

Calculus Limits

Images in this handout were obtained from the My Math Lab Briggs online e-book.

A limit is the value a function approaches as the input value gets closer to a specified quantity. Limits are used to define continuity, derivatives, and integrals. This handout focuses on determining limits analytically and determining limits by looking at a graph.

Determining a Limit Analytically

There are many methods to determine a limit analytically, and they are usually used in succession. First, see if the limit can be evaluated by direct substitution. Second, if direct substitution yields an undefined result, factor and reduce the fraction or multiply by the conjugate. Third, if the second method does not work, find the left and right sided limits.

- **Evaluate by Direct Substitution:** Take the value of the limit and evaluate the function at this value. There are two situations where direct substitution will be used, direct substitution with a numerical value and direct substitution with infinity.

Example 1: Limits at a Numerical Value.

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{\sqrt{2x^3 + 9} + 3x - 1}{4x + 1} &= \\ \frac{\sqrt{2(2)^3 + 9} + 3(2) - 1}{4(2) + 1} &= \\ \frac{\sqrt{25} + 5}{9} &= \frac{10}{9}\end{aligned}$$

Example 2: Limits at Infinity. Limits at infinity are solved by plugging in ∞ or $-\infty$ into the function for the given variable. When determining limits at infinity, think more about the trends of the function at infinity rather than the math.

$$\begin{aligned}\lim_{x \rightarrow -\infty} \left(2 + \frac{10}{x^2}\right) \\ 2 + \frac{10}{(-\infty)^2} = 2 + \frac{10}{\infty}\end{aligned}$$

Think about the decimal value of a fraction with a large number in the denominator. What is the trend? As the denominator gets larger, the fraction as a whole gets smaller until it ultimately reaches zero. Evaluating using arbitrary large numbers for infinity will show this trend:

$$\frac{10}{100} = 0.1$$

$$\frac{10}{100,000} = 0.0001$$

Therefore, it can be said that a constant numerator divided by infinity is equal to zero.

$$2 + 0 = 2$$

$$\lim_{x \rightarrow -\infty} \left(2 + \frac{10}{x^2} \right) = 2$$

Some Common Trends at Infinity		
$\frac{n}{\infty} = 0$	$\frac{\infty}{n} = \infty$ Where $n \neq 0$	$\frac{a(\infty)^n}{b(\infty)^n} = \frac{a}{b}$ Where $b \neq 0$
$(-\infty)^n = -\infty$ When n is odd and $n > 0$	$(\pm\infty)^n = \infty$ When n is even and $n > 0$	$(\pm\infty)^{-n} = 0$ Where $n > 0$

- **Factor and Reduce or Multiply by the Conjugate:** Use when the first method yields an undefined solution.

Example 1: Factoring and reducing.

$$\lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2}$$

If x were simply evaluated at 4 as shown in the first method, it would yield a zero in the denominator; therefore, the slope is undefined. One way to avoid this is to factor the numerator and denominator if applicable.

$$\lim_{x \rightarrow 4} \frac{(\sqrt{x} + 2)(\sqrt{x} - 2)}{\sqrt{x} - 2}$$

Reduce by $(\sqrt{x} - 2)$.

$$\lim_{x \rightarrow 4} (\sqrt{x} + 2)$$

Now, if x is evaluated at 4, the equation will not yield an undefined slope.

$$\sqrt{4} + 2 = 4$$

$$\lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2} = 4$$

Example 2: Multiplying by the Conjugate.

$$\lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2} * \frac{\sqrt{x} + 2}{\sqrt{x} + 2} = \lim_{x \rightarrow 4} \frac{2x - 8 + x\sqrt{x} - 4\sqrt{x}}{x - 4} =$$

$$\lim_{x \rightarrow 4} \frac{2(x - 4) + \sqrt{x}(x - 4)}{x - 4} = \lim_{x \rightarrow 4} \frac{(2 + \sqrt{x})(x - 4)}{x - 4}$$

Reduce by $(x - 4)$

$$2 + \sqrt{x} = 2 + \sqrt{4} = 4$$

$$\lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2} = 4$$

- Left and Right Sided Limits:** It is possible to determine the limit of a function that has an undefined answer by determining the left and right side of that limit. The left sided limit is denoted with a minus sign in the exponent, for example 3^- , because the limit is being approached from the negative side. The right sided limit is denoted with a plus sign in the exponent, for example 3^+ , because the limit is being approached from the positive side. Some common examples of functions that will need to be solved with left and right sided limits are piecewise functions and infinite limits.

Example 1: Piecewise functions.

$$f(x) = \begin{cases} -2x + 4, & x \leq 1 \\ \sqrt{x-1}, & x > 1 \end{cases}$$

To find the limit as x approaches 1 from the left side, the first equation must be used because it defines the function at values less than and equal to one. Since the equation will not yield an undefined result, direct substitution can be used.

$$\lim_{x \rightarrow 1^-} (-2x + 4) = -2(1) + 4 = 2$$

To find the limit as x approaches 1 from the right side, the second equation must be used because it defines the function at values greater than one. Since the equation will not yield an undefined result, direct substitution can be used.

$$\lim_{x \rightarrow 1^+} \sqrt{x-1} = \sqrt{(1)-1} = 0$$

The limit of this function does not exist (DNE) because the values for the left and right sided limits as x approaches 1 yields two different answers.

$$\lim_{x \rightarrow 1} \begin{cases} -2x + 4, & x \leq 1 \\ \sqrt{x-1}, & x > 1 \end{cases} = DNE$$

Example 2: Infinite Limits. Sometimes when computing limits, an answer of ∞ or $-\infty$ will be reached, resulting in an infinite limit.

$$\lim_{x \rightarrow 1} \frac{x-2}{(x-1)^2(x-3)}$$

Direct substitution will yield a denominator of zero and the function is already reduced to its simplest form so left and right sided limits must be used. The graph of this function is shown on the next page in Figure 2.27.

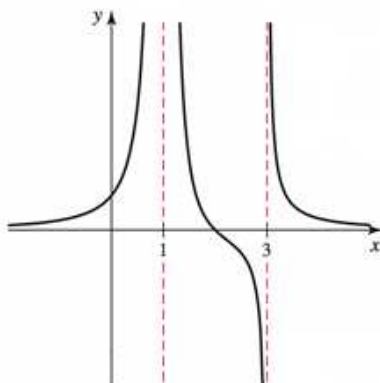


FIGURE 2.27

Figure 2.27 obtained from: http://media.pearsoncmg.com/aw/aw_briggs_calculus_et_1/ebook/bccalcet01_0204.nbp

To evaluate the left and right sided limits, evaluate the function for values very close to the limit. Think about the decimal value of a fraction with a small number in the denominator. What is the trend? As the denominator gets smaller the fraction as a whole gets larger until it ultimately reaches infinity. Evaluating using numbers close to the limit will show this trend:

To evaluate the limit as x approached one from the right, try evaluating with a number slightly greater than one.

$$\lim_{x \rightarrow 1^+} f(1.01) = \frac{(1.01) - 2}{[(1.01) - 1]^2[(1.01) - 3]} = \frac{-0.99}{(0.0001)(-1.99)} \approx 4,975$$

Now, try evaluating with a number closer to 1.

$$\lim_{x \rightarrow 1^+} f(1.000001) = \frac{(1.000001) - 2}{[(1.000001) - 1]^2[(1.000001) - 3]} \approx 502,512,060,300$$

To evaluate the limit as x approaches one from the left, try evaluating with a number slightly less than 1.

$$\lim_{x \rightarrow 1^-} f(0.9) = \frac{(0.9) - 2}{[(0.9) - 1]^2[(0.9) - 3]} \approx 52.381$$

Now, try evaluating with a number closer to 1.

$$\lim_{x \rightarrow 1^-} f(0.999999) = \frac{(0.999999) - 2}{[(0.999999) - 1]^2[(0.999999) - 3]} \approx 500,000,250,000$$

As the denominator gets closer to zero, the function becomes larger and gets closer to infinity. Therefore: $\lim_{x \rightarrow 1} = \infty$ since both the left and the right side of the limit approach infinity.

Determining a Limit Graphically

There are two types of conditions to be aware of when determining limits graphically, areas where a function is continuous and areas where a function is discontinuous. Suppose the following graph is used to determine various limits.

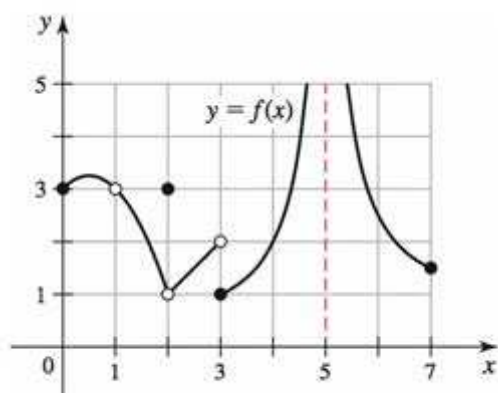


FIGURE 2.45

Figure 2.45 obtained from: http://media.pearsoncmg.com/aw/aw_briggs_calculus_et_1/ebook/bccalcet01_0206.nbp

- **Continuous Limits**

A function is continuous if the graph contains no abrupt changes in x and y values (i.e. no holes, asymptotes, jumps, or breaks).

Using Figure 2.45, look at the limit as x approaches 4. The graph does not have any holes or asymptotes at $x = 4$, therefore a limit exists and is equal to the y value of the function.

$$\lim_{x \rightarrow 4} f(x) = 2$$

- **Discontinuous Limits**

A function is discontinuous if there is an abrupt change in x and y values (i.e. holes, asymptotes, jumps, and/or breaks exist).

Example 1: The graph as x approaches 1 is discontinuous because there is a hole at $x = 1$ and therefore no value. However, evaluating the left and right sided limits will determine if a limit still exists at $x = 1$. Looking at the graph, as x approaches 1 from the left side the function approaches 3.

$$\lim_{x \rightarrow 1^-} f(x) = 3$$

Looking at the graph as x approaches 1 from the right side, the function approaches 3.

$$\lim_{x \rightarrow 1^+} f(x) = 3$$

Since both the left-sided and right-sided limits have the same value, a limit exists for this function.

$$\lim_{x \rightarrow 1} f(x) = 3$$

Example 2: Sometimes a discontinuity will not result in a limit. Using Figure 2.45 again, look at the limit as x approaches 3. As x approaches 3 from the left side, the function approaches 2.

$$\lim_{x \rightarrow 3^-} f(x) = 2$$

As x approaches 3 from the right side, the function (y value) approaches 1.

$$\lim_{x \rightarrow 3^+} f(x) = 1$$

The function does not have a limit at $x = 3$ because the left-sided and right-sided limits have different values at $x = 3$.

$$\lim_{x \rightarrow 3} f(x) = DNE$$

Example 3: Again, using Figure 2.45, look at the limit as x approaches 5. Since there is an asymptote at $x = 5$, the graph is discontinuous at $x = 5$. Left-sided and right-sided limits will have to be used to determine if there is a limit. As x approaches 5 from the left side, the function approaches positive infinity.

$$\lim_{x \rightarrow 5^-} f(x) = +\infty$$

As x approaches 5 from the left side, the function approaches positive infinity.

$$\lim_{x \rightarrow 5^+} f(x) = +\infty$$

Since both the left-sided and right-sided limits have the same value, a limit exists for this function.

$$\lim_{x \rightarrow 5} f(x) = +\infty$$