9.3

OBJECTIVES

- To define and find average rates of change
- To define the derivative as a rate of change
- To use the definition of derivative to find derivatives of functions
- To use derivatives to find slopes of tangents to curves

Average and Instantaneous Rates of Change: The Derivative

Application Preview

In Chapter 1, "Linear Equations and Functions," we studied linear revenue functions and defined the marginal revenue for a product as the rate of change of the revenue function. For linear revenue functions, this rate is also the slope of the line that is the graph of the revenue function. In this section, we will define **marginal revenue** as the rate of change of the revenue function, even when the revenue function is not linear.

Thus, if an oil company's revenue (in thousands of dollars) is given by

$$R = 100x - x^2, \quad x \geq 0$$

where x is the number of thousands of barrels of oil sold per day, we can find and interpret the marginal revenue when 20,000 barrels are sold (see Example 4).

We will discuss the relationship between the marginal revenue at a given point and the slope of the line tangent to the revenue function at that point. We will see how the derivative of the revenue function can be used to find both the slope of this tangent line and the marginal revenue.

Average Rates of Change

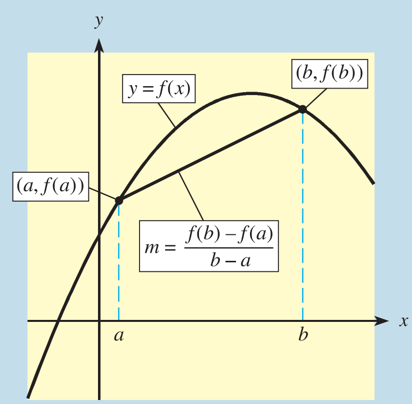
For linear functions, we have seen that the slope of the line measures the average rate of change of the function and can be found from any two points on the line. However, for a function that is not linear, the slope between different pairs of points no longer always gives the same number, but it can be interpreted as an average rate of change. We use this connection between average rates of change and slopes for linear functions to define the average rate of change for any function.

Average Rate of Change

The **average rate of change** of a function $y = f(x)$ from $x = a$ to $x = b$ is defined by

$$\text{Average rate of change} = \frac{f(b) - f(a)}{b - a}$$

The figure shows that this average rate is the same as the slope of the segment joining the points $(a, f(a))$ and $(b, f(b))$.



EXAMPLE 1 *Total Cost*

Suppose a company's total cost in dollars to produce x units of its product is given by

$$C(x) = 0.01x^2 + 25x + 1500$$

Find the average rate of change of total cost for (a) the first 100 units produced (from $x = 0$ to $x = 100$) and (b) the second 100 units produced.

Solution

(a) The average rate of change of total cost from $x = 0$ to $x = 100$ units is

$$\frac{C(100) - C(0)}{100 - 0} = \frac{(0.01(100)^2 + 25(100) + 1500) - (1500)}{100}$$

$$= \frac{4100 - 1500}{100} = \frac{2600}{100} = 26 \text{ dollars per unit}$$

(b) The average rate of change of total cost from $x = 100$ to $x = 200$ units is

$$\frac{C(200) - C(100)}{200 - 100} = \frac{(0.01(200)^2 + 25(200) + 1500) - (4100)}{100}$$

$$= \frac{6900 - 4100}{100} = \frac{2800}{100} = 28 \text{ dollars per unit}$$

EXAMPLE 2 *Elderly in the Work Force*

Figure 9.18 shows the percents of elderly men and of elderly women in the work force in selected census years from 1890 to 2000. For the years from 1950 to 2000, find and interpret the average rate of change of the percent of (a) elderly men in the work force and (b) elderly women in the work force. (c) What caused these trends?

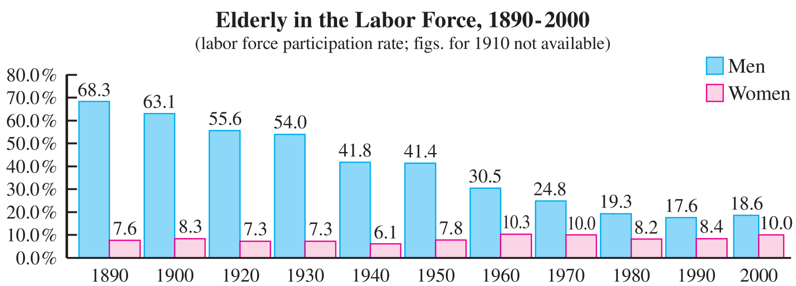


Figure 9.18

Source: Bureau of the Census, U.S. Department of Commerce

Solution

(a) From 1950 to 2000, the annual average rate of change in the percent of elderly men in the work force is

$$\frac{\text{Change in men's percent}}{\text{Change in years}} = \frac{18.6 - 41.4}{2000 - 1950} = \frac{-22.8}{50} = -0.456 \text{ percent per year}$$

This means that from 1950 to 2000, *on average,* the percent of elderly men in the work force dropped by 0.456% per year.

(b) Similarly, the average rate of change for women is

$$\frac{\text{Change in women's percent}}{\text{Change in years}} = \frac{10.0 - 7.8}{2000 - 1950} = \frac{2.2}{50} = 0.044 \text{ percent per year}$$

In like manner, this means that from 1950 to 2000, *on average,* the percent of elderly women in the work force increased by 0.044% each year.

(c) In general, from 1950 to 1990, people have been retiring earlier, but the number of women in the work force has increased dramatically.

Instantaneous Rates of Change: Velocity

Another common rate of change is velocity. For instance, if we travel 200 miles in our car over a 4-hour period, we know that we averaged 50 mph. However, during that trip there may have been times when we were traveling on an Interstate at faster than 50 mph and times when we were stopped at a traffic light. Thus, for the trip we have not only an average velocity but also instantaneous velocities (or instantaneous speeds as displayed on the speedometer). Let's see how average velocity can lead us to instantaneous velocity.

Suppose a ball is thrown straight upward at 64 feet per second from a spot 96 feet above ground level. The equation that describes the height y of the ball after x seconds is

$$y = f(x) = 96 + 64x - 16x^2$$

Figure 9.19 shows the graph of this function for $0 \le x \le 5$. The average velocity of the ball over a given time interval is the change in the height divided by the length of time that has passed. Table 9.4 shows some average velocities over time intervals beginning at $x = 1$.

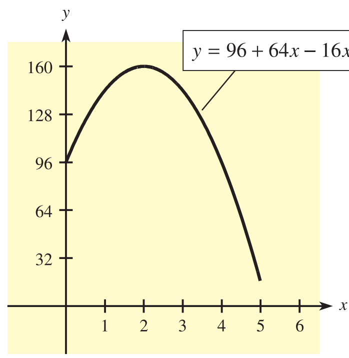


Figure 9.19

TABLE 9.4 Average Velocities

Time (seconds)			Height (feet)			Average Velocity (ft/sec)
Beginning	Ending	Change (Δx)	Beginning	Ending	Change (Δy)	($\Delta y/\Delta x$)
1	2	1	144	160	16	16/1 = 16
1	1.5	0.5	144	156	12	12/0.5 = 24
1	1.1	0.1	144	147.04	3.04	3.04/0.1 = 30.4
1	1.01	0.01	144	144.3184	0.3184	0.3184/0.01 = 31.84

In Table 9.4, the smaller the time interval, the more closely the average velocity approximates the instantaneous velocity at $x = 1$. Thus the instantaneous velocity at $x = 1$ is closer to 31.84 ft/s than to 30.4 ft/s.

If we represent the change in time by h, then the average velocity from $x = 1$ to $x = 1 + h$ approaches the instantaneous velocity at $x = 1$ as h approaches 0. (Note that h can be positive or negative.) This is illustrated in the following example.

● EXAMPLE 3 *Velocity*

Suppose a ball is thrown straight upward so that its height $f(x)$ (in feet) is given by the equation

$$f(x) = 96 + 64x - 16x^2$$

where x is time (in seconds).

(a) Find the average velocity from $x = 1$ to $x = 1 + h$.
(b) Find the instantaneous velocity at $x = 1$.

Solution

(a) Let h represent the change in x (time) from 1 to $1 + h$. Then the corresponding change in $f(x)$ (height) is

$$\begin{aligned} f(1 + h) - f(1) &= [96 + 64(1 + h) - 16(1 + h)^2] - [96 + 64 - 16] \\ &= 96 + 64 + 64h - 16(1 + 2h + h^2) - 144 \\ &= 16 + 64h - 16 - 32h - 16h^2 \\ &= 32h - 16h^2 \end{aligned}$$

The average velocity V_{av} is the change in height divided by the change in time.

$$\begin{aligned} V_{av} &= \frac{f(1 + h) - f(1)}{h} \\ &= \frac{32h - 16h^2}{h} \\ &= 32 - 16h \end{aligned}$$

(b) The instantaneous velocity V is the limit of the average velocity as h approaches 0.

$$\begin{aligned} V = \lim_{h \to 0} V_{av} &= \lim_{h \to 0} (32 - 16h) \\ &= 32 \text{ ft/s} \end{aligned}$$

Note that average velocity is found over a time interval. Instantaneous velocity is usually called **velocity,** and it can be found at any time x, as follows.

Velocity

Suppose that an object moving in a straight line has its position y at time x given by $y = f(x)$. Then the **velocity** of the object at time x is

$$V = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

provided that this limit exists.

The instantaneous rate of change of any function (commonly called *rate of change*) can be found in the same way we find velocity. The function that gives this instantaneous rate of change of a function f is called the **derivative** of f.

Derivative

If f is a function defined by $y = f(x)$, then the **derivative** of $f(x)$ at any value x, denoted $f'(x)$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

if this limit exists. If $f'(c)$ exists, we say that f is **differentiable** at c.

The following procedure illustrates how to find the derivative of a function $y = f(x)$ at any value x.

Derivative Using the Definition

Procedure	Example
To find the derivative of $y = f(x)$ at any value x:	Find the derivative of $f(x) = 4x^2$.
1. Let h represent the change in x from x to $x + h$.	1. The change in x from x to $x + h$ is h.
2. The corresponding change in $y = f(x)$ is $$f(x + h) - f(x)$$	2. The change in $f(x)$ is $$\begin{aligned} f(x + h) - f(x) &= 4(x + h)^2 - 4x^2 \\ &= 4(x^2 + 2xh + h^2) - 4x^2 \\ &= 4x^2 + 8xh + 4h^2 - 4x^2 \\ &= 8xh + 4h^2 \end{aligned}$$
3. Form the difference quotient $\frac{f(x + h) - f(x)}{h}$ and simplify.	3. $\frac{f(x + h) - f(x)}{h} = \frac{8xh + 4h^2}{h} = 8x + 4h$
4. Find $\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ to determine $f'(x)$, the derivative of $f(x)$.	4. $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ $f'(x) = \lim_{h \to 0} (8x + 4h) = 8x$

Note that in the example above, we could have found the derivative of the function $f(x) = 4x^2$ at a particular value of x, say $x = 3$, by evaluating the derivative formula at that value:

$$f'(x) = 8x \quad \text{so} \quad f'(3) = 8(3) = 24$$

In addition to $f'(x)$, the derivative at any point x may be denoted by

$$\frac{dy}{dx}, \quad y', \quad \frac{d}{dx}f(x), \quad D_x y, \quad \text{or} \quad D_x f(x)$$

We can, of course, use variables other than x and y to represent functions and their derivatives. For example, we can represent the derivative of the function defined by $p = 2q^2 - 1$ by dp/dq.

● Checkpoint

1. Find the average rate of change of $f(x) = 30 - x - x^2$ over [1, 4].
2. For the function $y = f(x) = x^2 - x + 1$, find
   (a) $f(x + h) - f(x)$
   (b) $\frac{f(x + h) - f(x)}{h}$
   (c) $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$
   (d) $f'(2)$

In Section 1.6, "Applications of Functions in Business and Economics," we defined the **marginal revenue** for a product as the rate of change of the total revenue function for the product. If the total revenue function for a product is not linear, we define the marginal revenue for the product as the instantaneous rate of change, or the derivative, of the revenue function.

Marginal Revenue

Suppose that the total revenue function for a product is given by $R = R(x)$, where x is the number of units sold. Then the **marginal revenue** at x units is

$$\overline{MR} = R'(x) = \lim_{h \to 0} \frac{R(x + h) - R(x)}{h}$$

provided that the limit exists.

Note that the marginal revenue (derivative of the revenue function) can be found by using the steps in the Procedure/Example table on the preceding page. These steps can also be combined, as they are in Example 4.

EXAMPLE 4 *Revenue (Application Preview)*

Suppose that an oil company's revenue (in thousands of dollars) is given by the equation

$$R = R(x) = 100x - x^2, \quad x \geq 0$$

where x is the number of thousands of barrels of oil sold each day.

(a) Find the function that gives the marginal revenue at any value of x.
(b) Find the marginal revenue when 20,000 barrels are sold (that is, at $x = 20$).

Solution

(a) The marginal revenue function is the derivative of $R(x)$.

$$\begin{aligned} R'(x) &= \lim_{h \to 0} \frac{R(x + h) - R(x)}{h} \\ &= \lim_{h \to 0} \frac{[100(x + h) - (x + h)^2] - (100x - x^2)}{h} \\ &= \lim_{h \to 0} \frac{100x + 100h - (x^2 + 2xh + h^2) - 100x + x^2}{h} \\ &= \lim_{h \to 0} \frac{100h - 2xh - h^2}{h} = \lim_{h \to 0}(100 - 2x - h) = 100 - 2x \end{aligned}$$

Thus, the marginal revenue function is $\overline{MR} = R'(x) = 100 - 2x$.

(b) The function found in (a) gives the marginal revenue at *any* value of x. To find the marginal revenue when 20 units are sold, we evaluate $R'(20)$.

$$R'(20) = 100 - 2(20) = 60$$

Hence the marginal revenue at $x = 20$ is \$60,000 per thousand barrels of oil. Because the marginal revenue is used to approximate the revenue from the sale of one additional unit, we interpret $R'(20) = 60$ to mean that the expected revenue from the sale of the next thousand barrels (after 20,000) will be approximately \$60,000. [*Note:* The actual revenue from this sale is $R(21) - R(20) = 1659 - 1600 = 59$ (thousand dollars).]

Tangent to a Curve

As mentioned earlier, the rate of change of revenue (the marginal revenue) for a linear revenue function is given by the slope of the line. In fact, the slope of the revenue curve gives us the marginal revenue even if the revenue function is not linear. We will show that the slope of the graph of a function at any point is the same as the derivative at that point. In order to show this, we must define the slope of a curve at a point on the curve. We will define the slope of a curve at a point as the slope of the line tangent to the curve at the point.

In geometry, a **tangent** to a circle is defined as a line that has one point in common with the circle. (See Figure 9.20(a).) This definition does not apply to all curves, as Figure 9.20(b) shows. Many lines can be drawn through the point A that touch the curve only at A. One of the lines, line l, looks like it is tangent to the curve.

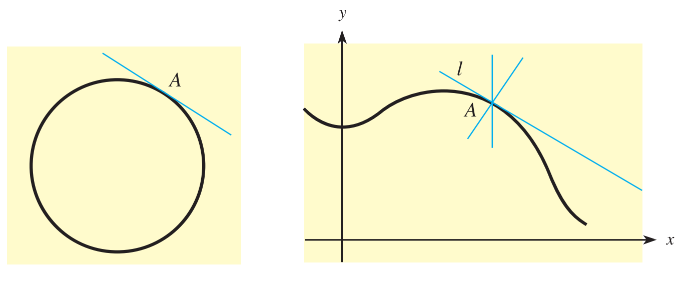


Figure 9.20

We can use **secant lines** (lines that intersect the curve at two points) to determine the tangent to a curve at a point. In Figure 9.21, we have a set of secant lines s_1, s_2, s_3, and s_4 that pass through a point A on the curve and points Q_1, Q_2, Q_3, and Q_4 on the curve near A. (For points and secant lines to the left of point A, there would be a similar figure and discussion.) The line l represents the tangent line to the curve at point A. We can get a secant line as close as we wish to the tangent line l by choosing a "second point" Q sufficiently close to point A.

As we choose points on the curve closer and closer to A (from both sides of A), the limiting position of the secant lines that pass through A is the **tangent line** to the curve at point A, and the slopes of those secant lines approach the slope of the tangent line at A. Thus we can find the slope of the tangent line by finding the slope of a secant line and taking the limit of this slope as the "second point" Q approaches A. To find the slope of the tangent to the graph of $y = f(x)$ at $A(x_1, f(x_1))$, we first draw a secant line from point A to a second point $Q(x_1 + h, f(x_1 + h))$ on the curve (see Figure 9.22).

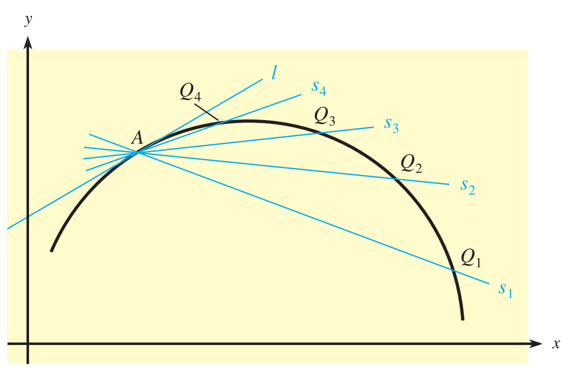


Figure 9.21

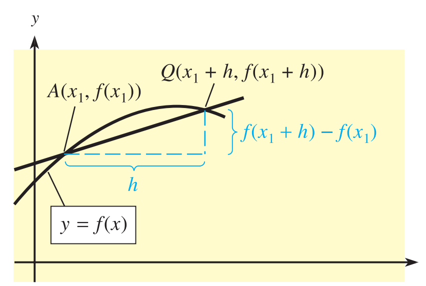


Figure 9.22

The slope of this secant line is

$$m_{AQ} = \frac{f(x_1 + h) - f(x_1)}{h}$$

As Q approaches A, we see that the difference between the x-coordinates of these two points decreases, so h approaches 0. Thus the slope of the tangent is given by the following.

Slope of the Tangent

The **slope of the tangent** to the graph of $y = f(x)$ at point $A(x_1, f(x_1))$ is

$$m = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$$

if this limit exists. That is, $m = f'(x_1)$, the derivative at $x = x_1$.

● EXAMPLE 5 *Slope of the Tangent*

Find the slope of $y = f(x) = x^2$ at the point $A(2, 4)$.

Solution

The formula for the slope of the tangent to $y = f(x)$ at (2, 4) is

$$m = f'(2) = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}$$

Thus for $f(x) = x^2$, we have

$$m = f'(2) = \lim_{h \to 0} \frac{(2 + h)^2 - 2^2}{h}$$

Taking the limit immediately would result in both the numerator and the denominator approaching 0. To avoid this, we simplify the fraction before taking the limit.

$$m = \lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h} = \lim_{h \to 0} \frac{4h + h^2}{h} = \lim_{h \to 0}(4 + h) = 4$$

Thus the slope of the tangent to $y = x^2$ at (2, 4) is 4 (see Figure 9.23).

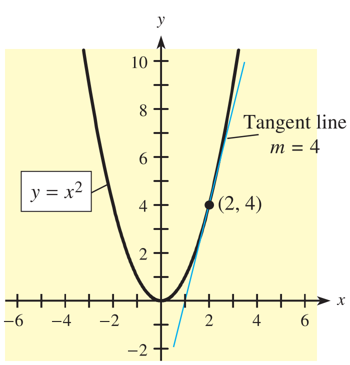


Figure 9.23

The statement "the slope of the tangent to the curve at (2, 4) is 4" is frequently simplified to the statement "the slope of the curve at (2, 4) is 4." Knowledge that the slope is a positive number on an interval tells us that the function is increasing on that interval, which means that a point moving along the graph of the function rises as it moves to the right on that interval. If the derivative (and thus the slope) is negative on an interval, the curve is decreasing on the interval; that is, a point moving along the graph falls as it moves to the right on that interval.

EXAMPLE 6 *Tangent Line*

Given $y = f(x) = 3x^2 + 2x + 11$, find

(a) the derivative of $f(x)$ at any point $(x, f(x))$.
(b) the slope of the curve at (1, 16).
(c) the equation of the line tangent to $y = 3x^2 + 2x + 11$ at (1, 16).

Solution

(a) The derivative of $f(x)$ at any value x is denoted by $f'(x)$ and is

$$\begin{aligned} y' = f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[3(x+h)^2 + 2(x+h) + 11] - (3x^2 + 2x + 11)}{h} \\ &= \lim_{h\to 0} \frac{3(x^2 + 2xh + h^2) + 2x + 2h + 11 - 3x^2 - 2x - 11}{h} \\ &= \lim_{h\to 0} \frac{6xh + 3h^2 + 2h}{h} \\ &= \lim_{h\to 0} (6x + 3h + 2) \\ &= 6x + 2 \end{aligned}$$

(b) The derivative is $f'(x) = 6x + 2$, so the slope of the tangent to the curve at (1, 16) is $f'(1) = 6(1) + 2 = 8$.
(c) The equation of the tangent line uses the given point (1, 16) and the slope $m = 8$. Using $y - y_1 = m(x - x_1)$ gives $y - 16 = 8(x - 1)$, or $y = 8x + 8$.

The discussion in this section indicates that the derivative of a function has several interpretations.

Interpretations of the Derivative

For a given function, each of the following means "find the **derivative.**"

1. Find the **velocity** of an object moving in a straight line.
2. Find the **instantaneous rate of change** of a function.
3. Find the **marginal revenue** for a given revenue function.
4. Find the **slope of the tangent** to the graph of a function.

That is, all the terms printed in boldface are mathematically the same, and the answers to questions about any one of them give information about the others. For example, if we know the slope of the tangent to the graph of a revenue function at a point, then we know the marginal revenue at that point.

Calculator Note

Note in Figure 9.23 that near the point of tangency at (2, 4), the tangent line and the function look coincident. In fact, if we graphed both with a graphing calculator and repeatedly zoomed in near the point (2, 4), the two graphs would eventually appear as one. Try this for yourself. Thus the derivative of $f(x)$ at the point where $x = a$ can be approximated by finding the slope between $(a, f(a))$ and a second point that is nearby. ■

In addition, we know that the slope of the tangent to $f(x)$ at $x = a$ is defined by

$$f'(a) = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}$$

Hence we could also estimate $f'(a)$—that is, the slope of the tangent at $x = a$—by evaluating

$$\frac{f(a + h) - f(a)}{h} \quad \text{when } h \approx 0 \text{ and } h \neq 0$$

● EXAMPLE 7 *Approximating the Slope of the Tangent Line*

(a) Let $f(x) = 2x^3 - 6x^2 + 2x - 5$. Use $\frac{f(a + h) - f(a)}{h}$ and two values of h to make estimates of the slope of the tangent to $f(x)$ at $x = 3$ on opposite sides of $x = 3$.

(b) Use the following table of values of x and $g(x)$ to estimate $g'(3)$.

x	1	1.9	2.7	2.9	2.999	3	3.002	3.1	4	5
$g(x)$	1.6	4.3	11.4	10.8	10.513	10.5	10.474	10.18	6	−5

Solution

The table feature of a graphing utility can facilitate the following calculations.

(a) We can use $h = 0.0001$ and $h = -0.0001$ as follows:

$$\text{With } h = 0.0001: \quad f'(3) \approx \frac{f(3 + 0.0001) - f(3)}{0.0001}$$

$$= \frac{f(3.0001) - f(3)}{0.0001} = 20.0012 \approx 20$$

$$\text{With } h = -0.0001: \quad f'(3) \approx \frac{f(3 + (-0.0001)) - f(3)}{-0.0001}$$

$$= \frac{f(2.9999) - f(3)}{-0.0001} = 19.9988 \approx 20$$

(b) We use the given table and measure the slope between (3, 10.5) and another point that is nearby (the closer, the better). Using (2.999, 10.513), we obtain

$$g'(3) \approx \frac{y_2 - y_1}{x_2 - x_1} = \frac{10.5 - 10.513}{3 - 2.999} = \frac{-0.013}{0.001} = -13$$

Calculator Note

Most graphing calculators have a feature called the **numerical derivative** (usually denoted by nDer or nDeriv) that can approximate the derivative of a function at a point. On most calculators this feature uses a calculation similar to our method in part (a) of Example 7 and produces the same estimate. The numerical derivative of $f(x) = 2x^3 - 6x^2 + 2x - 5$ with respect to x at $x = 3$ can be found as follows on many graphing calculators:

$$\text{nDeriv}(2x^3 - 6x^2 + 2x - 5, x, 3) = 20$$

■

Differentiability and Continuity

So far we have talked about how the derivative is defined, what it represents, and how to find it. However, there are functions for which derivatives do not exist at every value of x. Figure 9.24 shows some common cases where $f'(c)$ does not exist but where $f'(x)$ exists for all other values of x. These cases occur where there is a discontinuity, a corner, or a vertical tangent line.

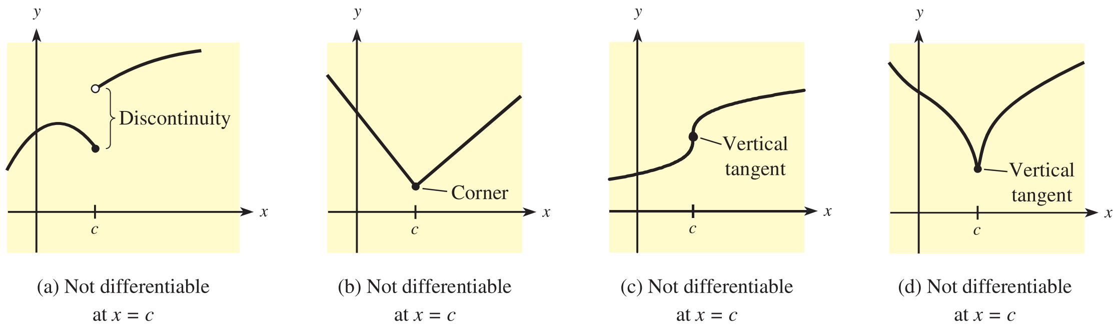


Figure 9.24

From Figure 9.24 we see that a function may be continuous at $x = c$ even though $f'(c)$ does not exist. Thus continuity does not imply differentiability at a point. However, differentiability does imply continuity.

Differentiability Implies Continuity

If a function f is differentiable at $x = c$, then f is continuous at $x = c$.

EXAMPLE 8 *Water Usage Costs*

The monthly charge for water in a small town is given by

$$y = f(x) = \begin{cases} 18 & \text{if } 0 \le x \le 20 \\ 0.1x + 16 & \text{if } x > 20 \end{cases}$$

(a) Is this function continuous at $x = 20$?
(b) Is this function differentiable at $x = 20$?

Solution

(a) We must check the three properties for continuity.

1. $f(x) = 18$ for $x \le 20$ so $f(20) = 18$
2. $$\left.\begin{aligned} \lim_{x\to 20^-} f(x) &= \lim_{x\to 20^-} 18 = 18 \\ \lim_{x\to 20^+} f(x) &= \lim_{x\to 20^+} (0.1x + 16) = 18 \end{aligned}\right\} \Rightarrow \lim_{x\to 20} f(x) = 18$$
3. $\lim_{x\to 20} f(x) = f(20)$

Thus $f(x)$ is continuous at $x = 20$.

(b) Because the function is defined differently on either side of $x = 20$, we need to test to see whether $f'(20)$ exists by evaluating both

(i) $\lim_{h\to 0^-} \dfrac{f(20 + h) - f(20)}{h}$ and (ii) $\lim_{h\to 0^+} \dfrac{f(20 + h) - f(20)}{h}$

and determining whether they are equal.

$$\text{(i)} \quad \lim_{h\to 0^-} \frac{f(20 + h) - f(20)}{h} = \lim_{h\to 0^-} \frac{18 - 18}{h}$$
$$= \lim_{h\to 0^-} 0 = 0$$

(ii) $$\lim_{h\to 0^+}\frac{f(20+h)-f(20)}{h}=\lim_{h\to 0^+}\frac{[0.1(20+h)+16]-18}{h}$$
$$=\lim_{h\to 0^+}\frac{0.1h}{h}$$
$$=\lim_{h\to 0^+}0.1=0.1$$

Because these limits are not equal, the derivative $f'(20)$ does not exist.

Checkpoint

3. Which of the following are given by $f'(c)$?
   (a) The slope of the tangent when $x = c$
   (b) The y-coordinate of the point where $x = c$
   (c) The instantaneous rate of change of $f(x)$ at $x = c$
   (d) The marginal revenue at $x = c$, if $f(x)$ is the revenue function
4. Must a graph that has no discontinuity, corner, or cusp at $x = c$ be differentiable at $x = c$?

Calculator Note

We can use a graphing calculator to explore the relationship between secant lines and tangent lines. For example, if the point (a, b) lies on the graph of $y = x^2$, then the equation of the secant line to $y = x^2$ from (1, 1) to (a, b) has the equation

$$y-1=\frac{b-1}{a-1}(x-1),\quad \text{or}\quad y=\frac{b-1}{a-1}(x-1)+1$$

Figure 9.25 illustrates the secant lines for three different choices for the point (a, b).

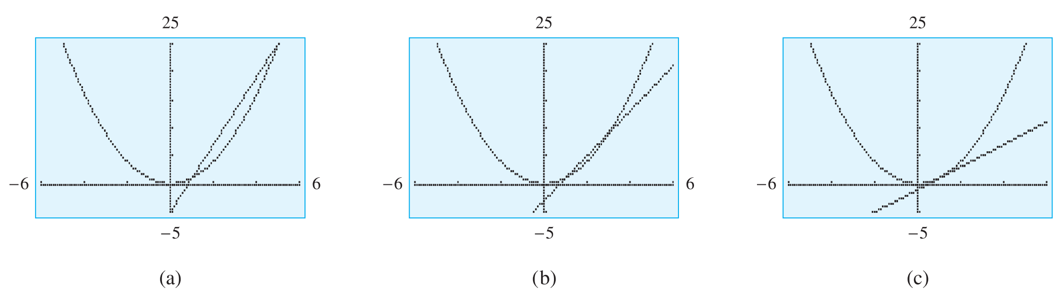


Figure 9.25

We see that as the point (a, b) moves closer to (1, 1), the secant line looks more like the tangent line to $y = x^2$ at (1, 1). Furthermore, (a, b) approaches (1, 1) as $a \to 1$, and the slope of the secant approaches the following limit.

$$\lim_{a\to 1}\frac{b-1}{a-1}=\lim_{a\to 1}\frac{a^2-1}{a-1}=\lim_{a\to 1}(a+1)=2$$

This limit, 2, is the slope of the tangent line at (1, 1). That is, the derivative of $y = x^2$ at (1, 1) is 2. [Note that a graphing utility's calculation of the numerical derivative of $f(x) = x^2$ with respect to x at $x = 1$ gives $f'(1) = 2$.] ■