In this section we extend the limit concept to *one-sided limits*, which are limits as x approaches the number c from the left-hand side (where $x < c$) or the right-hand side ($x > c$) only. These allow us to describe functions that have different limits at a point, depending on whether we approach the point from the left or from the right. One-sided limits also allow us to say what it means for a function to have a limit at an endpoint of an interval.

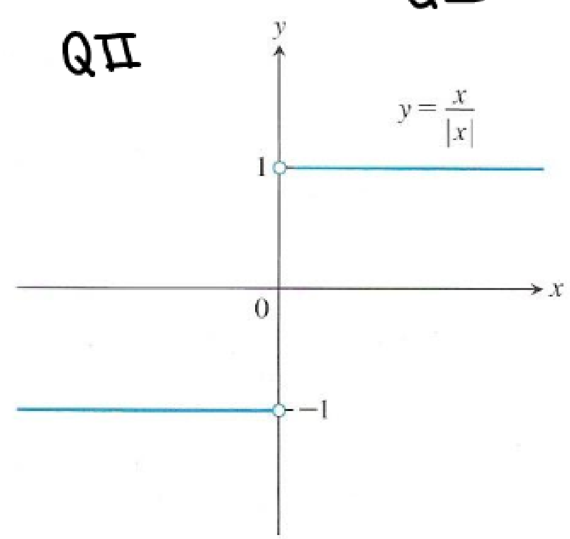
Approaching a Limit from One Side

Suppose a function f is defined on an interval that extends to both sides of a number c. In order for f to have a limit L as x approaches c, the values of $f(x)$ must approach the value L as x approaches c from either side. Because of this, we sometimes say that the limit is **two-sided**.

If f fails to have a two-sided limit at c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a **right-hand limit** or **limit from the right**. From the left, it is a **left-hand limit** or **limit from the left**.

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

QII QI



QII

$x < 0$

$|x| = -x$

$y = \frac{x}{-x} = -1$

QI

$x > 0$

$|x| = x$

$y = \frac{x}{x} = 1$

FIGURE 2.24 Different right-hand and left-hand limits at the origin.

The function $f(x) = x/|x|$ (Figure 2.24) has limit 1 as x approaches 0 from the right, and limit -1 as x approaches 0 from the left. Since these one-sided limit values are not the same, there is no single number that $f(x)$ approaches as x approaches 0. So $f(x)$ does not have a (two-sided) limit at 0.

Intuitively, if we only consider the values of $f(x)$ on an interval (c, b), where $c < b$, and the values of $f(x)$ become arbitrarily close to L as x approaches c from within that interval, then f has **right-hand limit** L at c. In this case we write

$$\lim_{x \to c^+} f(x) = L.$$

The notation "$x \to c^+$" means that we consider only values of $f(x)$ for x greater than c. We don't consider values of $f(x)$ for $x \leq c$.

Similarly, if $f(x)$ is defined on an interval (a, c), where $a < c$ and $f(x)$ approaches arbitrarily close to M as x approaches c from within that interval, then f has **left-hand limit** M at c. We write

$$\lim_{x \to c^-} f(x) = M.$$

The notation "$x \to c^+$" means that we consider only values of $f(x)$ for x greater than c. We don't consider values of $f(x)$ for $x \leq c$.

Similarly, if $f(x)$ is defined on an interval (a, c), where $a < c$ and $f(x)$ approaches arbitrarily close to M as x approaches c from within that interval, then f has **left-hand limit** M at c. We write

$$\lim_{x \to c^-} f(x) = M.$$

The symbol "$x \to c^-$" means that we consider the values of f only at x-values less than c.

These informal definitions of one-sided limits are illustrated in Figure 2.25. For the function $f(x) = x/|x|$ in Figure 2.24 we have

$$\lim_{x \to 0^+} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to 0^-} f(x) = -1.$$

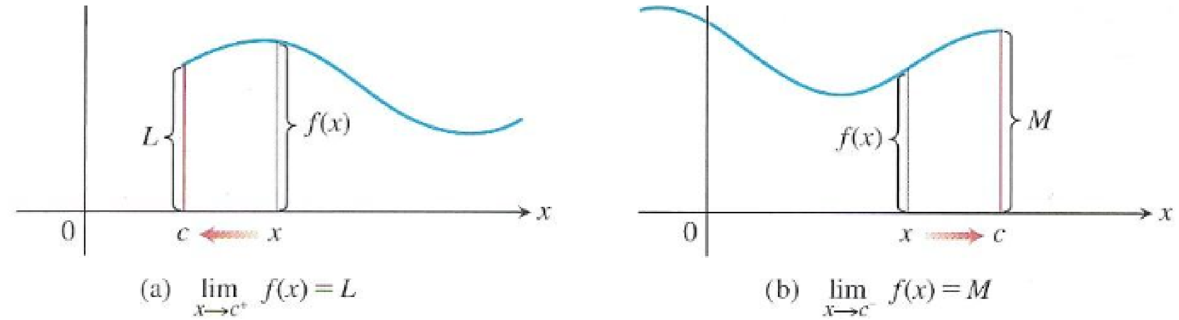


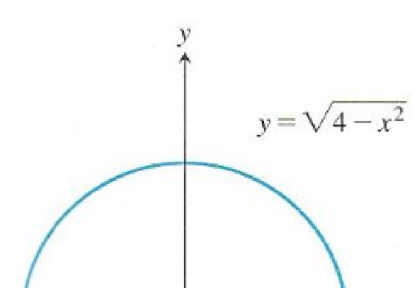
FIGURE 2.25 (a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.

We now give the definition of the limit of a function at a boundary point of its domain. This definition is consistent with limits at boundary points of regions in the plane and in space, as we will see in Chapter 14. When the domain of f is an interval lying to the left of c, such as $(a, c]$ or (a, c), then we say that f has a limit at c if it has a left-hand limit at c. Similarly, if the domain of f is an interval lying to the right of c, such as $[c, b)$ or (c, b), then we say that f has a limit at c if it has a right-hand limit at c.

EXAMPLE 1 The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2, 2]$; its graph is the semicircle in Figure 2.26. We have

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 \quad \text{and} \quad \lim_{x \to 2^-} \sqrt{4 - x^2} = 0.$$

This function has a two-sided limit at each point in $(-2, 2)$. It has a left-hand limit at $x = 2$ and a right-hand limit at $x = -2$. The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have a two-sided limit at either -2 or 2 because f is not defined on both sides of these points. At the domain boundary points, where the domain is an interval on one side of the point, we have $\lim_{x \to -2} \sqrt{4 - x^2} = 0$ and $\lim_{x \to 2} \sqrt{4 - x^2} = 0$. The function f does have a limit at $x = -2$ and at $x = 2$. ■



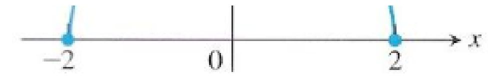


FIGURE 2.26 The function $f(x) = \sqrt{4 - x^2}$ has a right-hand limit 0 at $x = -2$ and a left-hand limit 0 at $x = 2$ (Example 1).

One-sided limits have all the properties listed in Theorem 1 in Section 2.2. The right-hand limit of the sum of two functions is the sum of their right-hand limits, and so on. The theorems for limits of polynomials and rational functions hold with one-sided limits, as does the Sandwich Theorem. One-sided limits are related to limits at interior points in the following way.

> **THEOREM 6** Suppose that a function f is defined on an open interval containing c, except perhaps at c itself. Then $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:
>
> $$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

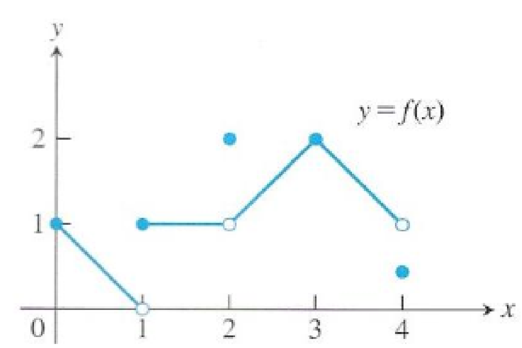


FIGURE 2.27 Graph of the function in Example 2.

Theorem 6 applies at interior points of a function's domain. At a boundary point of its domain, a function has a limit when it has an appropriate one-sided limit.

EXAMPLE 2 For the function graphed in Figure 2.27, At $x = 0$: $\lim_{x\to 0^-} f(x)$ does not exist, — f is not defined to the left of $x = 0$.

$\lim_{x\to 0^+} f(x) = 1$, — f has a right-hand limit at $x = 0$.

$\lim_{x\to 0} f(x) = 1$. — f has a limit at domain endpoint $x = 0$.

At $x = 1$: $\lim_{x\to 1^-} f(x) = 0$, — Even though $f(1) = 1$.

$\lim_{x\to 1^+} f(x) = 1$,

$\lim_{x\to 1} f(x)$ does not exist. — Right- and left-hand limits are not equal.

At $x = 2$: $\lim_{x\to 2^-} f(x) = 1$,

$\lim_{x\to 2^+} f(x) = 1$,

$\lim_{x\to 2} f(x) = 1$. — Even though $f(2) = 2$.

At $x = 3$: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = f(3) = 2$.

At $x = 4$: $\lim_{x\to 4^-} f(x) = 1$, — Even though $f(4) \neq 1$.

$\lim_{x\to 4^+} f(x)$ does not exist, — f is not defined to the right of $x = 4$.

$\lim_{x\to 4} f(x) = 1$. — f has a limit at domain endpoint $x = 4$.

At every other point c in $[0, 4]$, $f(x)$ has limit $f(c)$.

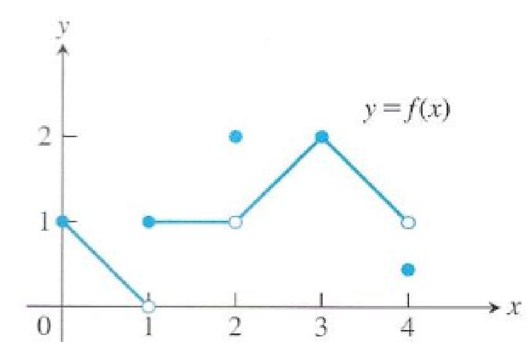
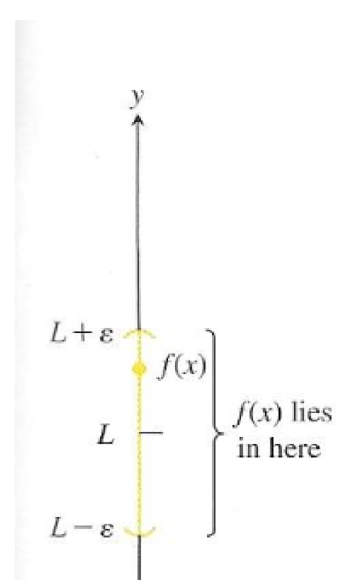


FIGURE 2.27 Graph of the function in Example 2.



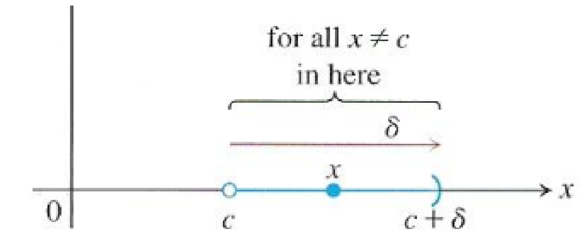
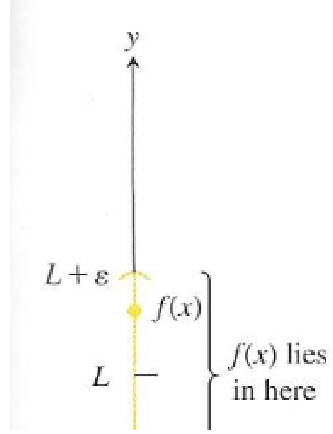


FIGURE 2.28 Intervals associated with the definition of right-hand limit.

Precise Definitions of One-Sided Limits

The formal definition of the limit in Section 2.3 is readily modified for one-sided limits.

> **DEFINITIONS** (a) Assume the domain of f contains an interval (c, d) to the right of c. We say that $f(x)$ has **right-hand limit L at c**, and write
>
> $$\lim_{x \to c^+} f(x) = L$$
>
> if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that
>
> $$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c < x < c + \delta.$$
>
> (b) Assume the domain of f contains an interval (b, c) to the left of c. We say that f has **left-hand limit L at c**, and write
>
> $$\lim_{x \to c^-} f(x) = L$$
>
> if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that
>
> $$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c - \delta < x < c.$$



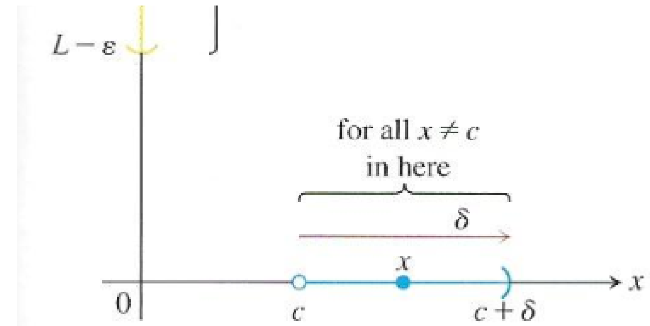


FIGURE 2.28 Intervals associated with the definition of right-hand limit.

The definitions are illustrated in Figures 2.28 and 2.29.

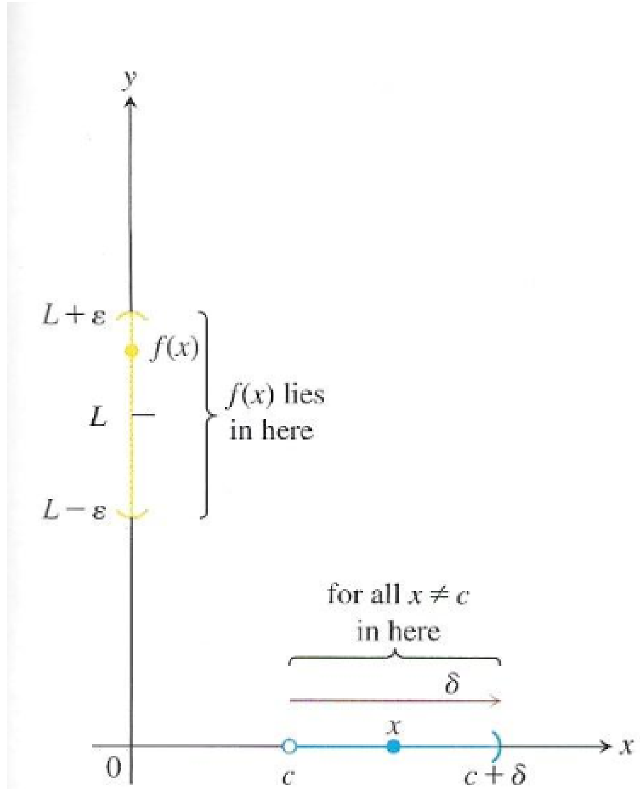


FIGURE 2.28 Intervals associated with the definition of right-hand limit.

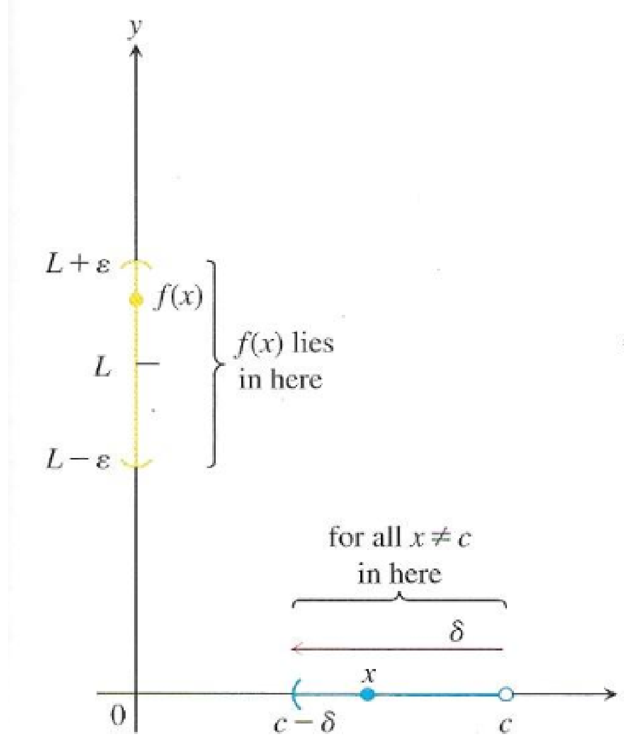
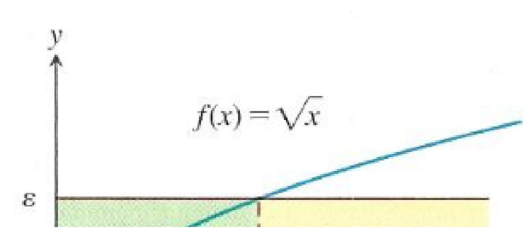


FIGURE 2.29 Intervals associated with the definition of left-hand limit.



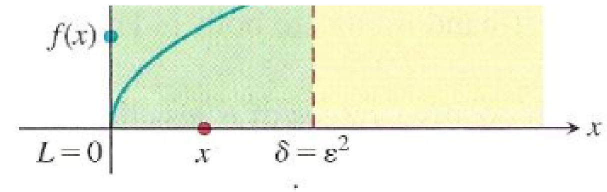


FIGURE 2.30 $\lim_{x \to 0^+} \sqrt{x} = 0$ in Example 3.

EXAMPLE 3 Prove that

$$\lim_{x \to 0^+} \sqrt{x} = 0.$$

Solution Let $\varepsilon > 0$ be given. Here $c = 0$ and $L = 0$, so we want to find a $\delta > 0$ such that

$$|\sqrt{x} - 0| < \varepsilon \qquad \text{whenever} \qquad 0 < x < \delta,$$

or

$$\sqrt{x} < \varepsilon \qquad \text{whenever} \qquad 0 < x < \delta.$$

$\sqrt{x} \geq 0$ so $|\sqrt{x}| = \sqrt{x}$

Squaring both sides of this last inequality gives

$$x < \varepsilon^2 \qquad \text{if} \qquad 0 < x < \delta.$$

If we choose $\delta = \varepsilon^2$ we have

$$\sqrt{x} < \varepsilon \qquad \text{whenever} \qquad 0 < x < \delta = \varepsilon^2,$$

or

$$|\sqrt{x} - 0| < \varepsilon \qquad \text{whenever} \qquad 0 < x < \varepsilon^2.$$

According to the definition, this shows that $\lim_{x \to 0^+} \sqrt{x} = 0$ (Figure 2.30). ■

The functions examined so far have had some kind of limit at each point of interest. In general, that need not be the case.

EXAMPLE 4 Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side (Figure 2.31).

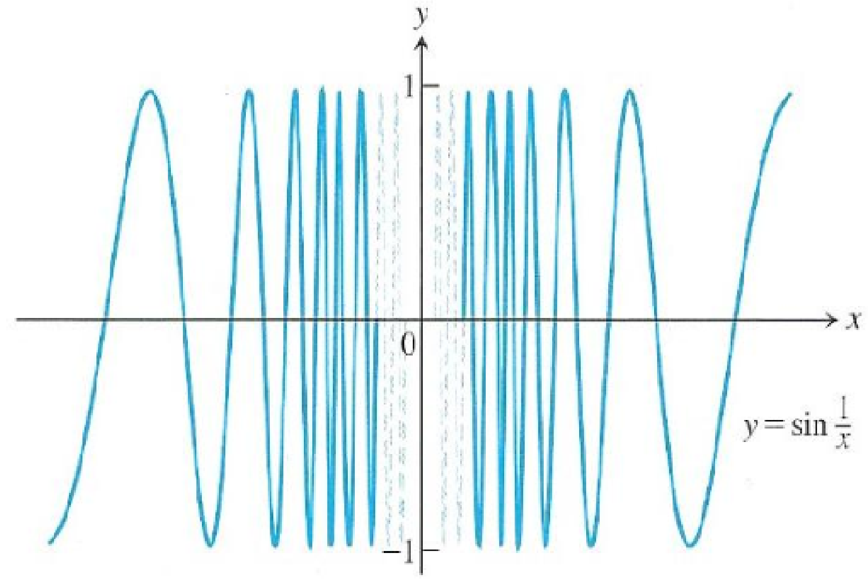
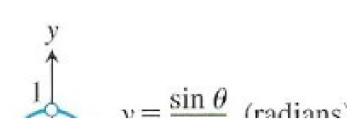


FIGURE 2.31 The function $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero (Example 4). The graph here omits values very near the y-axis.

Solution As x approaches zero, its reciprocal, $1/x$, grows without bound and the values of $\sin(1/x)$ cycle repeatedly from -1 to 1. There is no single number L that the function's values stay increasingly close to as x approaches zero. This is true even if we restrict x to positive values or to negative values. The function has neither a right-hand limit nor a left-hand limit at $x = 0$. ■

Limits Involving $(\sin\theta)/\theta$

A central fact about $(\sin\theta)/\theta$ is that in radian measure its limit as $\theta \to 0$ is 1. We can see this in Figure 2.32 and confirm it algebraically using the Sandwich Theorem. You will see the importance of this limit in Section 3.5, where instantaneous rates of change of the trigonometric functions are studied.



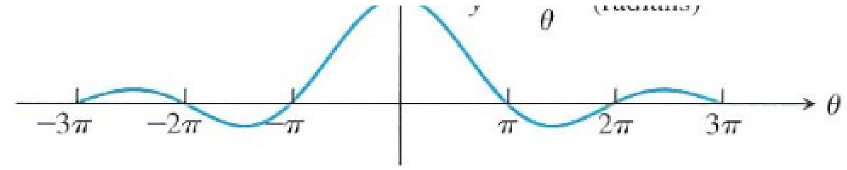


FIGURE 2.32 The graph of $f(\theta) = (\sin \theta)/\theta$ suggests that the right- and left-hand limits as θ approaches 0 are both 1.

THEOREM 7—Limit of the Ratio $\sin \theta/\theta$ as $\theta \to 0$

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \tag{1}$$

Proof The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well.

To show that the right-hand limit is 1, we begin with positive values of θ less than $\pi/2$ (Figure 2.33). Notice that

$$\text{Area } \Delta OAP < \text{ area sector } OAP < \text{ area } \Delta OAT.$$

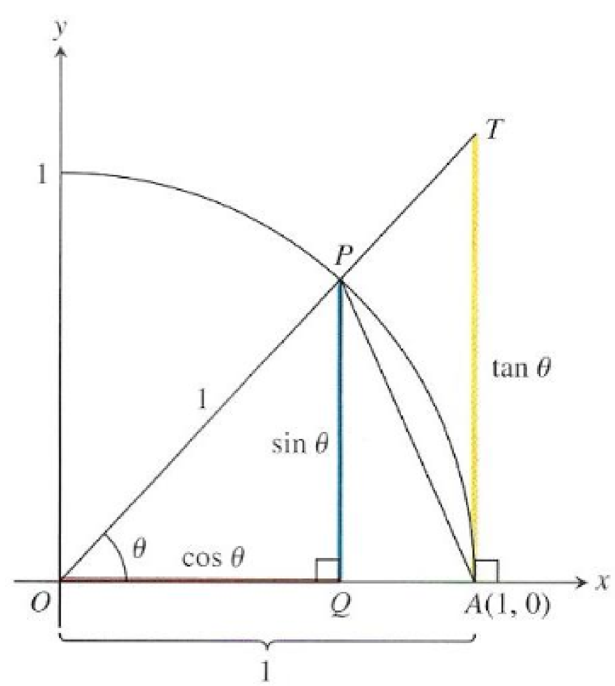


FIGURE 2.33 The ratio $TA/OA = \tan \theta$, and $OA = 1$, so $TA = \tan \theta$.

We can express these areas in terms of θ as follows:

$$\text{Area } \Delta OAP = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\sin\theta) = \frac{1}{2}\sin\theta$$

$$\text{Area sector } OAP = \frac{1}{2}r^2\theta = \frac{1}{2}(1)^2\theta = \frac{\theta}{2} \tag{2}$$

$$\text{Area } \Delta OAT = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta.$$

The use of radians to measure angles is essential in Equation (2): The area of sector OAP is $\theta/2$ only if θ is measured in radians.

Thus,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

This last inequality goes the same way if we divide all three terms by the number $(1/2)\sin\theta$, which is positive, since $0 < \theta < \pi/2$:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Taking reciprocals reverses the inequalities:

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim_{\theta\to 0^+}\cos\theta = 1$ (Example 11b, Section 2.2), the Sandwich Theorem gives

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1.$$

To consider the left-hand limit, we recall that $\sin\theta$ and θ are both *odd functions* (Section 1.1). Therefore, $f(\theta) = (\sin\theta)/\theta$ is an *even function*, with a graph symmetric about the y-axis (see Figure 2.32). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta \to 0^-} \frac{\sin \theta}{\theta} = 1 = \lim_{\theta \to 0^+} \frac{\sin \theta}{\theta},$$

so $\lim_{\theta \to 0} (\sin \theta)/\theta = 1$ by Theorem 6. ■

EXAMPLE 5 Show that **(a)** $\lim_{y \to 0} \frac{\cos y - 1}{y} = 0$ and **(b)** $\lim_{x \to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$.

Solution

(a) Using the half-angle formula $\cos y = 1 - 2\sin^2(y/2)$, we calculate

$$\begin{aligned}
\lim_{h \to 0} \frac{\cos y - 1}{y} &= \lim_{h \to 0} -\frac{2\sin^2(y/2)}{y} \\
&= -\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \sin \theta && \text{Let } \theta = y/2. \\
&= -(1)(0) = 0. && \text{Eq. (1) and Example 11a in Section 2.2}
\end{aligned}$$

(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin 2x}{5x} &= \lim_{x \to 0} \frac{(2/5) \cdot \sin 2x}{(2/5) \cdot 5x} \\
&= \frac{2}{5} \lim_{x \to 0} \frac{\sin 2x}{2x} && \text{Eq. (1) applies with } \theta = 2x. \\
&= \frac{2}{5}(1) = \frac{2}{5}.
\end{aligned}$$

■

$$\frac{\left(\frac{\sin 2x}{2x}\right)}{\left(\frac{5}{2}\right)} = \frac{1}{\left(\frac{5}{2}\right)} = \frac{2}{5}$$

EXAMPLE 6 Find $\lim_{t \to 0} \frac{\tan t \sec 2t}{3t}$.

Solution From the definition of $\tan t$ and $\sec 2t$, we have

$$\lim_{t \to 0} \frac{\tan t \sec 2t}{3t} = \lim_{t \to 0} \frac{1}{3} \cdot \frac{1}{t} \cdot \frac{\sin t}{\cos t} \cdot \frac{1}{\cos 2t}$$

$$= \frac{1}{3} \lim_{t \to 0} \frac{\sin t}{t} \cdot \frac{1}{\cos t} \cdot \frac{1}{\cos 2t}$$

Eq. (1) and Example 11b in Section 2.2

$$= \frac{1}{3}(1)(1)(1) = \frac{1}{3}.$$

EXAMPLE 7 Show that for nonzero constants A and B.

$$\lim_{\theta \to 0} \frac{\sin A\theta}{\sin B\theta} = \frac{A}{B}.$$

Solution

$$\lim_{\theta \to 0} \frac{\sin A\theta}{\sin B\theta} = \lim_{\theta \to 0} \frac{\sin A\theta}{A\theta} A\theta \frac{B\theta}{\sin B\theta} \frac{1}{B\theta}$$

Multiply and divide by $A\theta$ and $B\theta$.

$$= \lim_{\theta \to 0} \frac{\sin A\theta}{A\theta} \frac{B\theta}{\sin B\theta} \frac{A}{B}$$

$$\boxed{\lim_{B\theta \to 0} \frac{B\theta}{\sin B\theta} \Rightarrow 1}$$

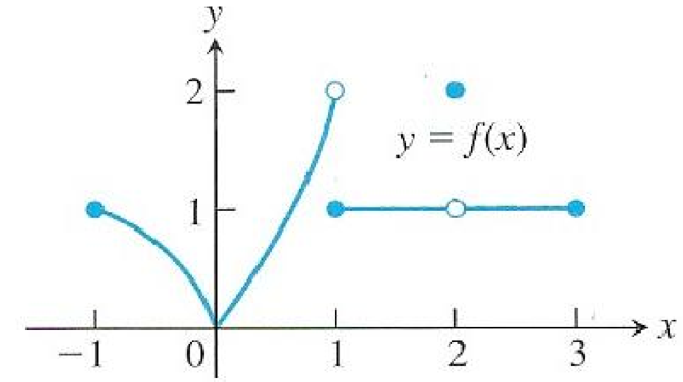
$$= \lim_{\theta \to 0} (1)(1) \frac{A}{B}$$

$\lim_{u \to 0} \frac{\sin u}{u} = 1$, with $u = A\theta$

$\lim_{v \to 0} \frac{v}{\sin v} = 1$, with $v = B\theta$

$$= \frac{A}{B}.$$

2. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a. $\lim_{x \to -1^+} f(x) = 1$

b. $\lim_{x \to 2} f(x)$ does not exist.

c. $\lim_{x \to 2} f(x) = 2$

d. $\lim_{x \to 1^-} f(x) = 2$

e. $\lim_{x \to 1^+} f(x) = 1$

f. $\lim_{x \to 1} f(x)$ does not exist.

g. $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$

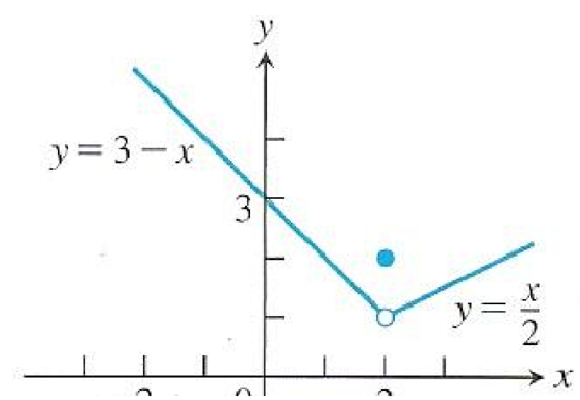
h. $\lim_{x \to c} f(x)$ exists at every c in the open interval $(-1, 1)$.

i. $\lim_{x \to c} f(x)$ exists at every c in the open interval $(1, 3)$.

j. $\lim_{x \to -1^-} f(x) = 0$

k. $\lim_{x \to 3^+} f(x)$ does not exist.

4. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ \dfrac{x}{2}, & x > 2. \end{cases}$



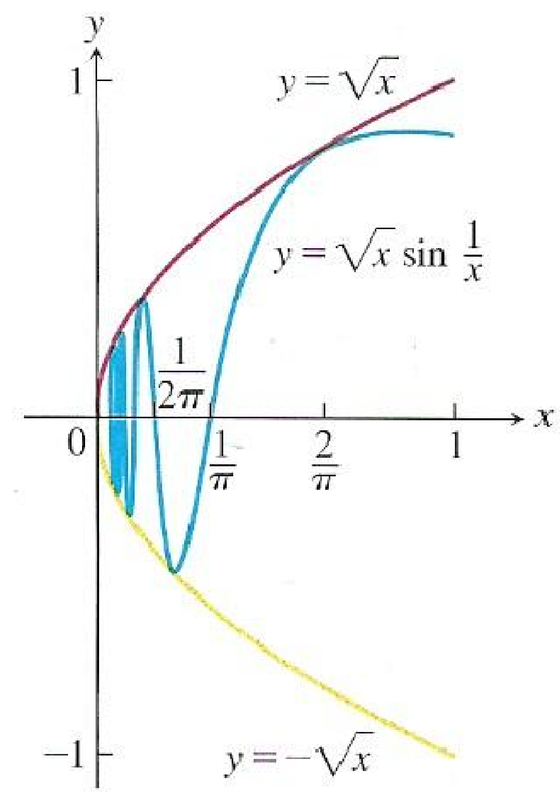
a. Find $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, and $f(2)$.

b. Does $\lim_{x\to 2} f(x)$ exist? If so, what is it? If not, why not?

c. Find $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to -1^+} f(x)$.

d. Does $\lim_{x\to -1} f(x)$ exist? If so, what is it? If not, why not?

6. Let $g(x) = \sqrt{x}\sin(1/x)$.



a. Does $\lim_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b. Does $\lim_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c. Does $\lim_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

7. a. Graph $f(x) = \begin{cases} x^3, & x \neq 1 \end{cases}$

7. **a.** Graph $f(x) = \Big\{\ 0, \quad x = 1.$

b. Find $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$.

c. Does $\lim_{x \to 1} f(x)$ exist? If so, what is it? If not, why not?

8. a. Graph $f(x) = \begin{cases} 1 - x^2, & x \neq 1 \\ 2, & x = 1. \end{cases}$

b. Find $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$.

c. Does $\lim_{x \to 1} f(x)$ exist? If so, what is it? If not, why not?

Graph the functions in Exercises 9 and 10. Then answer these questions.

a. What are the domain and range of f?

b. At what points c, if any, does $\lim_{x \to c} f(x)$ exist?

c. At what points does the left-hand limit exist but not the right-hand limit?

d. At what points does the right-hand limit exist but not the left-hand limit?

10. $f(x) = \begin{cases} x, & -1 \leq x < 0, \quad \text{or} \quad 0 < x \leq 1 \\ 1, & x = 0 \\ 0, & x < -1 \quad \text{or} \quad x > 1 \end{cases}$

Finding One-Sided Limits Algebraically

Find the limits in Exercises 11–20.

11. $$\lim_{x \to -0.5^-} \sqrt{\frac{x+2}{x+1}}$$

12. $$\lim_{x \to 1^+} \sqrt{\frac{x-1}{x+2}}$$

14. $$\lim_{x \to 1^-} \left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right)$$

16. $$\lim_{h \to 0^-} \frac{\sqrt{6} - \sqrt{5h^2 + 11h + 6}}{h}$$

18. **a.** $\displaystyle\lim_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|}$

b. $\displaystyle\lim_{x\to 1^-} \frac{\sqrt{2x}(x-1)}{|x-1|}$

19. **a.** $\displaystyle\lim_{x\to 0^+} \frac{|\sin x|}{\sin x}$

b. $\displaystyle\lim_{x\to 0^-} \frac{|\sin x|}{\sin x}$

20. **a.** $\displaystyle\lim_{x\to 0^+} \frac{1-\cos x}{|\cos x - 1|}$

$\cos x - 1$

b. $\lim\limits_{x \to 0^-} \dfrac{}{|\cos x - 1|}$

23. $\lim\limits_{\theta \to 0} \dfrac{\sin \sqrt{2}\theta}{\sqrt{2}\theta}$

24. $\lim\limits_{t \to 0} \dfrac{\sin kt}{t}$ (k constant)

25. $\lim\limits_{y \to 0} \dfrac{\sin 3y}{4y}$

26. $\lim\limits_{h \to 0^-} \dfrac{h}{\sin 3h}$

28. $\lim\limits_{t \to 0} \dfrac{2t}{\tan t}$

30. $\lim\limits_{x \to 0} 6x^2(\cot x)(\csc 2x)$

32. $\lim\limits_{x \to 0} \dfrac{x^2 - x + \sin x}{2x}$

34. $\lim\limits_{x \to 0} \dfrac{x - x\cos x}{\sin^2 3x}$

36. $\lim\limits_{h \to 0} \dfrac{\sin(\sin h)}{\sin h}$

38. $\lim\limits_{x \to 0} \dfrac{\sin 5x}{\sin 4x}$

40. $\lim\limits_{\theta \to 0} \sin\theta \cot 2\theta$

42. $\lim\limits_{y \to 0} \dfrac{\sin 3y \cot 5y}{y \cot 4y}$

43. $\lim \dfrac{\tan\theta}{\text{[illegible]}}$

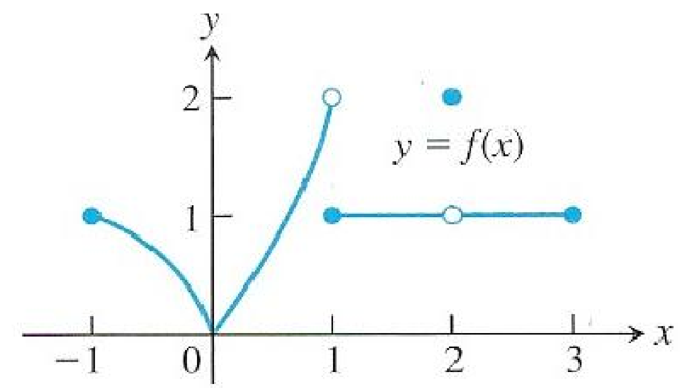
$\theta \to 0 \quad \theta^2 \cot 3\theta$

44. $\lim\limits_{\theta \to 0} \dfrac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta}$

45. $\lim\limits_{x \to 0} \dfrac{1 - \cos 3x}{2x}$

46. $\lim\limits_{x \to 0} \dfrac{\cos^2 x - \cos x}{x^2}$

2. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a. $\lim_{x \to -1^+} f(x) = 1$ True

b. $\lim_{x \to 2} f(x)$ does not exist. False

c. $\lim_{x \to 2} f(x) = 2$ False

d. $\lim_{x \to 1^-} f(x) = 2$ True

e. $\lim_{x \to 1^+} f(x) = 1$ True

f. $\lim_{x \to 1} f(x)$ does not exist. True

g. $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$ True

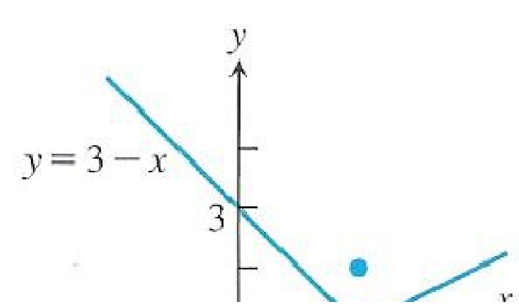
h. $\lim_{x \to c} f(x)$ exists at every c in the open interval $(-1, 1)$. True

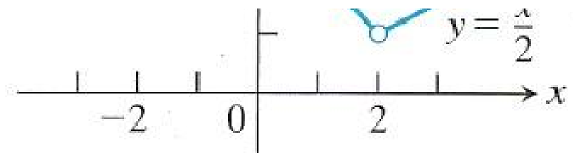
i. $\lim_{x \to c} f(x)$ exists at every c in the open interval $(1, 3)$. False

j. $\lim_{x \to -1^-} f(x) = 0$ False

k. $\lim_{x \to 3^+} f(x)$ does not exist. True

4. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ \dfrac{x}{2}, & x > 2. \end{cases}$



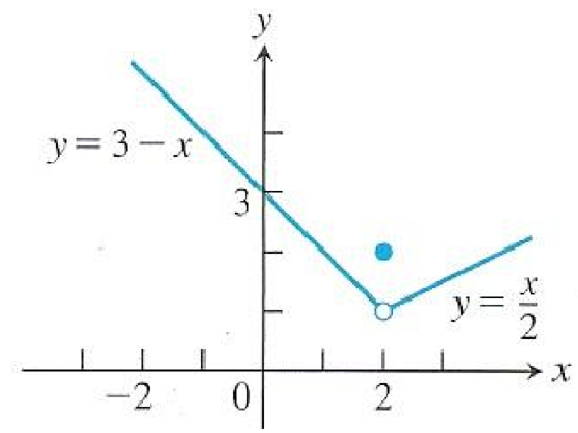


a. Find $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, and $f(2)$.

b. Does $\lim_{x\to 2} f(x)$ exist? If so, what is it? If not, why not?

c. Find $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to -1^+} f(x)$.

d. Does $\lim_{x\to -1} f(x)$ exist? If so, what is it? If not, why not?



a. $\lim_{x\to 2^-} f(x) = 1$; $\lim_{x\to 2^+} f(x) = 1$

$\lim_{x\to 2^-} f(x) = 1 = \lim_{x\to 2^+} f(x)$

b. $\therefore \lim_{x \to 2} f(x) = 1$

Despite that $f(2) = 2$

c.) $f(x) = 3 - x, \quad x < 2$

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (3 - x) = 3 - (-1) = 3 + 1 = 4$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (3 - x) = 3 - (-1) = 3 + 1 = 4$$

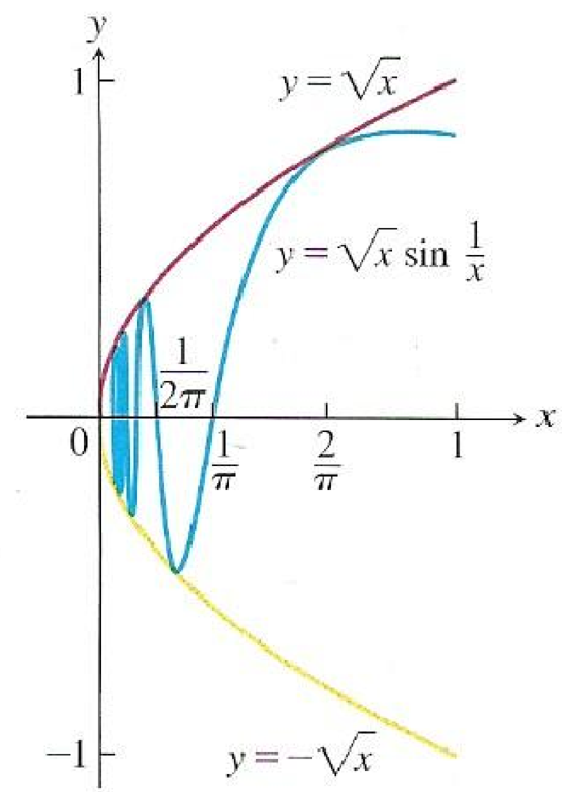
d. The $\lim_{x \to -1} f(x)$ does exist because

$$\lim_{x \to -1^-} f(x) = 4 = \lim_{x \to -1^+} f(x)$$

$$\therefore \lim_{x \to -1} f(x) = 4$$

---

6. Let $g(x) = \sqrt{x}\sin(1/x)$.



a. Does $\lim_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b. Does $\lim_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c. Does $\lim_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

(a) $\quad -1 \le \sin\left(\frac{1}{x}\right) \le 1$

$$-\sqrt{x} \leq \sqrt{x}\sin\left(\frac{1}{x}\right) \leq \sqrt{x}\sin\left(\frac{1}{x}\right)$$

$$\lim_{x\to 0^+} -\sqrt{x} = 0 \quad \& \quad \lim_{x\to 0^+} \sqrt{x} = 0$$

By the Sandwich Theorem or Squeeze Theorem

$$\lim_{x\to 0^+} \sqrt{x}\sin\left(\frac{1}{x}\right) = 0$$

b. $\lim\limits_{x\to 0^-} \sqrt{x}\sin\left(\frac{1}{x}\right)$ does not exist because

$\sqrt{x}$ does not exist when $x<0$

$\sqrt{x}$ does not exist when $x < 0$

Therefore

$$\lim_{x \to 0^-} \sqrt{x}\sin\left(\frac{1}{x}\right) \text{ is not defined}$$

because $\sqrt{x}$ does not exist when $x<0$

c. Does $\lim_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

$\lim_{x\to 0} \sqrt{x}\sin\left(\frac{1}{x}\right)$ does not exist because,

$\lim_{x\to 0^-} \sqrt{x}\sin\left(\frac{1}{x}\right)$ does not exist because

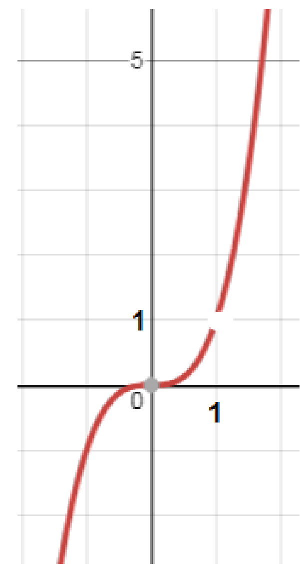
$\sqrt{x}$ does not exist when $x<0$

7. a. Graph $f(x) = \begin{cases} x^3, & x \neq 1 \\ 0, & x = 1. \end{cases}$

b. Find $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$.

c. Does $\lim_{x \to 1} f(x)$ exist? If so, what is it? If not, why not?

a.



b. $\lim_{x \to 1^-} f(x) = 1$ & $\lim_{x \to 1^+} f(x) = 1$

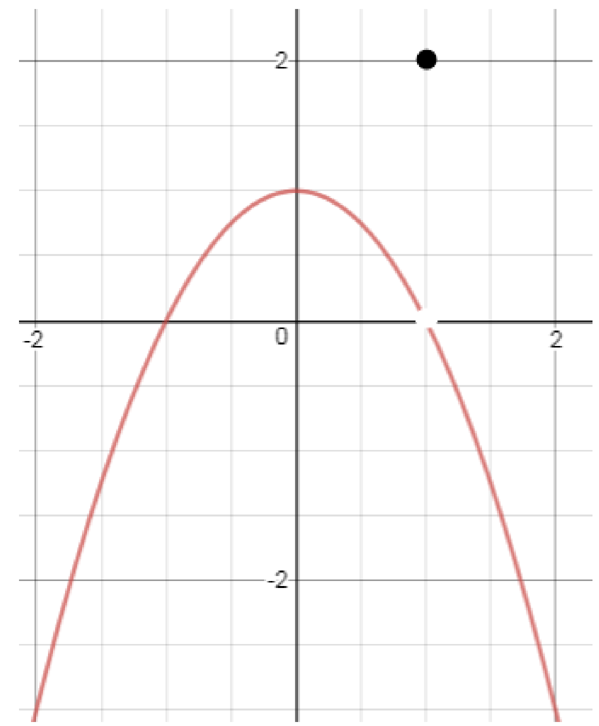
c. $\lim_{x \to 1} f(x) = 1$ because $\lim_{x \to 1^-} f(x) = 1 = \lim_{x \to 1^+} f(x)$

8. a. Graph $f(x) = \begin{cases} 1 - x^2, & x \neq 1 \\ 2, & x = 1. \end{cases}$

b. Find $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$.

c. Does $\lim_{x \to 1} f(x)$ exist? If so, what is it? If not, why not?

a.



b. $f(x) = 1 - x^2, \; x = 1$

$\lim_{x \to 1^-} f(x) = 0$ & $\lim_{x \to 1^+} f(x) = 0$

c.

$\lim_{x \to 1} f(x) = 0$ because $\lim_{x \to 1^-} f(x) = 0$ & $\lim_{x \to 1^+} f(x) = 0$

Graph the functions in Exercises 9 and 10. Then answer these questions.

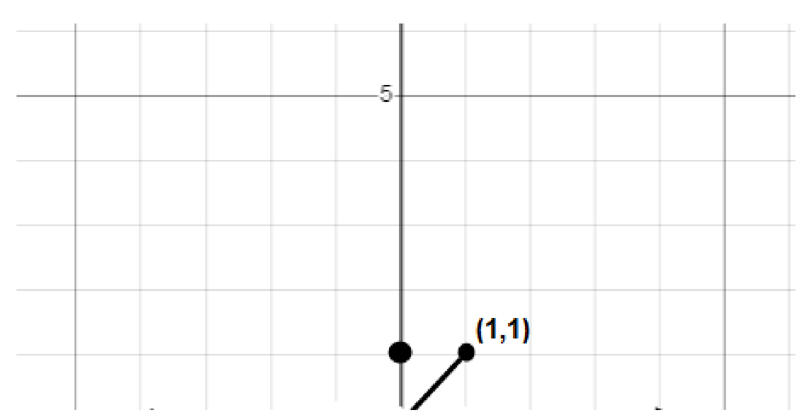
a. What are the domain and range of f?

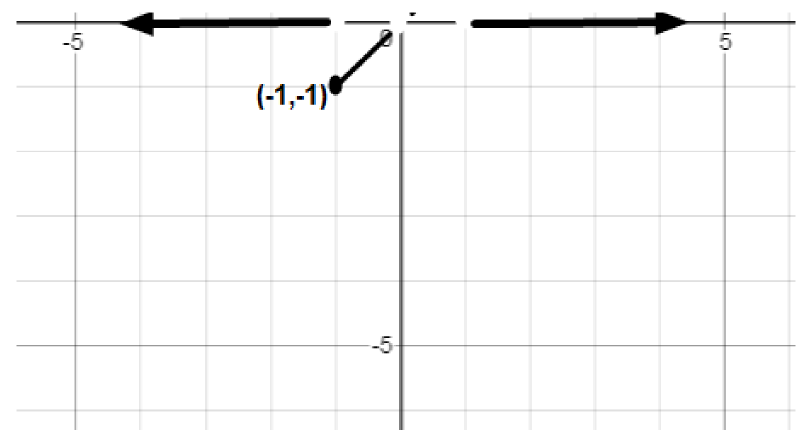
b. At what points c, if any, does $\lim_{x \to c} f(x)$ exist?

c. At what points does the left-hand limit exist but not the right-hand limit?

d. At what points does the right-hand limit exist but not the left-hand limit?

10. $f(x) = \begin{cases} x, & -1 \le x < 0, \quad \text{or} \quad 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1 \quad \text{or} \quad x > 1 \end{cases}$





a.

Domain: $-\infty < x < \infty$

Range: $-1 \leq y \leq 1$

b.

$\lim_{x \to c} f(x)$ exists for c belonging to

$(-\infty,-1) \cup (-1,1) \cup (1,\infty)$

c. None

$$\lim_{x\to -1^-} f(x)=0 \quad \& \quad \lim_{x\to -1^+} f(x)=-1$$

$$\lim_{x\to 1^-} f(x)=1 \quad \& \quad \lim_{x\to 1^+} f(x)=0$$

d. None

$$\lim_{x\to -1^+} f(x)=-1 \quad \& \quad \lim_{x\to -1^-} f(x)=0$$

$$\lim f(x)=0 \quad \& \quad \lim f(x)=1$$

$x \to 1^+$ $x \to 1$

11. $\lim_{x \to -0.5^-} \sqrt{\frac{x+2}{x+1}}$

$= \sqrt{\frac{\left(-\frac{1}{2}+2\right)}{-\frac{1}{2}+1}} = \sqrt{\frac{\left(\frac{3}{2}\right)}{\left(\frac{1}{2}\right)}} = \sqrt{\left(\frac{3}{2}\right)\left(\frac{2}{1}\right)} = \sqrt{3}$

12. $\lim_{x \to 1^+} \sqrt{\frac{x-1}{x+2}} = \sqrt{\frac{1-1}{1+2}}$

$= \sqrt{0} = 0$

$-\sqrt{3}$

14. $\lim_{x \to 1^-} \left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right)$

$= \left(\frac{1}{1+1}\right)\left(\frac{1+6}{1}\right)\left(\frac{3-1}{7}\right)$

$= \left(\frac{1}{\cancel{2}_{1}}\right)\left(\frac{\cancel{7}^{1}}{1}\right)\left(\frac{\cancel{2}^{1}}{\cancel{7}_{1}}\right) = 1$

$\sqrt{6} - \sqrt{5h^2 + 11h + 6}$

10. $\lim_{h\to 0^-} \frac{\quad}{h}$

$$= \frac{\sqrt{6} - \sqrt{5\cdot 0 + 11\cdot 0 + 6}}{0} = \frac{\sqrt{6}-\sqrt{6}}{0} = \frac{0}{0}$$

$\frac{0}{0}$ is an indeterminant form.

we can use algebra to rewrite given expression. We will use the difference of two squares concept:

$$\lim \frac{[\sqrt{6}-\sqrt{5h^2+11h+6}][\sqrt{6}+\sqrt{5h^2+11h+6}]}{}$$

$$\lim_{h\to 0^-}\left[\frac{\quad}{h}\right]\left[\frac{\quad}{\sqrt{6}+\sqrt{5h^2+11h+6}}\right]$$

$$\lim_{h\to 0^-}\frac{6-(5h^2+11h+6)}{h\left[\sqrt{6}+\sqrt{5h^2+11h+6}\right.}$$

$$\lim_{h\to 0^-}\frac{6-5h^2-11h-6}{h\left[\sqrt{6}+\sqrt{5h^2+11h+6}\right.}$$

$$\lim_{h\to 0^-}\frac{h(-5h-11)}{h\left[\sqrt{6}+\sqrt{5h^2+11h+6}\right]}$$

$$\lim_{h\to 0^-} \frac{\cancel{h}(-5h-11)}{\cancel{h}\left[\sqrt{6}+\sqrt{5h^2+11h+6}\right]}$$

$$\lim_{h\to 0^-} \frac{(-5h-11)}{\left[\sqrt{6}+\sqrt{5h^2+11h+6}\right]} = \frac{-11}{\sqrt{6}+\sqrt{6}} = \frac{-11}{2\sqrt{6}}$$

$x<1$	1	$x>1$
$x-1<0$		$x-1>0$

18. a. $\displaystyle\lim_{x\to 1^+} \frac{\sqrt{2x}\,(x-1)}{|x-1|}$

$x>1$, $|x-1| = (x-1)$

So we have

$$\lim_{x\to 1^+} \frac{\sqrt{2x}\,(x-1)}{(x-1)} = \lim_{x\to 1^+} \sqrt{2x} = \sqrt{2}$$

$x \to 1^+$ $\qquad$ $x \to 1^+$

b. $$\lim_{x \to 1^-} \frac{\sqrt{2x}(x-1)}{|x-1|}$$

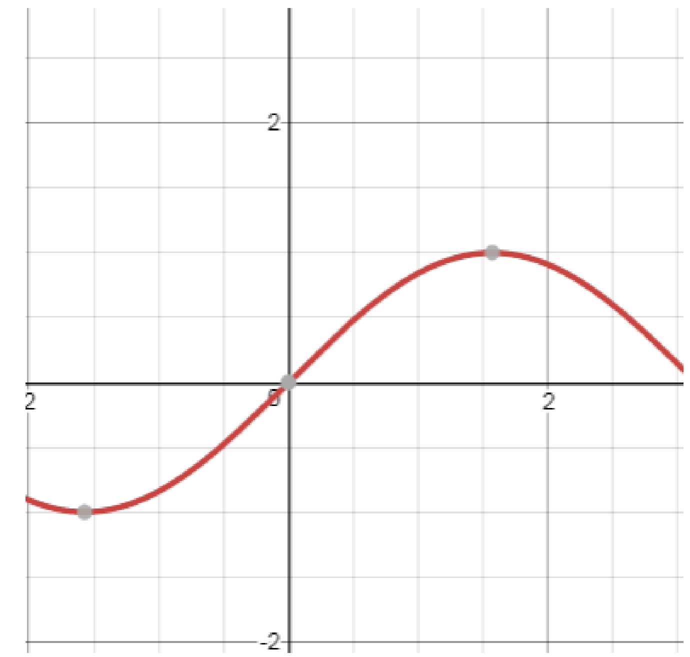
$|x-1| = -(x-1)$ when $x<1$

So we have

$$\lim_{x \to 1^-} \frac{\sqrt{2x}(x-1)}{-(x-1)} = \lim_{x \to 1^-} -\sqrt{2x} = -\sqrt{2}$$

19.)

y = sinx



19. a. $\lim\limits_{x \to 0^+} \dfrac{|\sin x|}{\sin x}$

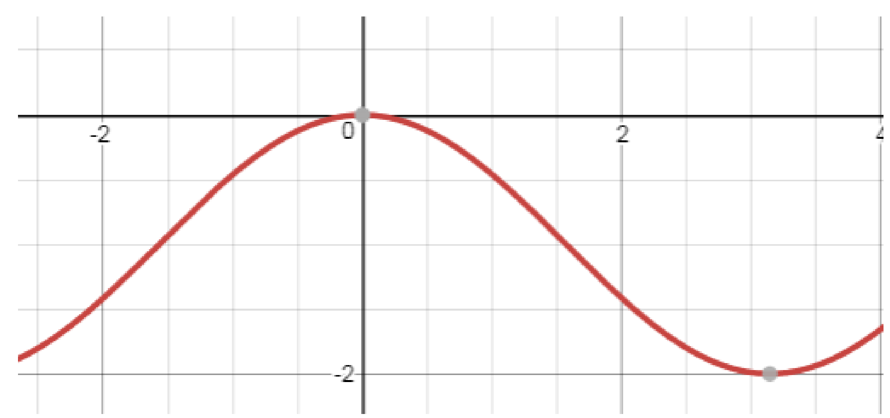
$$|\sin x| = \begin{cases} \sin x, & \text{when } 0 < x < \pi \\ -\sin x, & \text{when } -\pi < x < 0 \end{cases}$$

$$\lim_{x \to 0^+} \frac{|\sin x|}{\sin x} = \frac{\sin x}{\sin x} = 1$$

$x \to 0^+ \quad \sin x$

b. $\lim\limits_{x \to 0^-} \dfrac{|\sin x|}{\sin x} = \dfrac{-\sin x}{\sin x} = -1$

20.) $y = |\cos x - 1|$



$|\cos x - 1| = -(\cos x - 1)$ when $x \to 0^-$

&

when $x \to 0^+$

a. $\lim_{x \to 0^+} \dfrac{1 - \cos x}{|\cos x - 1|}$

$$= \frac{1-\cos x}{-(\cos x - 1)} = \frac{1-\cos x}{1-\cos x} = 1$$

b. $\lim_{x \to 0^-} \dfrac{\cos x - 1}{|\cos x - 1|}$

$$= \frac{\cos x - 1}{-(\cos x - 1)} = -1$$

23. $\lim_{\theta \to 0} \dfrac{\sin \sqrt{2}\theta}{\sqrt{2}\theta}$ $= 1$

24. $\lim_{t \to 0} \dfrac{\sin kt}{t}$ (k constant)

$\lim_{t \to 0} k\left(\dfrac{\sin kt}{kt}\right) = k \cdot 1 = k$

$\sin 3y$

25. $\lim_{y \to 0} \frac{\quad}{4y}$

$$\lim_{y \to 0} \left(\frac{3}{4}\right)\left(\frac{\sin 3y}{3y}\right) = \frac{3}{4} \cdot 1 = \frac{3}{4}$$

---

26. $\lim_{h \to 0^-} \frac{h}{\sin 3h}$

$$\lim_{h \to 0^-} \frac{1}{\left(\frac{\sin 3h}{h}\right)} = \lim_{h \to 0^-} \frac{1}{3\left(\frac{\sin 3h}{3h}\right)}$$

$$= \frac{1}{3 \cdot 1} = \frac{1}{3}$$

28. $\lim_{t \to 0} \frac{2t}{\tan t}$

$$= \lim_{t \to 0} (2t)\cot(t)$$

$$= \lim_{t \to 0} \frac{2t\cos(t)}{\sin t}$$

$$= \lim_{t\to 0} (2\cos(t))\left(\frac{1}{\left[\frac{\sin t}{t}\right]}\right)$$

$$= 2 \cdot \cos 0 \cdot \left(\frac{1}{1}\right)$$

$$= 2 \cdot 1 \cdot 1$$

$$= 2$$

30. $\lim_{x\to 0} 6x^2(\cot x)(\csc 2x)$

$$= \lim_{x\to 0} 6x^2\left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\sin 2x}\right)$$

$$= 3\left(\lim_{x\to 0} \cos x\right)\left(\lim_{x\to 0} \frac{x}{\sin x}\right)\left(\lim_{x\to 0} \frac{2x}{\sin 2x}\right)$$

$$= (3\cdot 1)\left(\lim_{x\to 0} \frac{1}{\left(\frac{\sin x}{x}\right)}\right)\left(\lim_{x\to 0} \frac{1}{\left(\frac{\sin 2x}{2x}\right)}\right)$$

$$= 3\cdot 1\cdot 1 = 3$$

32. $\lim_{x \to 0} \dfrac{x^2 - x + \sin x}{2x}$

$$\lim_{x \to 0} \frac{x^2}{2x} + \lim_{x \to 0} \frac{x}{2x} + \frac{1}{2}\left(\lim_{x \to 0} \frac{\sin x}{x}\right)$$

$$\lim_{x \to 0} \frac{x}{2} + \frac{1}{2} \lim_{x \to 0} 1 + \frac{1}{2} \cdot 1$$

$$0 + \frac{1}{2} + \frac{1}{2} = 1$$

34. $\displaystyle\lim_{x\to 0} \frac{x - x\cos x}{\sin^2 3x}$

$$\lim_{x\to 0} \frac{x(1-\cos x)}{[\sin 3x]^2} = \lim_{x\to 0} \frac{x(1-\cos x)}{\left[3x\left(\frac{\sin 3x}{3x}\right)\right]^2}$$

$$= \lim_{x\to 0} \frac{x(1-\cos x)}{9x^2}$$

$$= \frac{1}{9} \lim_{x \to 0} \frac{1-\cos x}{x}$$

By Example 5, $\lim_{x \to 0} \left(\frac{1-\cos x}{x}\right) = 0$

So $\quad \frac{1}{9} \lim_{x \to 0} \frac{1-\cos x}{x} = \frac{1}{9} \cdot 0 = 0$

$$\therefore \lim_{x \to 0} \frac{x - x\cos x}{\sin^2 3x} = 0$$

36 $\lim \dfrac{\sin(\sin h)}{} = 1$

$$\lim_{h \to 0} \quad \sin h$$

---

38. $\displaystyle \lim_{x \to 0} \frac{\sin 5x}{\sin 4x}$

$$\left(\lim_{x \to 0} \sin 5x\right)\left(\lim_{x \to 0} \frac{1}{\sin 4x}\right)$$

$$\left[\lim_{x \to 0} 5x\left(\frac{\sin 5x}{5x}\right)\right] \cdot \left[\lim_{x \to 0} \frac{1}{4x\left(\frac{\sin 4x}{4x}\right)}\right]$$

$$\left(\lim_{x\to 0} \frac{5x}{4x}\right)\left[\lim_{x\to 0} \frac{\sin 5x}{5x}\right]\left[\frac{1}{\lim_{x\to 0} \frac{\sin 4x}{4x}}\right]$$

$$\frac{5}{4}\cdot 1\cdot 1 = \frac{5}{4}$$

40. $\lim_{\theta\to 0} \sin\theta \cot 2\theta$

$$\left(\lim_{\theta\to 0} \sin\theta\right)\left(\lim_{\theta\to 0} \frac{\cos 2\theta}{\sin 2\theta}\right)$$

$$\left(\lim_{\theta\to 0} \sin\theta\right)\left(\lim_{\theta\to 0} \frac{1}{2\theta\left(\frac{\sin 2\theta}{2\theta}\right)}\right)\left(\lim_{\theta\to 0} \cos 2\theta\right)$$

$$\lim_{\theta\to 0}\left[\theta \cdot \frac{\sin\theta}{\theta}\right]\left[\lim_{\theta\to 0} \frac{1}{2\theta\left(\frac{\sin 2\theta}{2\theta}\right)}\right] \cdot 1$$

$$\lim_{\theta\to 0}\left[\cancel{\theta} \cdot \frac{\sin\theta}{\theta}\right]\left[\lim_{\theta\to 0} \frac{1}{2\cancel{\theta}\left(\frac{\sin 2\theta}{2\theta}\right)}\right]$$

$$\left[\lim_{\theta\to 0} \frac{\sin\theta}{\theta}\right] \cdot \left[\lim_{\theta\to 0} \frac{1}{2\left(\frac{\sin 2\theta}{2\theta}\right)}\right]$$

$$1 \cdot \frac{1}{2} = \frac{1}{2}$$

$$\textbf{42.}\ \lim_{y\to 0}\frac{\sin 3y\cot 5y}{y\cot 4y}$$

$$\underbrace{\left(\lim_{y\to 0}\frac{\sin y}{y}\right)}_{1}\left(\lim_{y\to 0}\tan 4y\right)\left(\lim_{y\to 0}\frac{\cos 5y}{\sin 5y}\right)$$

$$\left(\lim_{y\to 0}\frac{\sin 4y}{\cos 4y}\right)\underbrace{\left(\lim_{y\to 0}\cos 5y\right)}_{1}\left(\lim_{y\to 0}\frac{1}{5y\underbrace{\left[\frac{\sin 5y}{5y}\right]}_{1}}\right)$$

$$\lim_{y\to 0}4y\left(\frac{\sin 4y}{4y}\right)\left[\lim_{y\to 0}\frac{1}{\cos 4y}\right]\left[\lim_{y\to 0}\frac{1}{5y}\right]\left[\frac{1}{\lim_{y\to 0}\frac{\sin 5y}{5y}}\right]$$

$$\lim_{y\to 0} 4\cancel{y}\left(\frac{\sin 4y}{4y}\right)\left[\lim_{y\to 0}\frac{1}{\cos 4y}\right]\left[\lim_{y\to 0}\frac{1}{5\cancel{y}}\right]\left[\frac{1}{\lim_{y\to 0}\frac{\sin 5y}{5y}}\right]$$

$$\left[\lim_{y\to 0}\frac{\sin 4y}{4y}\right]\cdot 1\cdot\frac{1}{5}\cdot 1$$

$$1\cdot\frac{1}{5}=\frac{1}{5}$$

---

43. $\displaystyle\lim_{\theta\to 0}\frac{\tan\theta}{\theta^2\cot 3\theta}$

$$\lim_{\theta \to 0} \left(\frac{1}{\theta^2}\right)\left(\frac{\sin\theta}{\cos\theta}\right)(\tan 3\theta)$$

$$\lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta}\right) \cdot \left[\lim_{\theta \to 0} \frac{1}{\cos\theta}\right] \cdot \left[\lim_{\theta \to 0} 3\left(\frac{\sin 3\theta}{3\theta}\right)\right]\left[\lim_{\theta \to 0} \frac{1}{\cos 3\theta}\right]$$

$$1 \cdot 1 \cdot 3 \cdot 1 = 3$$

---

44. $\displaystyle \lim_{\theta \to 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta}$

$$\left[\theta \left(\frac{1}{\ldots}\right)\left(\frac{\cos 4\theta}{\ldots}\right) \tan^2 2\theta\right]$$

$$\lim_{\theta\to 0}\left\{\theta\,(\sin^2\theta)(\sin 4\theta)\tan 2\theta\right\}$$

$$\lim_{\theta\to 0}\left[\theta\left[\frac{1}{\left\{\theta\left(\frac{\sin\theta}{\theta}\right)\right\}^2}\right]\left[\frac{\cos 4\theta}{4\theta\left(\frac{\sin 4\theta}{4\theta}\right)}\right]\left[\left(\frac{\sin 2\theta}{\cos 2\theta}\right)^2\right]\right]$$

$$\lim_{\theta\to 0}\frac{\theta}{\theta^2}\left[\frac{1}{\left(\frac{\sin\theta}{\theta}\right)^2}\right]\left[\frac{\cos 4\theta}{4\theta}\right]\left[\frac{1}{\left(\frac{\sin 4\theta}{4\theta}\right)}\right]\left[\frac{4\theta^2}{\cos^2 2\theta}\right]\left(\frac{\sin 2\theta}{2\theta}\right)^2$$

$$\lim_{\theta \to 0} \left\{ \left(\frac{\theta}{\theta^2}\right)\left(\frac{4\theta^2}{4\theta}\right)\left[\frac{1}{\left(\frac{\sin\theta}{\theta}\right)^2}\right] \cos 4\theta \left[\frac{1}{\frac{\sin 4\theta}{4\theta}}\right]\left[\frac{1}{\cos^2 2\theta}\right]\left[\frac{\sin 2\theta}{2\theta}\right]^2 \right\}$$

$$\lim_{\theta \to 0} \left\{ \frac{1}{\left(\frac{\sin\theta}{\theta}\right)^2} \cdot \cos 4\theta \cdot \frac{1}{\left(\frac{\sin 4\theta}{4\theta}\right)} \cdot \frac{1}{\cos^2 2\theta} \cdot \left(\frac{\sin 2\theta}{2\theta}\right)^2 \right\}$$

$$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

---

45. $\lim\limits_{x \to 0} \dfrac{1 - \cos 3x}{2x}$

$$\left(\lim_{x \to 0} \frac{3}{2}\right)\left(\lim_{x \to 0} \frac{1 - \cos 3x}{3x}\right)$$

$$\frac{3}{2} \cdot 0 = 0$$

---

46. $\lim\limits_{x \to 0} \dfrac{\cos^2 x - \cos x}{x^2}$

$$\lim_{x\to 0} \frac{1-\sin^2 x - \cos x}{x^2} = \lim_{x\to 0} \frac{-\sin^2 x + (1-\cos x)}{x^2}$$

$$\lim_{x\to 0} -\frac{\sin^2 x}{x^2} + \lim_{x\to 0} \frac{1-\cos x}{x^2}$$

$$(-1)\cdot \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 + \lim_{x\to 0} \frac{1-\cos x}{x^2}\left[\frac{1+\cos x}{1+\cos x}\right]$$

$$(-1)\cdot 1 + \lim_{x\to 0}\left(\frac{1-\cos^2 x}{x^2}\right)\left(\frac{1}{1+\cos x}\right)$$

$$-1 + \lim_{x\to 0} \frac{\sin^2 x}{x^2} \cdot \lim_{x\to 0} \frac{1}{1+\cos x}$$

$$-1 + 1 \cdot \frac{1}{2}$$

$$-1 + \frac{1}{2} = -1\frac{1}{2} = \frac{-3}{2}$$