_REAL_ANALYSIS.PDF), p.39.