

1.5 LIMITS

- Find limits of functions graphically and numerically.
- Use the properties of limits to evaluate limits of functions.
- Use different analytic techniques to evaluate limits of functions.
- Evaluate one-sided limits.
- Recognize unbounded behavior of functions.

The Limit of a Function

In everyday language, people refer to a speed limit, a wrestler's weight limit, the limit of one's endurance, or stretching a spring to its limit. These phrases all suggest that a limit is a bound, which on some occasions may not be reached but on other occasions may be reached or exceeded.

Consider a spring that will break only if a weight of 10 pounds or more is attached. To determine how far the spring will stretch without breaking, you could attach increasingly heavier weights and measure the spring length s for each weight w , as shown in Figure 1.51. If the spring length approaches a value of L , then it is said that "the limit of s as w approaches 10 is L ." A mathematical limit is much like the limit of a spring. The notation for a limit is

$$\lim_{x \rightarrow c} f(x) = L$$

which is read as "the limit of $f(x)$ as x approaches c is L ."

EXAMPLE 1 Finding a Limit

Find the limit: $\lim_{x \rightarrow 1} (x^2 + 1)$.

SOLUTION Let $f(x) = x^2 + 1$. From the graph of f in Figure 1.52, it appears that $f(x)$ approaches 2 as x approaches 1 from either side, and you can write

$$\lim_{x \rightarrow 1} (x^2 + 1) = 2.$$

The table yields the same conclusion. Notice that as x gets closer and closer to 1, $f(x)$ gets closer and closer to 2.

	← x approaches 1.			← x approaches 1.			
x	0.900	0.990	0.999	1.000	1.001	1.010	1.100
$f(x)$	1.810	1.980	1.998	2.000	2.002	2.020	2.210
	← $f(x)$ approaches 2.			← $f(x)$ approaches 2.			

TRY IT 1

Find the limit: $\lim_{x \rightarrow 1} (2x + 4)$.

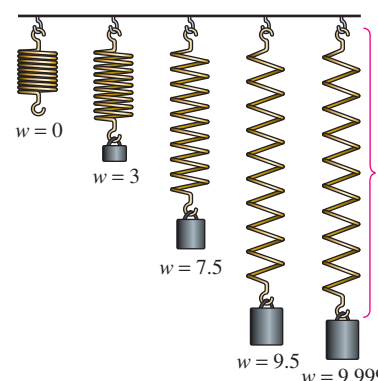


FIGURE 1.51 What is the limit of s as w approaches 10 pounds?

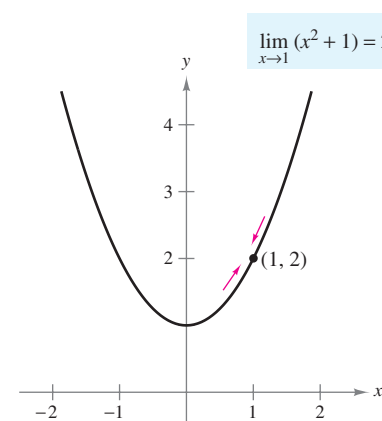


FIGURE 1.52

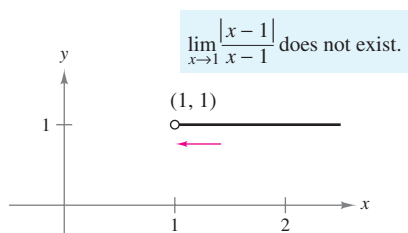
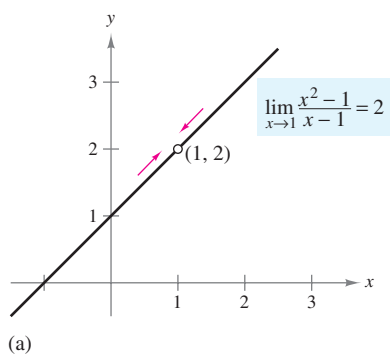
EXAMPLE 2 Finding Limits Graphically and Numerically

Find the limit: $\lim_{x \rightarrow 1} f(x)$.

(a) $f(x) = \frac{x^2 - 1}{x - 1}$ (b) $f(x) = \frac{|x - 1|}{x - 1}$ (c) $f(x) = \begin{cases} x, & x \neq 1 \\ 0, & x = 1 \end{cases}$

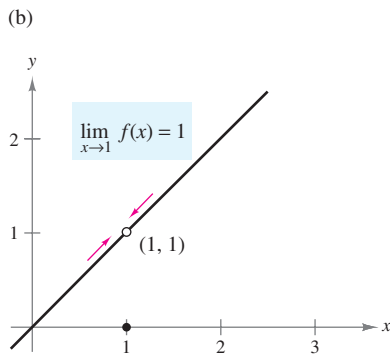
SOLUTION

(a) From the graph of f , in Figure 1.53(a), it appears that $f(x)$ approaches 2 as x approaches 1 from either side. A missing point is denoted by the open dot on the graph. This conclusion is reinforced by the table. Be sure you see that *it does not matter that $f(x)$ is undefined when $x = 1$* . The limit depends only on values of $f(x)$ near 1, not at 1.



	x approaches 1.			x approaches 1.			
x	0.900	0.990	0.999	1.000	1.001	1.010	1.100
f(x)	1.900	1.990	1.999	?	2.001	2.010	2.100
	f(x) approaches 2.			f(x) approaches 2.			

(b) From the graph of f , in Figure 1.53(b), you can see that $f(x) = -1$ for all values to the left of $x = 1$ and $f(x) = 1$ for all values to the right of $x = 1$. So, $f(x)$ is approaching a different value from the left of $x = 1$ than it is from the right of $x = 1$. In such situations, we say that the *limit does not exist*. This conclusion is reinforced by the table.



	x approaches 1.			x approaches 1.			
x	0.900	0.990	0.999	1.000	1.001	1.010	1.100
f(x)	-1.000	-1.000	-1.000	?	1.000	1.000	1.000
	f(x) approaches -1.			f(x) approaches 1.			

(c) From the graph of f , in Figure 1.53(c), it appears that $f(x)$ approaches 1 as x approaches 1 from either side. This conclusion is reinforced by the table. It does not matter that $f(1) = 0$. The limit depends only on values of $f(x)$ near 1, not at 1.

FIGURE 1.53

TRY IT 2

Find the limit: $\lim_{x \rightarrow 2} f(x)$.

(a) $f(x) = \frac{x^2 - 4}{x - 2}$

(b) $f(x) = \frac{|x - 2|}{x - 2}$

(c) $f(x) = \begin{cases} x^2, & x \neq 2 \\ 0, & x = 2 \end{cases}$

	x approaches 1.			x approaches 1.			
x	0.900	0.990	0.999	1.000	1.001	1.010	1.100
f(x)	0.900	0.990	0.999	0.000	1.001	1.010	1.100
	f(x) approaches 1.			f(x) approaches 1.			

There are three important ideas to learn from Examples 1 and 2.

1. Saying that the limit of $f(x)$ approaches L as x approaches c means that the value of $f(x)$ may be made *arbitrarily close* to the number L by choosing x closer and closer to c .
2. For a limit to exist, you must allow x to approach c from *either side* of c . If $f(x)$ approaches a different number as x approaches c from the left than it does as x approaches c from the right, then the limit *does not exist*. [See Example 2(b).]
3. The value of $f(x)$ when $x = c$ has no bearing on the existence or nonexistence of the limit of $f(x)$ as x approaches c . For instance, in Example 2(a), the limit of $f(x)$ exists as x approaches 1 even though the function f is not defined at $x = 1$.

Definition of the Limit of a Function

If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, then

$$\lim_{x \rightarrow c} f(x) = L$$

which is read as “the **limit** of $f(x)$ as x approaches c is L .”

Properties of Limits

Many times the limit of $f(x)$ as x approaches c is simply $f(c)$, as shown in Example 1. Whenever the limit of $f(x)$ as x approaches c is

$$\lim_{x \rightarrow c} f(x) = f(c) \quad \text{Substitute } c \text{ for } x.$$

the limit can be evaluated by **direct substitution**. (In the next section, you will learn that a function that has this property is *continuous at c*.) It is important that you learn to recognize the types of functions that have this property. Some basic ones are given in the following list.


Properties of Limits

Let b and c be real numbers, and let n be a positive integer.

1. $\lim_{x \rightarrow c} b = b$
2. $\lim_{x \rightarrow c} x = c$
3. $\lim_{x \rightarrow c} x^n = c^n$
4. $\lim_{x \rightarrow c} \sqrt[n]{x} = \sqrt[n]{c}$

In Property 4, if n is even, then c must be positive.

TECHNOLOGY

 Try using a graphing utility to determine the following limit.

$$\lim_{x \rightarrow 1} \frac{x^3 + 4x - 5}{x - 1}$$

You can do this by graphing

$$f(x) = \frac{x^3 + 4x - 5}{x - 1}$$

and zooming in near $x = 1$. From the graph, what does the limit appear to be?

By combining the properties of limits with the rules for operating with limits shown below, you can find limits for a wide variety of algebraic functions.

TECHNOLOGY



Symbolic computer algebra systems are capable of evaluating limits. Try using a computer algebra system to evaluate the limit given in Example 3.

Operations with Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = K$$

1. Scalar multiple: $\lim_{x \rightarrow c} [bf(x)] = bL$
2. Sum or difference: $\lim_{x \rightarrow c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x \rightarrow c} [f(x) \cdot g(x)] = LK$
4. Quotient: $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{K}$, provided $K \neq 0$
5. Power: $\lim_{x \rightarrow c} [f(x)]^n = L^n$
6. Radical: $\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L}$

In Property 6, if n is even, then L must be positive.

DISCOVERY

Use a graphing utility to graph $y_1 = 1/x^2$. Does y_1 approach a limit as x approaches 0? Evaluate $y_1 = 1/x^2$ at several positive and negative values of x near 0 to confirm your answer. Does $\lim_{x \rightarrow 1} 1/x^2$ exist?

EXAMPLE 3 Finding the Limit of a Polynomial Function

Find the limit: $\lim_{x \rightarrow 2} (x^2 + 2x - 3)$.

$$\begin{aligned} \lim_{x \rightarrow 2} (x^2 + 2x - 3) &= \lim_{x \rightarrow 2} x^2 + \lim_{x \rightarrow 2} 2x - \lim_{x \rightarrow 2} 3 \\ &= 2^2 + 2(2) - 3 \\ &= 4 + 4 - 3 \\ &= 5 \end{aligned}$$

Apply Property 2.

Use direct substitution.

Simplify.

TRY IT 3

Find the limit: $\lim_{x \rightarrow 1} (2x^2 - x + 4)$.

Example 3 is an illustration of the following important result, which states that the limit of a polynomial can be evaluated by direct substitution.

The Limit of a Polynomial Function

If p is a polynomial function and c is any real number, then

$$\lim_{x \rightarrow c} p(x) = p(c).$$

Techniques for Evaluating Limits

Many techniques for evaluating limits are based on the following important theorem. Basically, the theorem states that if two functions agree at all but a single point c , then they have identical limit behavior at $x = c$.

The Replacement Theorem

Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$. If the limit of $g(x)$ exists as $x \rightarrow c$, then the limit of $f(x)$ also exists and

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x).$$

To apply the Replacement Theorem, you can use a result from algebra which states that for a polynomial function p , $p(c) = 0$ if and only if $(x - c)$ is a factor of $p(x)$. This concept is demonstrated in Example 4.

EXAMPLE 4 Finding the Limit of a Function

Find the limit: $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1}$.

SOLUTION Note that the numerator and denominator are zero when $x = 1$. This implies that $x - 1$ is a factor of both, and you can divide out this like factor.

$$\begin{aligned} \frac{x^3 - 1}{x - 1} &= \frac{(x - 1)(x^2 + x + 1)}{x - 1} && \text{Factor numerator.} \\ &= \frac{\cancel{(x - 1)}(x^2 + x + 1)}{\cancel{x - 1}} && \text{Divide out like factor.} \\ &= x^2 + x + 1, \quad x \neq 1 && \text{Simplify.} \end{aligned}$$

So, the rational function $(x^3 - 1)/(x - 1)$ and the polynomial function $x^2 + x + 1$ agree for all values of x other than $x = 1$, and you can apply the Replacement Theorem.

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = \lim_{x \rightarrow 1} (x^2 + x + 1) = 1^2 + 1 + 1 = 3$$

Figure 1.54 illustrates this result graphically. Note that the two graphs are identical except that the graph of g contains the point $(1, 3)$, whereas this point is missing on the graph of f . (In the graph of f in Figure 1.54, the missing point is denoted by an open dot.)

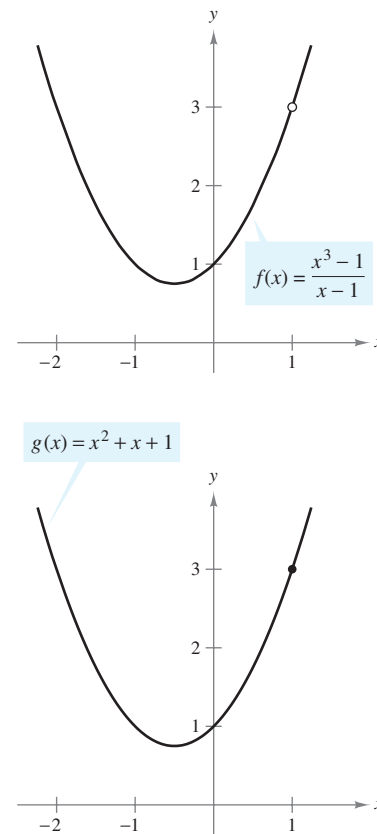


FIGURE 1.54

TRY IT 4

Find the limit: $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2}$.

The technique used to evaluate the limit in Example 4 is called the **dividing out** technique. This technique is further demonstrated in the next example.

DISCOVERY

Use a graphing utility to graph

$$y = \frac{x^2 + x - 6}{x + 3}.$$

Is the graph a line? Why or why not?

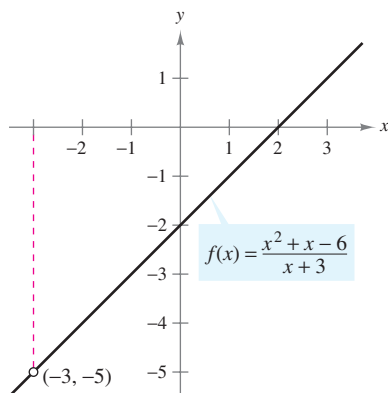


FIGURE 1.55 f is undefined when $x = -3$.

EXAMPLE 5 Using the Dividing Out Technique



Find the limit: $\lim_{x \rightarrow -3} \frac{x^2 + x - 6}{x + 3}$.

SOLUTION Direct substitution fails because both the numerator and the denominator are zero when $x = -3$.

$$\lim_{x \rightarrow -3} \frac{x^2 + x - 6}{x + 3} \quad \begin{array}{l} \leftarrow \lim_{x \rightarrow -3} (x^2 + x - 6) = 0 \\ \leftarrow \lim_{x \rightarrow -3} (x + 3) = 0 \end{array}$$

However, because the limits of both the numerator and denominator are zero, you know that they have a *common factor* of $x + 3$. So, for all $x \neq -3$, you can divide out this factor to obtain the following.

$$\begin{aligned} \lim_{x \rightarrow -3} \frac{x^2 + x - 6}{x + 3} &= \lim_{x \rightarrow -3} \frac{(x - 2)(x + 3)}{x + 3} && \text{Factor numerator.} \\ &= \lim_{x \rightarrow -3} \frac{(x - 2)\cancel{(x + 3)}}{\cancel{x + 3}} && \text{Divide out like factor.} \\ &= \lim_{x \rightarrow -3} (x - 2) && \text{Simplify.} \\ &= -5 && \text{Direct substitution} \end{aligned}$$

This result is shown graphically in Figure 1.55. Note that the graph of f coincides with the graph of $g(x) = x - 2$, except that the graph of f has a hole at $(-3, -5)$.

TRY IT 5

Find the limit: $\lim_{x \rightarrow 3} \frac{x^2 + x - 12}{x - 3}$.

STUDY TIP

When you try to evaluate a limit and both the numerator and denominator are zero, remember that you must rewrite the fraction so that the new denominator does not have 0 as its limit. One way to do this is to divide out like factors, as shown in Example 5. Another technique is to rationalize the numerator, as shown in Example 6.

TRY IT 6

Find the limit: $\lim_{x \rightarrow 0} \frac{\sqrt{x + 4} - 2}{x}$.

EXAMPLE 6 Finding a Limit of a Function

Find the limit: $\lim_{x \rightarrow 0} \frac{\sqrt{x + 1} - 1}{x}$.

SOLUTION Direct substitution fails because both the numerator and the denominator are zero when $x = 0$. In this case, you can rewrite the fraction by rationalizing the numerator.

$$\begin{aligned} \frac{\sqrt{x + 1} - 1}{x} &= \left(\frac{\sqrt{x + 1} - 1}{x} \right) \left(\frac{\sqrt{x + 1} + 1}{\sqrt{x + 1} + 1} \right) \\ &= \frac{(x + 1) - 1}{x(\sqrt{x + 1} + 1)} \\ &= \frac{\cancel{x}}{\cancel{x}(\sqrt{x + 1} + 1)} = \frac{1}{\sqrt{x + 1} + 1}, \quad x \neq 0 \end{aligned}$$

Now, using the Replacement Theorem, you can evaluate the limit as shown.

$$\lim_{x \rightarrow 0} \frac{\sqrt{x + 1} - 1}{x} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x + 1} + 1} = \frac{1}{1 + 1} = \frac{1}{2}$$

One-Sided Limits

In Example 2(b), you saw that one way in which a limit can fail to exist is when a function approaches a different value from the left of c than it approaches from the right of c . This type of behavior can be described more concisely with the concept of a **one-sided limit**.

$$\lim_{x \rightarrow c^-} f(x) = L \quad \text{Limit from the left}$$

$$\lim_{x \rightarrow c^+} f(x) = L \quad \text{Limit from the right}$$

The first of these two limits is read as “the limit of $f(x)$ as x approaches c from the left is L .” The second is read as “the limit of $f(x)$ as x approaches c from the right is L .”

EXAMPLE 7 Finding One-Sided Limits

Find the limit as $x \rightarrow 0$ from the left and the limit as $x \rightarrow 0$ from the right for the function

$$f(x) = \frac{|2x|}{x}.$$

SOLUTION From the graph of f , shown in Figure 1.56, you can see that $f(x) = -2$ for all $x < 0$. So, the limit from the left is

$$\lim_{x \rightarrow 0^-} \frac{|2x|}{x} = -2. \quad \text{Limit from the left}$$

Because $f(x) = 2$ for all $x > 0$, the limit from the right is

$$\lim_{x \rightarrow 0^+} \frac{|2x|}{x} = 2. \quad \text{Limit from the right}$$

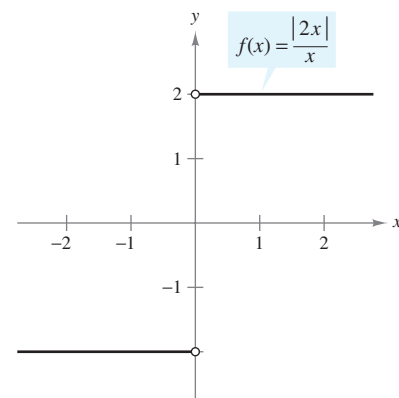


FIGURE 1.56

TRY IT 7

Find each limit. (a) $\lim_{x \rightarrow 2^-} \frac{|x-2|}{x-2}$ (b) $\lim_{x \rightarrow 2^+} \frac{|x-2|}{x-2}$

In Example 7, note that the function approaches different limits from the left and from the right. In such cases, the limit of $f(x)$ as $x \rightarrow c$ does not exist. For the limit of a function to exist as $x \rightarrow c$, *both* one-sided limits must exist and must be equal.

Existence of a Limit

If f is a function and c and L are real numbers, then

$$\lim_{x \rightarrow c} f(x) = L$$

if and only if both the left and right limits are equal to L .

TECHNOLOGY

On most graphing utilities, the absolute value function is denoted by *abs*. You can verify the result in Example 7 by graphing

$$y = \frac{\text{abs}(2x)}{x}$$

in the viewing window $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$.

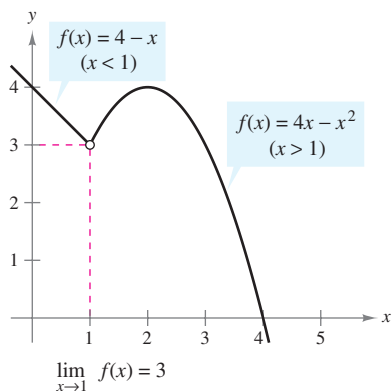


FIGURE 1.57

TRY IT 8

Find the limit of $f(x)$ as x approaches 0.

$$f(x) = \begin{cases} x^2 + 1, & x < 0 \\ 2x + 1, & x > 0 \end{cases}$$

EXAMPLE 8 Finding One-Sided Limits

Find the limit of $f(x)$ as x approaches 1.

$$f(x) = \begin{cases} 4 - x, & x < 1 \\ 4x - x^2, & x > 1 \end{cases}$$

SOLUTION Remember that you are concerned about the value of f near $x = 1$ rather than at $x = 1$. So, for $x < 1$, $f(x)$ is given by $4 - x$, and you can use direct substitution to obtain

$$\begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} (4 - x) \\ &= 4 - 1 = 3. \end{aligned}$$

For $x > 1$, $f(x)$ is given by $4x - x^2$, and you can use direct substitution to obtain

$$\begin{aligned} \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} (4x - x^2) \\ &= 4(1) - 1^2 = 4 - 1 = 3. \end{aligned}$$

Because both one-sided limits exist and are equal to 3, it follows that

$$\lim_{x \rightarrow 1} f(x) = 3.$$

The graph in Figure 1.57 confirms this conclusion.

EXAMPLE 9 Comparing One-Sided Limits

An overnight delivery service charges \$8 for the first pound and \$2 for each additional pound. Let x represent the weight of a parcel and let $f(x)$ represent the shipping cost.

$$f(x) = \begin{cases} 8, & 0 < x \leq 1 \\ 10, & 1 < x \leq 2 \\ 12, & 2 < x \leq 3 \end{cases}$$

Show that the limit of $f(x)$ as $x \rightarrow 2$ does not exist.

SOLUTION The graph of f is shown in Figure 1.58. The limit of $f(x)$ as x approaches 2 from the left is

$$\lim_{x \rightarrow 2^-} f(x) = 10$$

whereas the limit of $f(x)$ as x approaches 2 from the right is

$$\lim_{x \rightarrow 2^+} f(x) = 12.$$

Because these one-sided limits are not equal, the limit of $f(x)$ as $x \rightarrow 2$ does not exist.

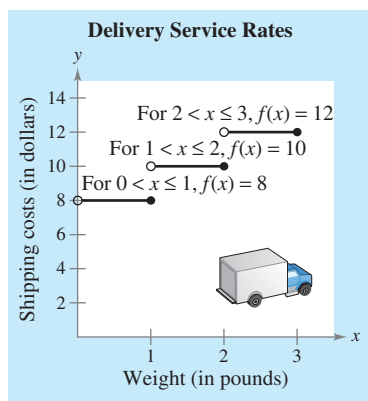


FIGURE 1.58

TRY IT 9

Show that the limit of $f(x)$ as $x \rightarrow 1$ does not exist in Example 9.

Unbounded Behavior

Example 9 shows a limit that fails to exist because the limits from the left and right differ. Another important way in which a limit can fail to exist is when $f(x)$ increases or decreases without bound as x approaches c .

EXAMPLE 10 An Unbounded Function

Find the limit (if possible).

$$\lim_{x \rightarrow 2} \frac{3}{x - 2}$$

SOLUTION From Figure 1.59, you can see that $f(x)$ decreases without bound as x approaches 2 from the left and $f(x)$ increases without bound as x approaches 2 from the right. Symbolically, you can write this as

$$\lim_{x \rightarrow 2^-} \frac{3}{x - 2} = -\infty$$

and

$$\lim_{x \rightarrow 2^+} \frac{3}{x - 2} = \infty.$$

Because f is unbounded as x approaches 2, the limit does not exist.

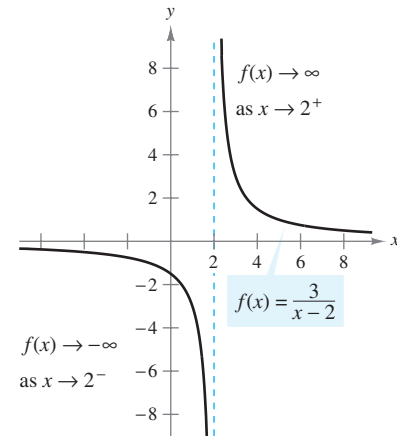


FIGURE 1.59

TRY IT 10

Find the limit (if possible): $\lim_{x \rightarrow -2} \frac{5}{x + 2}$

STUDY TIP

The equal sign in the statement $\lim_{x \rightarrow c^+} f(x) = \infty$ does not mean that the limit exists. On the contrary, it tells you how the limit *fails to exist* by denoting the unbounded behavior of $f(x)$ as x approaches c .

TAKE ANOTHER LOOK

Evaluating a Limit

Consider the limit from Example 6.

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x}$$

- Approximate the limit graphically, using a graphing utility.
- Approximate the limit numerically, by constructing a table.
- Find the limit analytically, using the Replacement Theorem, as shown in Example 6.

Of the three methods, which do you prefer? Explain your reasoning.

**PREREQUISITE
REVIEW 1.5**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–4, evaluate the expression and simplify.

1. $f(x) = x^2 - 3x + 3$

(a) $f(-1)$ (b) $f(c)$ (c) $f(x + h)$

2. $f(x) = \begin{cases} 2x - 2, & x < 1 \\ 3x + 1, & x \geq 1 \end{cases}$

(a) $f(-1)$ (b) $f(3)$ (c) $f(t^2 + 1)$

3. $f(x) = x^2 - 2x + 2$ $\frac{f(1+h) - f(1)}{h}$

4. $f(x) = 4x$ $\frac{f(2+h) - f(2)}{h}$

In Exercises 5–8, find the domain and range of the function and sketch its graph.

5. $h(x) = -\frac{5}{x}$

6. $g(x) = \sqrt{25 - x^2}$

7. $f(x) = |x - 3|$

8. $f(x) = \frac{|x|}{x}$

In Exercises 9 and 10, determine whether y is a function of x .

9. $9x^2 + 4y^2 = 49$

10. $2x^2y + 8x = 7y$

EXERCISES 1.5

In Exercises 1–8, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.

1. $\lim_{x \rightarrow 2} (5x + 4)$

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$?			

2. $\lim_{x \rightarrow 2} (x^2 - 3x + 1)$

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$?			

3. $\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 4}$

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$?			

4. $\lim_{x \rightarrow 2} \frac{x^5 - 32}{x - 2}$

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$?			

5. $\lim_{x \rightarrow 0} \frac{\sqrt{x+3} - \sqrt{3}}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$?			

6. $\lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$?			

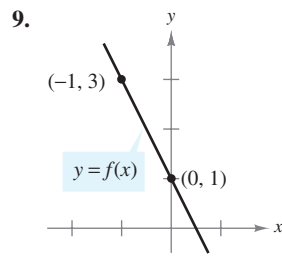
7. $\lim_{x \rightarrow 0^-} \frac{\frac{1}{x+4} - \frac{1}{4}}{x}$

x	-0.5	-0.1	-0.01	-0.001	0
$f(x)$?

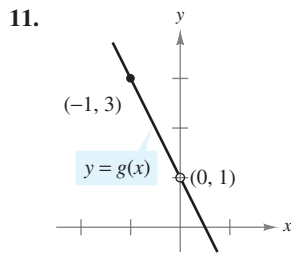
8. $\lim_{x \rightarrow 0^+} \frac{\frac{1}{2+x} - \frac{1}{2}}{2x}$

x	0.5	0.1	0.01	0.001	0
$f(x)$?

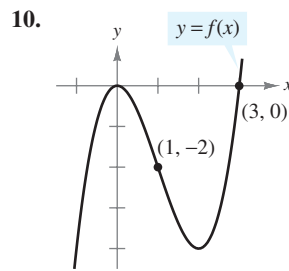
In Exercises 9–12, use the graph to find the limit (if it exists).



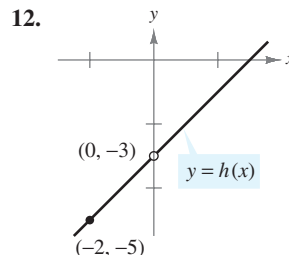
- (a) $\lim_{x \rightarrow 0} f(x)$
 (b) $\lim_{x \rightarrow -1} f(x)$



- (a) $\lim_{x \rightarrow 0} g(x)$
 (b) $\lim_{x \rightarrow -1} g(x)$



- (a) $\lim_{x \rightarrow 1} f(x)$
 (b) $\lim_{x \rightarrow 3} f(x)$



- (a) $\lim_{x \rightarrow -2} h(x)$
 (b) $\lim_{x \rightarrow 0} h(x)$

In Exercises 13 and 14, find the limit of (a) $f(x) + g(x)$, (b) $f(x)g(x)$, and (c) $f(x)/g(x)$ as x approaches c .

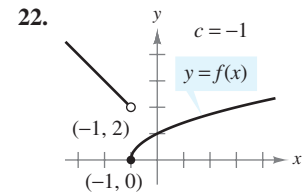
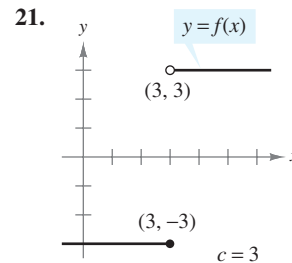
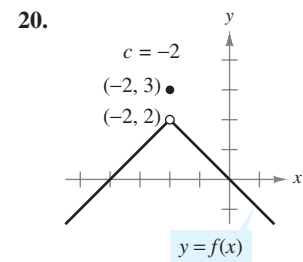
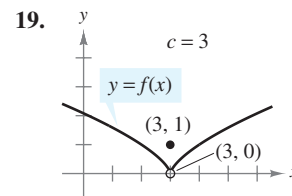
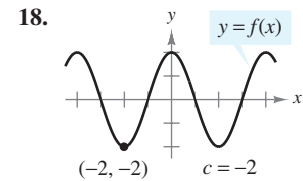
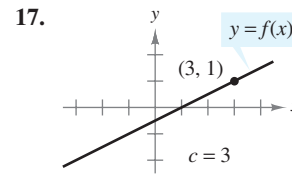
13. $\lim_{x \rightarrow c} f(x) = 3$ 14. $\lim_{x \rightarrow c} f(x) = \frac{3}{2}$
 $\lim_{x \rightarrow c} g(x) = 9$ $\lim_{x \rightarrow c} g(x) = \frac{1}{2}$

In Exercises 15 and 16, find the limit of (a) $\sqrt{f(x)}$, (b) $[3f(x)]$, and (c) $[f(x)]^2$ as x approaches c .

15. $\lim_{x \rightarrow c} f(x) = 16$ 16. $\lim_{x \rightarrow c} f(x) = 9$

In Exercises 17–22, use the graph to find the limit (if it exists).

- (a) $\lim_{x \rightarrow c^+} f(x)$
 (b) $\lim_{x \rightarrow c^-} f(x)$
 (c) $\lim_{x \rightarrow c} f(x)$



In Exercises 23–40, find the limit.

23. $\lim_{x \rightarrow 2} x^4$ 24. $\lim_{x \rightarrow -2} x^3$
 25. $\lim_{x \rightarrow -3} (3x + 2)$ 26. $\lim_{x \rightarrow 0} (2x - 3)$
 27. $\lim_{x \rightarrow 1} (1 - x^2)$ 28. $\lim_{x \rightarrow 2} (-x^2 + x - 2)$
 29. $\lim_{x \rightarrow 3} \sqrt{x + 1}$ 30. $\lim_{x \rightarrow 4} \sqrt[3]{x + 4}$
 31. $\lim_{x \rightarrow -3} \frac{2}{x + 2}$ 32. $\lim_{x \rightarrow -2} \frac{3x + 1}{2 - x}$
 33. $\lim_{x \rightarrow -2} \frac{x^2 - 1}{2x}$ 34. $\lim_{x \rightarrow -1} \frac{4x - 5}{3 - x}$
 35. $\lim_{x \rightarrow 7} \frac{5x}{x + 2}$ 36. $\lim_{x \rightarrow 3} \frac{\sqrt{x + 1}}{x - 4}$
 37. $\lim_{x \rightarrow 3} \frac{\sqrt{x + 1} - 1}{x}$ 38. $\lim_{x \rightarrow 5} \frac{\sqrt{x + 4} - 2}{x}$
 39. $\lim_{x \rightarrow 1} \frac{\frac{1}{x+4} - \frac{1}{4}}{x}$ 40. $\lim_{x \rightarrow 2} \frac{\frac{1}{x+2} - \frac{1}{2}}{x}$

60 CHAPTER 1 Functions, Graphs, and Limits

In Exercises 41–58, find the limit (if it exists).

41. $\lim_{x \rightarrow -1} \frac{x^2 - 1}{x + 1}$

42. $\lim_{x \rightarrow -1} \frac{2x^2 - x - 3}{x + 1}$

43. $\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 4x + 4}$

44. $\lim_{x \rightarrow 2} \frac{2 - x}{x^2 - 4}$

45. $\lim_{t \rightarrow 5} \frac{t - 5}{t^2 - 25}$

46. $\lim_{t \rightarrow 1} \frac{t^2 + t - 2}{t^2 - 1}$

47. $\lim_{x \rightarrow -2} \frac{x^3 + 8}{x + 2}$

48. $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1}$

49. $\lim_{x \rightarrow -2} \frac{|x + 2|}{x + 2}$

50. $\lim_{x \rightarrow 2} \frac{|x - 2|}{x - 2}$

51. $\lim_{x \rightarrow 3} f(x)$, where $f(x) = \begin{cases} \frac{1}{3}x - 2, & x \leq 3 \\ -2x + 5, & x > 3 \end{cases}$

52. $\lim_{s \rightarrow 1} f(s)$, where $f(s) = \begin{cases} s, & s \leq 1 \\ 1 - s, & s > 1 \end{cases}$

53. $\lim_{\Delta x \rightarrow 0} \frac{2(x + \Delta x) - 2x}{\Delta x}$


54. $\lim_{\Delta x \rightarrow 0} \frac{4(x + \Delta x) - 5 - (4x - 5)}{\Delta x}$

55. $\lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + 2 + \Delta x} - \sqrt{x + 2}}{\Delta x}$

56. $\lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}$

57. $\lim_{\Delta t \rightarrow 0} \frac{(t + \Delta t)^2 - 5(t + \Delta t) - (t^2 - 5t)}{\Delta t}$

58. $\lim_{\Delta t \rightarrow 0} \frac{(t + \Delta t)^2 - 4(t + \Delta t) + 2 - (t^2 - 4t + 2)}{\Delta t}$


 **Graphical, Numerical, and Analytic Analysis** In Exercises 59–62, use a graphing utility to graph the function and estimate the limit. Use a table to reinforce your conclusion. Then find the limit by analytic methods.

59. $\lim_{x \rightarrow 1^-} \frac{2}{x^2 - 1}$

60. $\lim_{x \rightarrow 1^+} \frac{5}{1 - x}$

61. $\lim_{x \rightarrow -2^-} \frac{1}{x + 2}$

62. $\lim_{x \rightarrow 0^-} \frac{x + 1}{x}$

 In Exercises 63–66, use a graphing utility to estimate the limit (if it exists).

63. $\lim_{x \rightarrow 2} \frac{x^2 - 5x + 6}{x^2 - 4x + 4}$

64. $\lim_{x \rightarrow 1} \frac{x^2 + 6x - 7}{x^3 - x^2 + 2x - 2}$

65. $\lim_{x \rightarrow -4} \frac{x^3 + 4x^2 + x + 4}{2x^2 + 7x - 4}$

66. $\lim_{x \rightarrow -2} \frac{4x^3 + 7x^2 + x + 6}{3x^2 - x - 14}$

67. The limit of


$$f(x) = (1 + x)^{1/x}$$

is a natural base for many business applications, as you will see in Section 4.2.

$$\lim_{x \rightarrow 0} (1 + x)^{1/x} = e \approx 2.718$$

(a) Show the reasonableness of this limit by completing the table.

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$f(x)$							

 (b) Use a graphing utility to graph f and to confirm the answer in part (a).

(c) Find the domain and range of the function.

68. Find $\lim_{x \rightarrow 0} f(x)$, given

$$4 - x^2 \leq f(x) \leq 4 + x^2, \text{ for all } x.$$

69. **Environment** The cost (in dollars) of removing $p\%$ of the pollutants from the water in a small lake is given by

$$C = \frac{25,000p}{100 - p}, \quad 0 \leq p < 100$$

where C is the cost and p is the percent of pollutants.

(a) Find the cost of removing 50% of the pollutants.


(b) What percent of the pollutants can be removed for \$100,000?

(c) Evaluate $\lim_{p \rightarrow 100^-} C$. Explain your results.

70. **Compound Interest** You deposit \$1000 in an account that is compounded quarterly at an annual rate of r (in decimal form). The balance A after 10 years is

$$A = 1000 \left(1 + \frac{r}{4} \right)^{40}.$$

Does the limit of A exist as the interest rate approaches 6%? If so, what is the limit?

 71. **Compound Interest** Consider a certificate of deposit that pays 10% (annual percentage rate) on an initial deposit of \$500. The balance A after 10 years is

$$A = 500(1 + 0.1x)^{10/x}$$

where x is the length of the compounding period (in years).

(a) Use a graphing utility to graph A , where $0 \leq x \leq 1$.

(b) Use the *zoom* and *trace* features to estimate the balance for quarterly compounding and daily compounding.

(c) Use the *zoom* and *trace* features to estimate

$$\lim_{x \rightarrow 0^+} A.$$

What do you think this limit represents? Explain your reasoning.